International Journal of Pure and Applied Mathematics
Volume 91 No. 4 2014, 541-544
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v91i4.11
ijpam.eu

ON THE DIOPHANTINE EQUATION $5^x + 63^y = z^2$

Banyat Sroysang
Department of Mathematics and Statistics
Faculty of Science and Technology
Thammasat University
Rangsit Center, Pathumthani 12121, THAILAND

Abstract: In this paper, we prove that the Diophantine equation $5^{\mathsf{x}}+63^{\mathsf{y}} = z^2$ has a unique non-negative integer solution where x, y and z are non-negative integers. The solution (x, y, z) is $(0, 1, 8)$.

AMS Subject Classification: 11D61
Key Words: exponential Diophantine equation

1. Introduction

In 2007, Acu [1] proved that $(3, 0, 3)$ and $(2, 1, 3)$ are only two solutions (x, y, z) for the Diophantine equation $2^{\mathsf{x}} + 5^{\mathsf{y}} = z^2$ where x, y and z are non-negative integers.

In 2012, Sroysang [7] proved that $(1, 0, 2)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $3^{\mathsf{x}} + 5^{\mathsf{y}} = z^2$ where x, y and z are non-negative integers.

In 2013, Sroysang [9, 10] proved that (i) the Diophantine equation $5^{\mathsf{x}} + 7^{\mathsf{y}} = z^2$ has no non-negative integer solution, and (ii) the Diophantine equation $5^{\mathsf{x}} + 23^{\mathsf{y}} = z^2$ has no non-negative integer solution, where x, y and z are non-negative integers.

Received: January 14, 2014

In the same year, Rabago [4] proved that $(1,0,3)$, $(1,1,5)$, $(2,1,9)$ and $(3,1,23)$ are only four solutions (x,y,z) for the Diophantine equation $8^x+17^y = z^2$ where x,y and z are non-negative integers.

In the same year, Chotchaisthit [2] proved that the Diophantine equation $2^x + 11^y = z^2$ has a unique non-negative integer solution where x, y and z are non-negative integers. The solution (x,y,z) is $(3,0,3)$.

Recently, we [5, 6, 8, 11, 12, 13, 14, 15] solved some Diophantine equations. In this paper, we prove that the Diophantine equation $5^x+63^y = z^2$ has a unique non-negative integer solution where x, y and z are non-negative integers. The solution (x,y,z) is $(0,1,8)$.

2. Preliminaries

Proposition 2.1. [3] (**Catalan's conjecture**) $(3,2,2,3)$ *is a unique solution* (a,b,x,y) *for the Diophantine equation* $a^x - b^y = 1$ *where* a,b,x *and* y *are integers such that* $\min\{a,b,x,y\} > 1$.

Lemma 2.2. [9] *The Diophantine equation* $5^x + 1 = z^2$ *has no non-negative integer solution where* x *and* z *are non-negative integers.*

Lemma 2.3. $(1,8)$ *is a unique solution* (y,z) *for the Diophantine equation* $1 + 63^y = z^2$ *where* y *and* z *are non-negative integers.*

Proof. Let y and z be non-negative integers such that $1+63^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. It follows that $y \geq 1$. Thus, $z^2 = 1 + 63^y \geq 1 + 63^1 = 64$. We obtain that $z \geq 8$. Now, we consider on the equation $z^2 - 63^y = 1$. By Proposition 2.1, we obtain that $y = 1$ and then $z = 8$. Hence, $(1,8)$ is a unique solution (y,z) for the equation $1 + 63^y = z^2$. □

3. Main Results

Theorem 3.1. $(0,1,8)$ *is a unique non-negative integer solution* (x,y,z) *for the Diophantine equation* $5^x + 63^y = z^2$ *where* x,y *and* z *are non-negative integers.*

Proof. Let x,y and z be non-negative integers such that $5^x + 63^y = z^2$. By Lemma 2.2, we obtain that $y \geq 1$. Note that z is even. It follows that $z^2 \equiv 0 \pmod 4$. Note that $5^x \equiv 1 \pmod 4$. Then $63^y \equiv 3 \pmod 4$. Thus, y is odd. Now, we will divide the number x into two cases.

