

**ON SUMMATION OF SUBSERIES OF
THE RIEMANN ZETA FUNCTION**

Hongwei Chen

Department of Mathematics
Christopher Newport University
Newport News, VA 23606, USA

Abstract: Three classes of subseries of the Riemann zeta function are evaluated in closed form. All the results are expressed in terms of the Riemann zeta function itself and powers of π . By using the Euler totient function, we prove that these three classes are unique subseries that exhibit this kind of closed form.

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1. Introduction

Let $\zeta(s)$ be the *Riemann zeta function* defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The series converges for all $s > 1$. There are many formulas related the subseries

of the Riemann zeta function

$$\zeta(a, b, s) := \sum_{k=0}^{\infty} \frac{1}{(ak + b)^s}, \quad (a, b, s \in \mathbb{N}, \gcd(a, b) = 1) \quad (1)$$

to $\zeta(s)$ itself, for example, from the well-known expression

$$\zeta(2, 1, s) = \sum_{k=0}^{\infty} \frac{1}{(2k + 1)^s} = (1 - 2^{-s}) \zeta(s), \quad (2)$$

to more involved results such as

$$\zeta(3, 1, 3) = \sum_{k=0}^{\infty} \frac{1}{(3k + 1)^3} = \frac{1}{2}(1 - 3^{-3})\zeta(3) + \frac{2}{3^5} \sqrt{3} \pi^3, \quad (3)$$

$$\zeta(3, 1, 5) = \sum_{k=0}^{\infty} \frac{1}{(3k + 1)^5} = \frac{1}{2}(1 - 3^{-5})\zeta(5) + \frac{2}{3^7} \sqrt{3} \pi^5. \quad (4)$$

Motivated by such intriguing results, it is natural for us to ask

Q1. When does a given series $\zeta(a, b, s)$ have a closed form?

Q2. Is the closed form always a linear combination of π^s and $\zeta(s)$?

Notice that these series are all subseries of the Riemann zeta function. Appealing to the polylogarithm of Lewin, $\text{Li}_s(z)$ (see [3, p.189] or [4, p.611]) defined by

$$\text{Li}_s(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^s}. \quad (5)$$

the *series multisection formula* (see [2, Theorem 10.1, p.110]) yields, for example,

$$\zeta(3, 1, 3) = \sum_{k=0}^{\infty} \frac{1}{(3k + 1)^3} = \frac{1}{3} \left(\text{Li}_3(1) + \frac{1}{\omega} \text{Li}_3(\omega) + \frac{1}{\omega^2} \text{Li}_3(\omega^2) \right), \quad (6)$$

where $\omega = e^{2\pi i/3}$, the primitive cubic root. However, due to the limited knowledge of $\text{Li}_s(z)$ for complex arguments (see [3]), we can't recover (3) from (6).

Using the contour integral, Zucker and Robertson [7] evaluated some Dirichlet L -series exactly, which are alternating series of (1). The main results are summarized in the Appendix A of [1]. In this paper, we present another method to study $\zeta(a, b, s)$ in closed form. We will show that the affirmative answers to Q1 and Q2 only hold for $a = 3, 4, 6$ and s odd. Our approach is in purely Eulerian style, and our all manipulation of series and appearance of identities have the familiar Eulerian flair.

2. Proof of (3)

Before we proceed to develop the closed form for $\zeta(a, b, s)$, we give an elementary proof of (3), and will see that the proof of the extensions follows by analogy.

First, inspired by Euler’s infinite product formula

$$\zeta(s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1},$$

we find that

$$\begin{aligned} \zeta(3)(1 - 3^{-3}) &= 1 + \frac{1}{2^3} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{8^3} + \frac{1}{10^3} + \frac{1}{11^3} + \dots \\ &= \left(1 + \frac{1}{4^3} + \frac{1}{7^3} + \dots\right) + \left(\frac{1}{2^3} + \frac{1}{5^3} + \frac{1}{8^3} + \dots\right) \\ &= \sum_{k=0}^{\infty} \frac{1}{(3k + 1)^3} + \sum_{k=0}^{\infty} \frac{1}{(3k + 2)^3}. \end{aligned} \tag{7}$$

On the other hand, recall that *Hurwitz zeta function* (see [4, Chapter 25.11, p.607]), which is defined by

$$\zeta(s, a) := \sum_{k=0}^{\infty} \frac{1}{(k + a)^s}.$$

First we will need a lemma that converts the series into an integral.

