International Journal of Pure and Applied Mathematics
Volume 91 No. 4 2014, 495-504
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v91i4.6
ijpam.eu

PRODUCT CORDIAL LABELING OF GRAPHS RELATED TO HELM, CLOSED HELM AND GEAR GRAPH

G.V. Ghodasara[1 §], S.M. Vaghasiya[2]
[1]H.&H.B. Kotak Institute of Science
Rajkot, Gujarat, INDIA
[2]R.K. University
Rajkot, Gujarat, INDIA

Abstract: We prove that path union of finite copies of helm, closed helm, gear graph are product cordial graphs. Further we prove that the graph obtained by joining two copies of helm by a path of arbitrary length is product cordial graph. We prove similar results for closed helm and gear graph.

AMS Subject Classification: 05C78
Key Words: product cordial labeling, helm, closed helm, gear graph

1. Introduction

We consider simple, finite, undirected graph $G = (V, E)$. In this paper P_n denotes path with n vertices. For all other terminology and notations we follow Harary[1]. First we will provide some definitions useful for the present work.

Definition 1. Let G be a graph and $G_1, G_2, \dots, G_n$, $n \geq 2$ be n copies of graph G. The graph obtained by adding an edge from G_i to G_{i+1}(for $i = 1, 2, \dots, n-1$) is called *path union of G*.

Definition 2. If the vertices of the graph are assigned values subject to certain conditions is known as *graph labeling*.

Detailed survey on graph labeling is given and updated by Gallian[2].

Received: December 4, 2013

§Correspondence author

Definition 3. Let $G = (V, E)$ be a graph. A mapping $f : V(G) \to \{0, 1\}$ is called *binary vertex labeling* of G and $f(v)$ is called *label* of the vertex v of G under f.

Definition 4. A binary vertex labeling $f : V(G) \to \{0, 1\}$ of graph G with induced edge labeling $f^* : E(G) \longrightarrow \{0, 1\}$ defined by $f^*(uv) = f(u)f(v)$ is called a *product cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$, where $v_f(0)$, $v_f(1)$ denote the number of vertices of G having labels 0, 1 respectively under f and $e_f(0)$, $e_f(1)$ denote the number of edges of G having labels 0, 1 respectively under f^*.

A graph G is *product cordial* if it admits product cordial labeling.

Definition 5. A *helm* H_n, $n \geq 3$ is the graph obtained from the wheel W_n by adding a pendant edge at each vertex on the rim of the wheel W_n.

Definition 6. A *closed helm* CH_n is the graph obtained by taking a helm H_n and adding edges between the pendant vertices.

Definition 7. A *gear graph* is obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices of rim of the wheel W_n.

The concept of product cordial labeling was introduced by R. Ponraj, M. Sivakumar, M. Sundaram[3]. Vaidya and Barasara[4] proved that friendship graph, cycle with one chord, cycle with twin chord are product cordial graphs. They also proved middle graph of path P_n is product cordial graph. Vaidya and Dani[5] proved that the graph obtained by joining apex vertices of two stars is product cordial. They also proved similar results for shell and wheel. In [6], same authors proved that path union of k copies of cycle C_n, the graph obtained by joining two copies of cycle C_n by path P_k, the path union of k copies of $D_2(C_n)$ are product cordial graphs. Vaidya and Vyas[7] proved that the graphs obtained by joining the connected components of respective graphs by a path of arbitrary length is product cordial.

In the present paper we discuss cordial labeling for path union of helm, path union of closed helm and path union of gear graph. Further we prove that the graph obtained by joining two copies of helm by path of arbitrary length is product cordial. Similar results are investigated for closed helm and gear graph.

2. Main Results

Theorem 1. *The path union of k copies of helm H_n admits product cordial labeling.*

Proof. Let G be the path union of k copies $G_1, G_2, \ldots, G_k$ of helm H_n. Let $\{v_{i0}, v_{i1}, \ldots, v_{in}, v_{i1}^{'}, v_{i2}^{'}, \ldots, v_{in}^{'}\}$ denote the vertices of G. Let $e_i = v_{i0}v_{(i+1)0}$ be the edge joining G_i and G_{i+1}. where v_{i0} is apex vertex, $\{v_{i1}, v_{i2}, \ldots, v_{in}\}$ are internal vertices and $\{v_{i1}^{'}, v_{i2}^{'}, \ldots, v_{in}^{'}\}$ are external vertices. We define labeling function $f : V(G) \to \{0, 1\}$ as follows.