Case $x = 0$. By Lemma 2.3, we obtain that $y = 1$ and $z = 8$.

Case $x \geq 1$. Note that $5^x \equiv 0 \pmod 5$. Note that $3^y \equiv 2 \pmod 5$ or $3^y \equiv 3 \pmod 5$. We obtain that $z^2 \equiv 2 \pmod 5$ or $z^2 \equiv 3 \pmod 5$. This is a contradiction since $z^2 \equiv 0 \pmod 5$ or $z^2 \equiv 1 \pmod 5$ or $z^2 \equiv 4 \pmod 5$.

Hence, $(0, 1, 8)$ is a unique non-negative integer solution (x, y, z) for the equation $5^x + 63^y = z^2$. □

Corollary 3.2. *The Diophantine equation* $5^x + 63^y = w^4$ *has no non-negative integer solution where* x, y *and* w *are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and w such that $5^x + 63^y = w^4$. Let $z = w^2$. Then $5^x + 63^y = z^2$. By Theorem 3.1, we obtain that $(x, y, z) = (0, 1, 8)$. Hence, $w^2 = z = 8$. This is a contradiction. □

Corollary 3.3. $(0, 1, 2)$ *is a unique non-negative integer solution* (x, y, z) *for the Diophantine equation* $5^x + 63^y = u^6$ *where* x, y *and* u *are non-negative integers.*

Proof. Let x, y and u be non-negative integers such that $5^x + 63^y = u^6$. Let $z = u^3$. Then $5^x + 63^y = z^2$. By Theorem 3.1, we obtain that$(x, y, z) = (0, 1, 8)$. Then $u^3 = z = 8$. Hence, $u = 2$. □

References

[1] D. Acu, On a Diophantine equation $2^x + 5^y = z^2$, *Gen. Math.*, **15** (2007), 145–148.

[2] S. Chotchaisthit, On the Diophantine equation $2^x + 11^y = z^2$, *Maejo Int. J. Sci. Technol.*, **7** (2013), 291–293.

[3] P. Mihailescu, Primary cycolotomic units and a proof of Catalan's conjecture, *J. Reine Angew. Math.*, **27** (2004), 167–195.

[4] J. F. T. Rabago, On an open problem by B. Sroysang, *Konuralp J. Math.*, **1** (2013), 30–32.

[5] B. Sroysang, More on the Diophantine equation $2^x + 3^y = z^2$, *Int. J. Pure Appl. Math.*, **84** (2013), 133–137.

[6] B. Sroysang, More on the Diophantine equation $8^x + 19^y = z^2$, *Int. J. Pure Appl. Math.*, **81** (2012), 601–604.

[7] B. Sroysang, On the Diophantine equation $3^{\mathsf{x}} + 5^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **81** (2012), 605−608.

[8] B. Sroysang, On the Diophantine equation $3^{\mathsf{x}} + 17^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 111−114.

[9] B. Sroysang, On the Diophantine equation $5^{\mathsf{x}} + 7^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 115−118.

[10] B. Sroysang, On the Diophantine equation $5^{\mathsf{x}} + 23^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 119−122.

[11] B. Sroysang, On the Diophantine equation $7^{\mathsf{x}} + 8^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **84** (2013), 111−114.

[12] B. Sroysang, On the Diophantine equation $23^{\mathsf{x}} + 32^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **84** (2013), 231−234.

[13] B. Sroysang, On the Diophantine equation $31^{\mathsf{x}} + 32^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **81** (2012), 609−612.

[14] B. Sroysang, On the Diophantine equation $47^{\mathsf{x}} + 49^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 279−282.

[15] B. Sroysang, On the Diophantine equation $89^{\mathsf{x}} + 91^{\mathsf{y}} = z^2$, *Int. J. Pure Appl. Math.*, **89** (2013), 283−286.