Lemma 1. *For $s > 1$ we have*

$$\zeta(s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1} e^{-ax}}{1 - e^{-x}} dx, \tag{8}$$

where Γ is the *Euler gamma function* defined by

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt.$$

Proof. Note that

$$\frac{1}{1 - e^{-x}} = \sum_{k=0}^{\infty} e^{-kx}. \tag{9}$$

Therefore,

$$\int_0^{\infty} \frac{x^{s-1} e^{-ax}}{1 - e^{-x}} dx = \sum_{k=0}^{\infty} \int_0^{\infty} x^{s-1} e^{-(k+a)x} dx.$$

The change of variable $t = (k + a)x$ yields

$$\int_0^\infty \frac{x^{s-1}e^{-ax}}{1 - e^{-x}} dx = \sum_{k=0}^\infty \frac{1}{(k + a)^s} \int_0^\infty t^{s-1}e^{-t} dt = \zeta(s, a)\Gamma(s),$$

which is equivalent to (8). □

Next, in terms of the integral representation (8), we establish a lemma connecting the reflection sum of the Hurwitz zeta function:

Lemma 2. For $0 < \mu < 1, n \in \mathbb{N}$ we have

$$\zeta(n, \mu) - (-1)^{n-1}\zeta(n, 1 - \mu) = \frac{(-1)^{n-1}\pi}{(n - 1)!} (\cot(\mu\pi))^{(n-1)}, \tag{10}$$

where $f^{(k)}$ denotes the k -th derivative.

Proof. Applying (8) we see that

$$\begin{aligned} \zeta(n, \mu) - (-1)^{n-1}\zeta(n, 1 - \mu) &= \frac{1}{(n - 1)!} \int_0^\infty \frac{x^{n-1}(e^{-\mu x} - (-1)^{n-1}e^{-(1-\mu)x})}{1 - e^{-x}} dx. \end{aligned} \tag{11}$$

Via the parametric differentiation, this can be related to

$$f(\mu) = \int_0^\infty \frac{e^{-\mu x} - e^{-(1-\mu)x}}{1 - e^{-x}} dx.$$

In terms of (9), we have

$$\begin{aligned} f(\mu) &= \int_0^\infty \frac{e^{-\mu x}}{1 - e^{-x}} dx - \int_0^\infty \frac{e^{-(1-\mu)x}}{1 - e^{-x}} dx \\ &= \sum_{k=0}^\infty \frac{1}{k + \mu} - \sum_{k=0}^\infty \frac{1}{k + 1 - \mu} \\ &= \sum_{k=0}^\infty \frac{1}{k + \mu} + \sum_{k=0}^\infty \frac{1}{-(k + 1) + \mu} \\ &= \sum_{k=-\infty}^\infty \frac{1}{k + \mu}. \end{aligned}$$

Appealing to Euler’s elegant partial fraction decomposition (see [2, Eq. 6.27, p.68])

$$\cot x = \sum_{k=-\infty}^{\infty} \frac{1}{k\pi + x},$$

we find that

$$f(\mu) = \pi \cot(\mu\pi).$$

Taking $(n - 1)$ th derivative of $f(\mu)$ gives the desired result (10). □

Now, returning to (3), notice that

$$\sum_{k=0}^{\infty} \frac{1}{(3k + 1)^3} - \sum_{k=0}^{\infty} \frac{1}{(3k + 2)^3} = \frac{1}{3^3} (\zeta(3, 1/3) - \zeta(3, 2/3)).$$

Putting $n = 3$ in (10) yields

$$\sum_{k=0}^{\infty} \frac{1}{(3k + 1)^3} - \sum_{k=0}^{\infty} \frac{1}{(3k + 2)^3} = \frac{\pi}{2! \cdot 3^3} (\cot(\mu\pi))''|_{\mu=1/3} = \frac{4}{3^5} \sqrt{3} \pi^3. \tag{12}$$