Case 1: k is odd.

Subcase 1: n is odd.

$f(v_{ij}) = 1; 1 \le i \le \frac{k-1}{2}, 0 \le j \le n$
$\quad = 0; \frac{k+3}{2} \le i \le k \le, 0 \le j \le n$
$f(v_{ij})^{'} = 1; 1 \le i \le \frac{k-1}{2}, 1 \le j \le n$
$\quad = 0; \frac{k+3}{2} \le i \le k, 1 \le j \le n$
$f(v_{(\frac{k+1}{2})1}) = 0,\ f(v_{(\frac{k+1}{2})1})^{'} = 0,\ f(v_{(\frac{k+1}{2})2})^{'} = 0$
$f(v_{(\frac{k+1}{2})j}) = 1; 0 \le j \le \frac{n+3}{2}$
$\quad = 0; \frac{n+5}{2} \le j \le n$
$f(v_{\frac{k+1}{2})j})^{'} = 1; 3 \le j \le \frac{n+3}{2}$
$\quad = 0; \frac{n+5}{2} \le j \le n$

It can be easily seen that the vertex conditions and edge conditions of product cordial labeling are satisfied in this case, i.e. $|v_f(1) - v_f(0)| \le 1$ and $|e_f(1) - e_f(0)| \le 1$.

Subcase 2: n is even.

$f(v_{ij}) = 1; 1 \le i \le \frac{k-1}{2}, 0 \le j \le n$
$\quad = 0; \frac{k+3}{2} \le i \le k \le, 0 \le j \le n$
$f(v_{ij})^{'} = 1; 1 \le i \le \frac{k-1}{2}, 1 \le j \le n$
$\quad = 0; \frac{k+3}{2} \le i \le k, 1 \le j \le n$
$f(v_{(\frac{k+1}{2})1}) = 0,\ f(v_{(\frac{k+1}{2})1})^{'} = 0,\ f(v_{(\frac{k+1}{2})2})^{'} = 0,\ f(v_{(\frac{k+1}{2})3})^{'} = 0$
$f(v_{(\frac{k+1}{2})j}) = 1; 0 \le j \le \frac{n+4}{2}, j \ne 1$
$\quad = 0; \frac{n+6}{2} \le j \le n$
$f(v_{\frac{k+1}{2})j})^{'} = 1; 4 \le j \le \frac{n+4}{2}$
$\quad = 0; \frac{n+6}{2} \le j \le n$

It can be easily seen that the vertex conditions and edge conditions of product cordial labeling are satisfied in this case, i.e. $|v_f(1) - v_f(0)| \le 1$ and $|e_f(1) - e_f(0)| \le 1$.

Case 2: k is even, $\forall n$

$f(v_{ij}) = 1; 1 \le i \le \frac{k}{2}, 0 \le j \le n$

$= 0; \frac{k}{2}+1 \le i \le k, 0 \le j \le n$
$f(v_{ij})' = 1; 1 \le i \le \frac{k}{2}, 1 \le j \le n$
$= 0; \frac{k}{2}+1 \le i \le k, 1 \le j \le n$
It can be easily seen that the vertex conditions and edge conditions of product cordial labeling are satisfied in this case, i.e. $|v_f(1) - v_f(0)| \le 1$ and $|e_f(1) - e_f(0)| \le 1$.

Hence in each case the graph G under consideration satisfies the conditions for product cordial labeling. Hence G is a product cordial graph. □

Example 1. The product cordial labeling of path union of 3-copies of helm H_8 is shown in *Figure 1.* It is the case related to k is odd, n is even.

