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ON THE DIOPHANTINE EQUATION $5^x + 43^y = z^2$

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Abstract: In this paper, we show that the Diophantine equation $5^x+43^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.

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1. Introduction

In [1], Acu solved the Diophantine equation $2^x + 5^y = z^2$ where x, y and z are non-negative integers. The solutions (x, y, z) are $(3, 0, 3)$ and $(2, 1, 3)$.

In [16], Suvarnamani, Singta and Chotchaisthit proved that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution where x, y and z are non-negative integers.

In [7], Sroysang solved the Diophantine equation $3^x + 5^y = z^2$ where x, y and z are non-negative integers. The solution (x, y, z) is $(1, 0, 2)$. In [9], Sroysang proved that the Diophantine equation $5^x + 7^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers. In [10], Sroysang proved that the Diophantine equation $5^x + 23^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers. Moreover, we refer to [5, 6, 8, 11, 12, 13, 14, 15].

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In [4], Rabago proved that the two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$ have exactly two solutions (x, y, z) where x, y and z are non-negative integers. The solutions are in $\{(1, 0, 2), (4, 1, 10)\}$ and $\{(1, 0, 2), (2, 1, 10)\}$, respectively.

In [2], Chotchaisthit solved the Diophantine equation $2^x + 11^y = z^2$ where x, y and z are non-negative integers. The solution (x, y, z) is $(3, 0, 3)$.

In this paper, we show that the Diophantine equation $5^x + 43^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.

2. Preliminaries

Proposition 2.1. [3] (**Catalan's conjecture**) $(3, 2, 2, 3)$ *is a unique solution* (a, b, x, y) *for the Diophantine equation* $a^x - b^y = 1$ *where* a, b, x *and* y *are integers such that* $\min\{a, b, x, y\} > 1$.

Lemma 2.2. [10] *The Diophantine equation* $5^x + 1 = z^2$ *has no non-negative integer solution where* x *and* z *are non-negative integers.*

Lemma 2.3. *The Diophantine equation* $1 + 43^y = z^2$ *has no non-negative integer solution where* y *and* z *are non-negative integers.*

Proof. Suppose that there are non-negative integers y and z such that $1 + 43^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. It follows that $y \geq 1$. Then $z^2 = 1 + 43^y \geq 1 + 43^1 = 44$. Thus, $z \geq 7$. Now, we consider on the equation $z^2 - 43^y = 1$. By Proposition 2.1, we obtain that $y = 1$. Thus, $z^2 = 44$. This is a contradiction. Hence, the equation $1 + 43^y = z^2$ has no non-negative integer solution. □

3. Main Results

Theorem 3.1. *The Diophantine equation* $5^x + 43^y = z^2$ *has no non-negative integer solution where* y *and* z *are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and z such that $5^x + 43^y = z^2$. By Lemma 2.2, we have $y \geq 1$. Note that z is even. Then $z^2 \equiv 0 \pmod 4$. Since $5^x \equiv 1 \pmod 4$, it follows that $43^y \equiv 3 \pmod 4$. Then $3^y \equiv 3 \pmod 4$. This implies that y is odd. Note that $43^y \equiv 2 \pmod 5$ or $43^y \equiv 3 \pmod 5$. By Lemma 2.3, we have $x \geq 1$. Since $5^x \equiv 0 \pmod 5$, it follows that

$z^2 \equiv 0 \pmod 5$ or $z^2 \equiv 1 \pmod 5$ or $z^2 \equiv 4 \pmod 5$. This is a contradiction. Hence, the equation $5^x + 43^y = z^2$ has no non-negative integer solution. □

Corollary 3.2. *Let k be a positive integer. The Diophantine equation $5^x + 43^y = w^{2k+2}$ has no non-negative integer solution where x, y and w are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and w such that $5^x + 43^y = w^{2k+2}$. Let $z = w^{k+1}$. Then $5^x + 43^y = z^2$. This is a contradiction with Theorem 3.1. Hence, the equation $5^x + 43^y = w^{2k+2}$ has no non-negative integer solution. □

Corollary 3.3. *The Diophantine equation $25^u + 43^y = z^2$ has no non-negative integer solution where u, y and z are non-negative integers.*

Proof. Suppose that there are non-negative integers u, y and z such that $25^u + 43^y = z^2$. Let $x = 2u$. Then $5^x + 43^y = z^2$. This is a contradiction with Theorem 3.1. Hence, the equation $25^u + 43^y = z^2$ has no non-negative integer solution. □

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