In conjunction with (7), this yields (3) as expected. As a byproduct, we also find the closed form for its reflection sum

$$\sum_{k=0}^{\infty} \frac{1}{(3k + 2)^3} = \frac{1}{2}(1 - 3^{-3})\xi(3) - \frac{2}{3^5} \sqrt{3} \pi^3. \tag{13}$$

In general, observe that if $n = 2m + 1$, applying (10) yields

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{1}{(3k + 1)^{2m+1}} - \sum_{k=0}^{\infty} \frac{1}{(3k + 2)^{2m+1}} \\ &= \frac{1}{3^{2m+1}} (\zeta(2m + 1, 1/3) - \zeta(2m + 1, 2/3)) = \frac{\pi}{3^{2m+1}(2m)!} (\cot(\mu\pi))^{(2m)}|_{\mu=1/3}. \end{aligned}$$

Coupling with

$$\sum_{k=0}^{\infty} \frac{1}{(3k + 1)^n} + \sum_{k=0}^{\infty} \frac{1}{(3k + 2)^n} = (1 - 3^{-n})\zeta(n), \tag{14}$$

we obtain

$$\sum_{k=0}^{\infty} \frac{1}{(3k+1)^{2m+1}} = \frac{1}{2}(1-3^{-(2m+1)})\zeta(2m+1) + \frac{\pi}{2 \cdot 3^{2m+1} (2m)!} (\cot(\mu\pi))^{(2m)}|_{\mu=1/3}, \tag{15}$$

$$\sum_{k=0}^{\infty} \frac{1}{(3k+2)^{2m+1}} = \frac{1}{2}(1-3^{-(2m+1)})\zeta(2m+1) - \frac{\pi}{2 \cdot 3^{2m+1} (2m)!} (\cot(\mu\pi))^{(2m)}|_{\mu=1/3}. \tag{16}$$

In particular, setting $m = 2$ and using that $(\cot(\mu\pi))^{(4)}|_{\mu=1/3} = 32\sqrt{3}\pi^4/3$, we arrive at (4).

However, if $n = 2m$, (10) yields the same equation as (14). In contrast to (2) where $a = 2$, we are unable to find the closed form for

$$\zeta(3, 1, 2m) = \sum_{k=0}^{\infty} \frac{1}{(3k+1)^{2m}}$$

in terms of $\zeta(2m)$ and π^{2m} , even for $m = 1$.

As we end this section, for rational μ , we show how the values of $\cot(\mu\pi)^{(n)}$ are related to the *Bernoulli polynomials* — another favorite of Euler. This lemma will enable us to answer Q2.

Lemma 3. *If $a, b, n \in \mathbb{N}, 1 \leq b < a$ and $(a, b) = 1$, then*

$$(\cot(\mu\pi))^{(2n)}|_{\mu=b/a} = (-1)^{n-1} \frac{2(2a\pi)^{2n}}{2n+1} \sum_{j=1}^a B_{2n+1} \left(\frac{j}{a}\right) \sin\left(\frac{2bj\pi}{a}\right), \tag{17}$$

$$(\cot(\mu\pi))^{(2n-1)}|_{\mu=b/a} = (-1)^n \frac{(2a\pi)^{2n-1}}{n} \sum_{j=1}^a B_{2n} \left(\frac{j}{a}\right) \cos\left(\frac{2bj\pi}{a}\right). \tag{18}$$

Here $B_n(x)$ is the Bernoulli polynomial of degree n (see [4, Chapter 24]).

Proof. The approach is based on another way to evaluate $\zeta(n, \mu) \pm \zeta(n, 1 - \mu)$. Notice that

$$\zeta(n, b/a) = a^n \zeta(a, b, n).$$

Applying the series multisection formula and (5), we have

$$\zeta(n, b/a) = a^{n-1} \sum_{j=1}^a \omega^{-jb} \text{Li}_n(\omega^j),$$

where $\omega = e^{2\pi i/a}$. Simplification yields

$$\zeta(n, b/a) + \zeta(n, 1 - b/a) = 2a^{n-1} \sum_{j=1}^a \left(\sum_{k=1}^{\infty} \frac{\cos(2kj\pi/a)}{k^n} \right) \cos\left(\frac{2bj\pi}{a}\right), \quad (19)$$

$$\zeta(n, b/a) - \zeta(n, 1 - b/a) = 2a^{n-1} \sum_{j=1}^a \left(\sum_{k=1}^{\infty} \frac{\sin(2kj\pi/a)}{k^n} \right) \sin\left(\frac{2bj\pi}{a}\right). \quad (20)$$