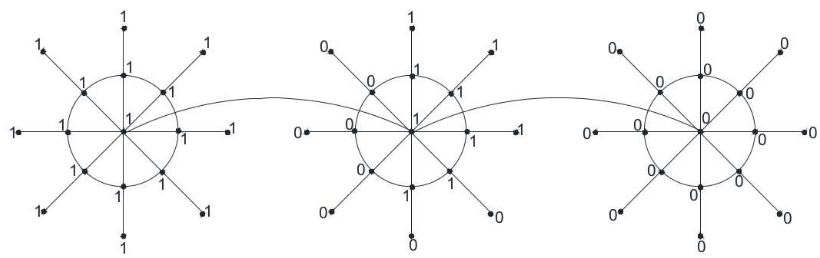

Figure 1: Product cordial labeling of path union of 3 copies of H_8

Theorem 2. *The path union of k copies of closed helm CH_n admits product cordial labeling.*

Proof. Let G be the path union of k copies $G_1, G_2, \ldots, G_k$ of closed helm CH_n. Let $\{v_{i0}, v_{i1}, \ldots, v_{in}, v_{i1}', v_{i2}', \ldots, v_{in}'\}$ denote the vertices of G. Let $e_i = v_{i0}v_{(i+1)0}$ be the edge joining G_i and G_{i+1} where v_{i0} is apex vertex, $\{v_{i1}, \ldots, v_{in}\}$ are internal vertices and $\{v_{i1}', v_{i2}', \ldots, v_{in}'\}$ are external vertices. We define labeling function $f : V(G) \to \{0, 1\}$ as follows.

Case 1: k is odd.
$f(v_{ij}) = 1; 1 \le i \le \frac{k-1}{2}, 0 \le j \le n$
$= 0; \frac{k+3}{2} \le i \le k, 0 \le j \le n$
$f(v_{ij})' = 1; 1 \le i \le \frac{k-1}{2}, 1 \le j \le n$
$= 0; \frac{k+3}{2} \le i \le k, 1 \le j \le n$
$f(v_{(\frac{k+1}{2})j}) = 1; 0 \le j \le n$
$f(v_{(\frac{k+1}{2})j})' = 0; 1 \le j \le n.$

Case 2: k is even.
$f(v_{ij}) = 1; 1 \leq i \leq \frac{k}{2}, 0 \leq j \leq n$
$= 0; \frac{k}{2}+1 \leq i \leq k, 0 \leq j \leq n$
$f(v_{ij})' = 1; 1 \leq i \leq \frac{k}{2}, 1 \leq j \leq n$
$= 0; \frac{k}{2}+1 \leq i \leq k, 1 \leq j \leq n$
The graph G under consideration satisfies the conditions $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$ in each case. Hence the graph G is product cordial graph. □

Example 2. The product cordial labeling of the path union of 3-copies of closed helm CH_5 is shown in *Figure 2*. It is the case related to k is odd.

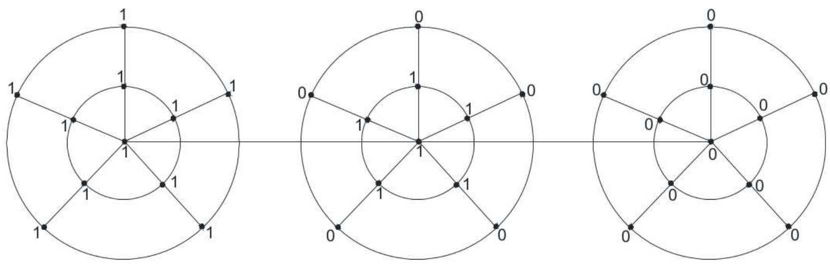

Figure 2: Product cordial labeling of path union of 3 copies of closed helm CH_5.

Theorem 3. *The path union of k copies of gear graph G_n admits product cordial labeling, when n is odd.*

Proof. Let G be the path union of k copies $G_1, G_2, \ldots, G_k$ of gear graph G_n. Let $\{v_{i0}, v_{i1}, \ldots, v_{i2n}\}$ denote the vertices of G. where v_{i0} is apex vertex. Let $e_i = v_{i0}v_{(i+1)0}$ be the edge joining G_i and G_{i+1}. We define labeling function $f : V(G) \to \{0, 1\}$ as follows.