The proposed formulas (17) and (18) now follow from the Fourier series for $B_n(x)$ (see [4, Eq. 24.8.1 and Eq. 24.8.2, p.592]):

$$B_{2n}(x) = (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos(2kx\pi)}{k^{2n}},$$

$$B_{2n-1}(x) = (-1)^n \frac{2(2n-1)!}{(2\pi)^{2n-1}} \sum_{k=1}^{\infty} \frac{\sin(2kx\pi)}{k^{2n-1}}. \quad \square$$

When $a = 3, b = 1$, in view of the well-known fact that $B_{2m+1}(1) = 0$ and the formulas

$$B_n(0) = (-1)^n B_n(1) = B_n, \quad (\text{see [4, Eq. 24.4.25, p.590]}),$$

$$B_n(x) = (-1)^n B_n(1-x), \quad (\text{see [4, Eq. 24.4.3, p.589]}), \quad (21)$$

$$B_{2m}(1/3) = \frac{1}{2}(3^{1-2m} - 1)B_{2m}, \quad (\text{see [4, Eq. 24.4.29, p.590]}), \quad (22)$$

where B_n is the Bernoulli number, applying (17) and (18) we have

$$(\cot(\mu\pi))^{(2m)}|_{\mu=1/3} = (-1)^{m-1} \frac{2 \cdot 6^{2m} B_{2m+1}(1/3)}{2m+1} \sqrt{3} \pi^{2m}, \quad (23)$$

$$(\cot(\mu\pi))^{(2m-1)}|_{\mu=1/3} = (-1)^m \frac{2^{2m-1}(3^{2m} - 1)}{2m} B_{2m} \pi^{2m-1}. \quad (24)$$

In conjunction with (14), by (24) we recover another Euler's famous result:

$$\zeta(2m) = \frac{1}{1 - 3^{-2m}} \left(\sum_{k=0}^{\infty} \frac{1}{(3k+1)^{2m}} + \sum_{k=0}^{\infty} \frac{1}{(3k+2)^{2m}} \right)$$

$$\begin{aligned}
 &= \frac{3^{2m}}{3^{2m}-1} \frac{1}{3^{2m}} (\zeta(2m, 1/3) + \zeta(2m, 2/3)) \\
 &= -\frac{\pi}{(3^{2m}-1)(2m-1)!} (\cot(\mu\pi))^{(2m-1)}|_{\mu=1/3} \\
 &= (-1)^{m+1} \frac{2^{2m-1}}{(2m)!} B_{2m} \pi^{2m}.
 \end{aligned}$$

Furthermore, (23) and (24) enable us to rewrite (15) and (16) as

$$\begin{aligned}
 \sum_{k=0}^{\infty} \frac{1}{(3k+1)^{2m+1}} &= \frac{1}{2} (1 - 3^{-(2m+1)}) \zeta(2m+1) \\
 &\quad + (-1)^{m-1} \frac{2^{2m}}{3 \cdot (2m+1)!} B_{2m+1}(1/3) \sqrt{3} \pi^{2m+1}, \tag{25}
 \end{aligned}$$

$$\begin{aligned}
 \sum_{k=0}^{\infty} \frac{1}{(3k+2)^{2m+1}} &= \frac{1}{2} (1 - 3^{-(2m+1)}) \zeta(2m+1) \\
 &\quad + (-1)^m \frac{2^{2m}}{3 \cdot (2m+1)!} B_{2m+1}(1/3) \sqrt{3} \pi^{2m+1}. \tag{26}
 \end{aligned}$$

These formulas yield the affirmative answer to Q1 and Q2 for $a = 3, s = 2m + 1$.