Case 1: k is odd, n is odd.
$f(v_{ij}) = 1; 1 \leq i \leq \frac{k-1}{2}, 0 \leq j \leq 2n$
$= 0; \frac{k+3}{2} \leq i \leq k, 0 \leq j \leq 2n$
$f(v_{(\frac{k+1}{2})j}) = 1; 0 \leq j \leq 2n$
$= 0; n+1 \leq j \leq 2n.$

Case 2: k is even, n is odd.
$f(v_{ij}) = 1; 1 \leq i \leq \frac{k}{2}, 0 \leq j \leq 2n$
$= 0; \frac{k}{2}+1 \leq i \leq k, 0 \leq j \leq 2n$

The graph G under consideration satisfies the condition $|v_f(1)-v_f(0)| \leq 1$ and $|e_f(1)-e_f(0)| \leq 1$ in each case.

Hence the graph G is product cordial graph. □

Example 3. The product cordial labeling of the path union of 3-copies of gear graph G_3 is shown in *Figure 3.* It is the case related to k is odd.

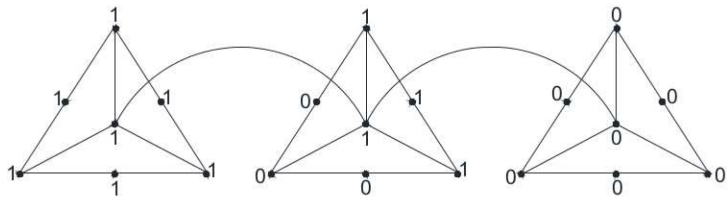

Figure 3: Product cordial labeling of path union of 3 copies of gear graph G_3.

Theorem 4. *The graph obtained by joining two copies of helm H_n by a path of arbitrary length admits product cordial labeling.*

Proof. Let G be the graph obtained by joining two copies of helm H_n by path P_k of length $k-1$. Let $\{u_0, u_1, \ldots, u_n, u_1', u_2', \ldots, u_n'\}$ denote the consecutive vertices of first copy of helm H_n. Here u_0 is the apex vertex, $\{u_0, u_1, \ldots, u_n\}$ be the internal vertices and $\{u_0', u_1', \ldots, u_n'\}$ be the external(pendant) vertices. Similarly let $\{w_0, w_1, \ldots, w_n, w_1', w_2', \ldots, w_n'\}$ denote the consecutive vertices of second copy of helm H_n, where w_0 is the apex vertex, $\{w_0, w_1, \ldots, w_n\}$ be the internal vertices and $\{w_0', w_1', \ldots, w_n'\}$ be the external(pendant) vertices. Let $\{v_1, v_2, \ldots, v_k\}$ denote the vertices of path P_k with $u_0 = v_1$ and $v_k = w_0$. First we label the vertices of first copy of H_n by label 1 and label the vertices of second copy of H_n by label 0. At this stage the vertex conditions and the edge conditions of product cordial labeling are satisfied. Now the remaining task is to label the vertices of path P_k for which we define labeling function $f : V(G) \to \{0,1\}$ as follows.

Case 1: k is even. In this case we label the vertices as:

$f(v_i) = 1; 1 \leq i \leq \frac{k}{2}$

$\quad\quad = 0; \frac{k}{2}+1 \leq i \leq k$

It can easily seen that the vertex condition and edge conditions of product cordial labeling are satisfied in this case.

Case 2: k is odd. In this case we label the vertices as:

$f(v_i) = 1; 1 \leq i \leq \frac{k+1}{2}$

$= 0; \frac{k+3}{2} \leq i \leq k$

It can easily seen that the vertex condition and edge conditions of product cordial labeling are satisfied in this case. Hence the graph G is product cordial graph. □

Example 4. Product cordial labeling of the graph obtaining by joining two copies of helm H_4 by path P_5 is shown in *Figure 4*. It is the case related to k is odd.