3. Extensions to $a = 4, 6$

Observe that finding the closed form of $\zeta(3, 1, n)$ is based on a system of two equations for $\zeta(3, 1, n)$ and its reflection sum $\zeta(3, 2, n)$. Here, when n is odd, (10) provides one equation, while the decomposition (14) supplies another. Along these lines, in this section we establish closed formulas of $\zeta(a, b, n)$ for $a = 4, 6$ and n odd. First, similar to (7), we have

$$\begin{aligned}
 \zeta(n)(1 - 2^{-n}) &= 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{15^n} + \dots \\
 &= \left(1 + \frac{1}{5^n} + \frac{1}{9^n} + \dots\right) + \left(\frac{1}{3^n} + \frac{1}{7^n} + \frac{1}{11^n} + \dots\right) \\
 &= \sum_{k=0}^{\infty} \frac{1}{(4k+1)^n} + \sum_{k=0}^{\infty} \frac{1}{(4k+2)^n}. \tag{27}
 \end{aligned}$$

On the other hand, using (10), we find that

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)^n} - (-1)^{n-1} \sum_{k=0}^{\infty} \frac{1}{(4k+2)^n} = \frac{(-1)^{n-1} \pi}{4^n (n-1)!} (\cot(\mu x))^{(n-1)}|_{\mu=1/4}. \tag{28}$$

If $n = 2m + 1$, solving the coupled system of (27) and (28) yields

$$\sum_{k=0}^{\infty} \frac{1}{(4k + 1)^{2m+1}} = \frac{1}{2}(1 - 2^{-(2m+1)})\zeta(2m + 1) + \frac{\pi}{2 \cdot 4^{2m+1} (2m)!} (\cot(\mu\pi))^{(2m)}|_{\mu=1/4}, \tag{29}$$

$$\sum_{k=0}^{\infty} \frac{1}{(4k + 3)^{2m+1}} = \frac{1}{2}(1 - 2^{-(2m+1)})\zeta(2m + 1) - \frac{\pi}{2 \cdot 4^{2m+1} (2m)!} (\cot(\mu\pi))^{(2m)}|_{\mu=1/4}. \tag{30}$$

By (17) and (21), we have

$$\begin{aligned} (\cot(\mu\pi))^{(2m)}|_{\mu=1/4} &= (-1)^{m-1} \frac{2 \cdot 8^{2m}}{2m + 1} \pi^{2m} (B_{2m+1}(1/4) - B_{2m+1}(3/4)) \\ &= (-1)^{m-1} \frac{4 \cdot 8^{2m} B_{2m+1}(1/4)}{2m + 1} \pi^{2m}. \end{aligned}$$

This transforms (29) and (30) into

$$\sum_{k=0}^{\infty} \frac{1}{(4k + 1)^{2m+1}} = \frac{1}{2}(1 - 2^{-(2m+1)})\zeta(2m + 1) + (-1)^{m-1} \frac{2^{2m+1}}{4 \cdot (2m + 1)!} B_{2m+1}(1/4) \pi^{2m+1}, \tag{31}$$

$$\sum_{k=0}^{\infty} \frac{1}{(4k + 3)^{2m+1}} = \frac{1}{2}(1 - 2^{-(2m+1)})\zeta(2m + 1) + (-1)^m \frac{2^{2m+1}}{4 \cdot (2m + 1)!} B_{2m+1}(1/4) \pi^{2m+1}. \tag{32}$$

Therefore, when $a = 4, n = 2m + 1$, we obtain the positive answers to Q1 and Q2 again. In particular, letting $m = 1, 2$, we have

$$\zeta(4, 1, 3) = \sum_{k=0}^{\infty} \frac{1}{(4k + 1)^3} = \frac{1}{2}(1 - 2^{-3})\zeta(3) + \frac{1}{26}\pi^3, \tag{33}$$

$$\zeta(4, 3, 3) = \sum_{k=0}^{\infty} \frac{1}{(4k + 3)^3} = \frac{1}{2}(1 - 2^{-3})\zeta(3) - \frac{1}{26}\pi^3, \tag{34}$$

$$\zeta(4, 1, 5) = \sum_{k=0}^{\infty} \frac{1}{(4k + 1)^5} = \frac{1}{2}(1 - 2^{-5})\zeta(5) + \frac{5}{3 \cdot 2^{10}}\pi^5, \tag{35}$$

$$\zeta(4, 3, 3) = \sum_{k=0}^{\infty} \frac{1}{(4k + 3)^5} = \frac{1}{2}(1 - 2^{-5})\zeta(3) - \frac{5}{3 \cdot 2^{10}}\pi^5. \tag{36}$$

On the other hand, if n is even, for example, we have

$$\zeta(4, 1, 2) = \sum_{k=0}^{\infty} \frac{1}{(4k + 1)^2} = \frac{1}{2}(1 - 2^{-2})\zeta(2) + \frac{1}{2}G, \tag{37}$$

where G is the well-known *Catalan number*, and so Q2 does not valid. Moreover, we are unable to find the closed form for $\zeta(4, 1, 2m)$ for any $m > 1$.

We now turn to find the closed form for $\zeta(a, b, n)$ when $a = 6$. Applying (10) yields

$$\sum_{k=0}^{\infty} \frac{1}{(6k + 1)^n} - (-1)^{n-1} \sum_{k=0}^{\infty} \frac{1}{(6k + 5)^n} = \frac{(-1)^{n-1}\pi}{6^n (n - 1)!} (\cot(\mu\pi))^{(n-1)}|_{\mu=1/6}. \tag{38}$$

On the other hand, stemming from Euler’s infinite product formula again gives

$$\begin{aligned} \zeta(n)(1 - 2^{-n})(1 - 3^{-n}) &= 1 + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \dots \\ &= \left(1 + \frac{1}{7^n} + \frac{1}{13^n} + \dots\right) + \left(\frac{1}{5^n} + \frac{1}{11^n} + \frac{1}{17^n} + \dots\right) \\ &= \sum_{k=0}^{\infty} \frac{1}{(6k + 1)^n} + \sum_{k=0}^{\infty} \frac{1}{(6k + 5)^n}. \end{aligned} \tag{39}$$

When $n = 2m + 1$, combining (38) and (39) we have

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{(6k + 1)^{2m+1}} &= \frac{1}{2} (1 - 2^{-(2m+1)})(1 - 3^{-(2m+1)})\zeta(2m + 1) \\ &\quad + \frac{\pi}{2 \cdot 6^{2m+1} (2m)!} (\cot(\mu\pi))^{(2m)}|_{\mu=1/6}, \end{aligned} \tag{40}$$

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{(6k + 5)^{2m+1}} &= \frac{1}{2} (1 - 2^{-(2m+1)})(1 - 3^{-(2m+1)})\zeta(2m + 1) \\ &\quad - \frac{\pi}{2 \cdot 6^{2m+1} (2m)!} (\cot(\mu\pi))^{(2m)}|_{\mu=1/6}. \end{aligned} \tag{41}$$

Applying (17) and (21) once more we have

$$(\cot(\mu\pi))^{(2m)}|_{\mu=1/6} = (-1)^{m-1} \frac{2 \cdot 12^{2m} \sqrt{3}}{2m + 1} (B_{2m+1}(1/6) + B_{2m+1}(1/3))\pi^{2m}.$$

Setting $m = 2$ in the *Raabe formula* (see [4, Eq. 24.4.18, p.590])

$$B_n(x) = \frac{1}{m^n} \sum_{k=0}^{m-1} B_n \left(x + \frac{k}{m} \right)$$

gives

$$B_{2m+1}(1/6) = (1 + 2^{-2m})B_{2m+1}(1/3),$$

and thus

$$(\cot(\mu\pi))^{(2m)}|_{\mu=1/6} = (-1)^{m-1} \frac{2 \cdot 12^{2m} \sqrt{3}}{2m + 1} (2 + 2^{-2m})B_{2m+1}(1/3)\pi^{2m}.$$

This converts (40) and (41) into

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{(6k + 1)^{2m+1}} &= \frac{1}{2} (1 - 2^{-(2m+1)})(1 - 3^{-(2m+1)})\zeta(2m + 1) \\ &\quad + (-1)^{m-1} \frac{(1 + 2^{2m+1})}{6 \cdot (2m + 1)!} B_{2m+1}(1/3) \sqrt{3} \pi^{2m+1}, \end{aligned} \tag{42}$$