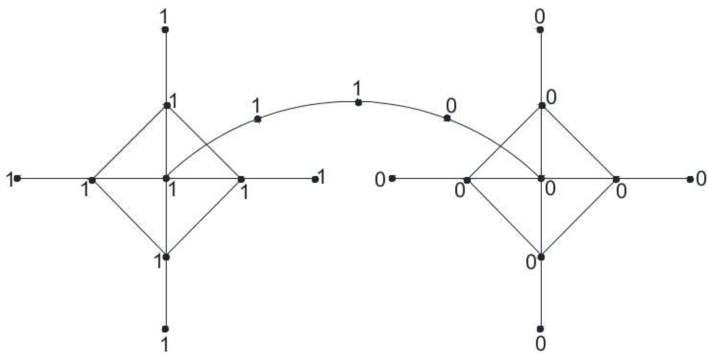

Figure 4: Product cordial labeling of the graph obtained by joining two copies of helm H_4 by path P_5.

Theorem 5. *The graph obtaining by joining two copies of closed helm CH_n by path of arbitrary length admits product cordial labeling.*

Proof. Let G be the graph obtained by joining two copies of closed helm CH_n by path P_k of $k-1$ length. Let $\{u_0, u_1, \ldots, u_n, u_1^{'}, u_2^{'}, \ldots, u_n^{'}\}$ denote the consecutive vertices of first copy of closed helm CH_n. Here u_0 is the apex vertex, $\{u_0, u_1, \ldots, u_n\}$ be the internal vertices and $\{u_0^{'}, u_1^{'}, \ldots, u_n^{'}\}$ be the external(pendant) vertices. Similarly let $\{w_0, w_1, \ldots, w_n, w_1^{'}, w_2^{'}, \ldots, w_n^{'}\}$ denote the consecutive vertices of second copy of closed helm CH_n, where w_0 is the apex vertex, $\{w_0, w_1, \ldots, w_n\}$ be the internal vertices and $\{w_0^{'}, w_1^{'}, \ldots, w_n^{'}\}$ be the external(pendant) vertices. Let $\{v_1, v_2, \ldots, v_k\}$ denote the vertices of the path P_k with $u_0 = v_1$ and $v_k = w_0$. First we label the vertices of first copy of CH_n by label 1 and label the vertices of second copy of CH_n by label 0. At this stage the vertex condition and the edge condition of product cordial labeling are satisfied. Now the remaining task is to label the vertices of path P_k for which we define labeling function $f : V(G) \to \{0, 1\}$ as follows.

Case 1: k is even. In this case we label the vertices as:
$f(v_i) = 1; 1 \leq i \leq \frac{k}{2}$
$= 0; \frac{k}{2} + 1 \leq i \leq k$
It can easily seen that the vertex condition and edge conditions of product cordial labeling are satisfied in this case.

Case 2: k is odd. In this case we label the vertices as:
$f(v_i) = 1; 1 \leq i \leq \frac{k+1}{2}$
$= 0; \frac{k+3}{2} \leq i \leq k.$

It can easily seen that the vertex condition and edge conditions of product cordial labeling are satisfied in this case. Hence the graph G is product cordial graph. □

Example 5. Product cordial labeling of the graph obtained by joining two copies of closed helm CH_5 by path P_3 is shown in *Figure 5.* It is the case related to k is odd.

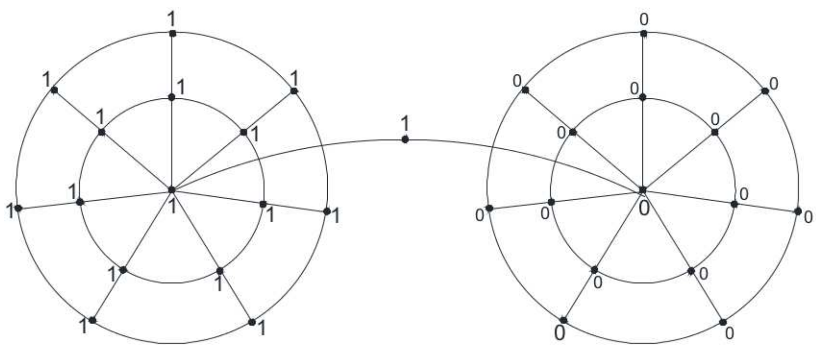

Figure 5: Product cordial labeling of the graph obtained by joining two copies of closed helm CH_7 by path P_3.