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{(6k + 5)^{2m+1}} &= \frac{1}{2} (1 - 2^{-(2m+1)})(1 - 3^{-(2m+1)})\zeta(2m + 1) \\ &\quad + (-1)^m \frac{(1 + 2^{2m+1})}{6 \cdot (2m + 1)!} B_{2m+1}(1/3) \sqrt{3} \pi^{2m+1}. \end{aligned} \tag{43}$$

These yield a positive answer to Q1 and Q2 again. In particular, if $m = 1, 2$, we have

$$\sum_{k=0}^{\infty} \frac{1}{(6k + 1)^3} = \frac{1}{2} (1 - 2^{-3})(1 - 3^{-3})\zeta(3) + \frac{1}{2^2 \cdot 3^3} \sqrt{3} \pi^3, \tag{44}$$

$$\sum_{k=0}^{\infty} \frac{1}{(6k + 1)^5} = \frac{1}{2} (1 - 2^{-5})(1 - 3^{-5})\zeta(5) + \frac{11}{2^4 \cdot 3^6} \sqrt{3} \pi^5. \tag{45}$$

The last one was recently proposed as the Monthly Problem 11715 [5].

4. Uniqueness

In this section, we show that $\zeta(a, b, n)$ admitting a closed form in terms of π^n and $\zeta(n)$ only occurs for $a = 3, 4, 5$ and n odd.

For Q1 and Q2, (37) offers a counterexample when n is even. Thus, we assume that n is odd in what follows. The reflection formula (10) and Lemma 3 demonstrate that $\zeta(a, b, n) - \zeta(a, a - b, n)$ is related to the sine and cosine at rational multiples of π . To obtain a closed form for $\zeta(a, b, n)$ alone, we need to have another approach to sum $\zeta(a, b, n) + \zeta(a, a - b, n)$ in terms of $\zeta(n)$. This is equivalent to decomposing $\zeta(n)$ as a sum of $\zeta(a, b, n)$ and $\zeta(a, a - b, n)$. We show this occurs only for $a = 3, 4, 6$. To this end, we recall another tool from Euler. Let $\phi(a)$ be the Euler totient function, i.e., $\phi(a)$ equals the number of positive integers less than a that are relatively prime to a . Observe that

$$\zeta(a, b, n) + \zeta(a, a - b, n) = \sum_{k=0}^{\infty} \left(\frac{1}{(ak + b)^n} + \frac{1}{(ak + (a - b))^n} \right) = c\zeta(n)$$

if and only if $\phi(a) = 2$. Thus, the desired result follows from the simple fact:

$$\phi(a) = 2 \iff a = 3, 4, 6.$$

As an illustrative example, by (10) we have

$$\zeta(12, 5, 3) - \zeta(12, 7, 3) = \frac{7 - 4\sqrt{3}}{3^3 \cdot 4^2} \pi^3. \tag{46}$$

However, since $\phi(12) = 4$, using (39), the decomposition of $\zeta(3)$ becomes

$$(1 - 2^{-3})(1 - 3^{-3})\zeta(3) = \zeta(12, 1, 3) + \zeta(12, 5, 3) + \zeta(12, 7, 3) + \zeta(12, 11, 3),$$

instead of $\zeta(12, 5, 3) + \zeta(12, 7, 3)$ as expected. So we can't find $\zeta(12, 5, 3)$ in closed form alone.

Investigating further, the problem of determining the rational numbers b/a such that the sine and cosine values at $b\pi/a$ are radicals was considered by Gauss ([7]). He associated this problem to constructible regular polygon by compass and ruler, and proved that a must be of the form

$$a = 2^k p_1 p_2 \cdots p_l$$

where $k \in \mathbb{N}$ and p_i are distinct Fermat primes (a prime of the form $2^{2^n} + 1$). In conjunction with (10) and Lemma 3, we see that Q2 is true only for $a = 3, 4, 6$ again.

Based on Euler's idea and tools, we have found a class of closed forms for $\zeta(a, b, n)$. It is interesting to notice that today we have resources not available to Euler. For example, Mathematica. It is easy to verify our results (3), (4), (33) — (36), (44) and (45) via Mathematica. However, to evaluate (46), Mathematica only returns

$$-(\text{PolyGamma}[2, 5/12]/3456) + \text{PolyGamma}[2, 7/12]/3456$$

It is hard to see how to reduce this answer to match the result in (46).

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