Theorem 6. *The graph obtaining by joining two copies of gear G_n by path of arbitrary length admits product cordial labeling.*

Proof. Let G be the graph obtained by joining two copies of gear graph G_n by path P_k of $k-1$ length. Let $\{u_0, u_1, \ldots, u_{2n}\}$ denote the consecutive vertices of first copy of gear graph G_n and let $\{w_0, w_1, \ldots, w_{2n}\}$ denote the consecutive vertices of second copy of gear graph G_n. Let $\{v_1, v_2, \ldots, v_k\}$ denote the vertices of the path P_k with $u_0 = v_1$ and $v_k = w_0$. First we label the vertices of first copy of G_n by label 1 and label the vertices of second copy of G_n by label 0.

At this stage the vertex conditions and the edge conditions of product cordial labeling are satisfied. Now the remaining task is to label the vertices of path P_k for which we define labeling function $f : V(G) \rightarrow \{0, 1\}$ as follows.

Case 1: k is even

In this case we label the vertices as:

$f(v_i) = 1; 1 \leq i \leq \frac{k}{2}$

$\quad = 0; \frac{k}{2} + 1 \leq i \leq k$

It can easily seen that the vertex condition and edge condition of product cordial labeling are satisfied in this case.

Case 2: k is odd

In this case we label the vertices as:

$f(v_i) = 1; 1 \leq i \leq \frac{k+1}{2}$

$\quad = 0; \frac{k+3}{2} \leq i \leq k$

It can easily seen that the vertex condition and edge condition of product cordial labeling are satisfied in this case.

Hence in each case the graph G under consideration satisfies the conditions of product cordial labeling i.e. $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$.

Hence G is product cordial graph. □

Example 6. The product cordial labeling of two copies of gear G_6 joined by path P_6 is shown in *Figure 6.* It is the case related to k is even.

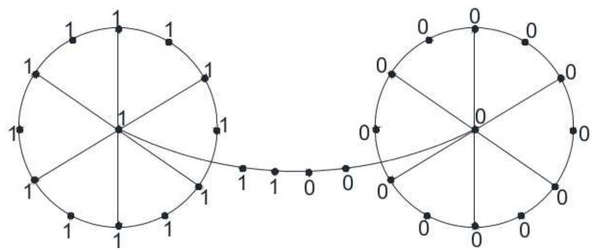

Figure 6: Product cordial labeling of the graph obtaining by joining two copies of gear G_6 by path P_6.

3. Conclusion

In this paper we investigated six new product cordial graphs. The results proved in this paper are novel. Illustrations are provided at the end of each theorem for better understanding of the labeling pattern defined in each theorem.

References

[1] F. Harary, *Graph theory*, Addision-wesley, Reading, MA (1969).

[2] J. A. Gallian, A dynamic survey of graph labeling, *The Electronics Journal of Combinatorics*, **19** (2012), $\sharp DS6$ $1 - 260$.

[3] R. Ponraj, M. Sundaram, M. Sivakumar, k-product cordial labeling of graphs, *International Journal of contemp. mathematical sciences*, **7**, No.15(2012), $733 - 742$.

[4] S. K. Vaidya, C. M. Barasara, Product Cordial Labeling for some New Graphs, *Journal of mathematics Research*, **3**, No.2(2011), $206 - 211$, **doi**:10.5539/jmr.V3n2p206.

[5] S. K. Vaidya, N. A. Dani, Some new product cordial graphs, *Journal of Applied computer science and Mathematics*, **4**, No.8(2010), $62 - 65$.

[6] S. K. Vaidya, K. K. Kanani, Some cycle related product cordial graphs, *International Journal of Algorithms, computing and Mathematics*, **3**, No.1(2010), $109 - 116$.

[7] S. K. Vaidya, N. B. Vyas, Product Cordial Labeling in the context of Tensor Product of Graphs, *Journal of mathematics Research*, **3**, No.3(2011), $83 - 88$, **doi**:10.5539/jmr.V3n3p83.