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DET-NORM ON FUZZY MATRICES

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Abstract: In this paper we introduce fuzzy det-norm ordering with fuzzy matrices using the structure of $M_n(F)$, the set of $(n\times n)$ fuzzy det-norm ordering with fuzzy matrices is introduced. From this row and column, determinant of the fuzzy norm has been obtained by imposing an equivalence relation on $M_n(F)$. We know that the comparability relation on fuzzy matrices is a partial ordering. We prove that det-norm ordering is a partitions ordering on the set of all idempotent matrices in $M_n(F)$. We begin with the det-norm ordering on fuzzy matrices as analogue of the ordering on real matrices. Several properties of these orderings are derived. Discuss their relationship between these ordering with det-norm ordering and Also we introduce the concept of Fuzzy norm and partitions of $M_n(F)$, Properties of fuzzy det-norm ordering.

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Key Words: fuzzy matrix, fuzzy m-norm matrix, t-ordering, determinant of a square fuzzy matrix

1. Introduction

The concept of fuzzy set was introduced by Zadeh[8] in 1965. Jian Miao

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Chen. [2] introduced the Fuzzy matrix partial ordering and generalized inverse. Bertoiuzza [1] introduced the distributivity of t-norm and t-conorms. In 1995, Ragab .M. Z. and Emam E. G [7] introduced the determinant and adjoint of a square fuzzy matrix. Meenakshi A.R. and Cokilavany R. [3] introduced the concept of fuzzy 2-normed linear spaces. Nagoorgani A. and Kalyani G. [5] Introduced the Binormed sequences in fuzzy matrices. Nagoorgani A. and Kalyani G. [6] Introduced the Fuzzy matrix m-ordering. ZHOU Min - na [9] Introduced the Characterizations of the Minus Ordering in Fuzzy Matrix Set.Nagoorgani A. and Manikandan A. R. [4] introduced the properties of fuzzy det-norm matrices.

In this paper, we introduce the concept of fuzzy det-norm ordering with fuzzy matrices. The purpose of the introduction is to explain det-norm ordering with fuzzy matrices and partitions of $M_n(F)$. In Section 2, fuzzy det-norm ordering with fuzzy matrices is introduced in $M_n(F)$. In Section 3,Properties of det-norm ordering with fuzzy matrices.

2. Preliminaries

We consider $F = [0, 1]$ the fuzzy algebra with operation $[+, \cdot]$ and the standard order " $\leq$ " where $a + b = maxa, b$, $a \cdot b = mina, b$ for all a, b in F. F is a commutative semi-ring with additive and multiplicative identities 0 and 1 respectively. Let $M_{mn}(F)$ denote the set of all $m \times n$ fuzzy matrices over F. In short $M_n(F)$ is the set of all fuzzy matrices of order n. define $'+'$ and scalar multiplication in $M_n(F)$ as $A + B = [a_{ij} + b_{ij}]$, where $A = [a_{ij}]$ and $B = [b_{ij}]$ and $cA = [ca_{ij}]$, where c is in $[0, 1]$, with these operations $M_n(F)$ forms a linear space.

3. Fuzzy Det-Norm and Partitions of $M_n(F)$

To analysis more properties of $M_n(F)$ we introduce the concept of norm in $M_n(F)$ and thus we have defined for every A in $M_n(F)$ a non-negative quantity say det-norm is defined in the following way.

Definition 1. An $m \times n$ matrix $A = [a_{ij}]$ whose components are in the unit interval $[0, 1]$ is called a fuzzy matrix.

Definition 2. The determinant $|A|$ of an $m \times n$ fuzzy matrix A is defined as follows; $|A| = \Sigma_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$ Where S_n denotes the symmetric group of all permutations of the indices $(1, 2, \cdots, n)$

Definition 3. For every A in $M_n(F)$ the det-norm of A is defined as $\|A\| = det[A]$, where $A = [a_{ij}]$.

Definition 4. A matrix A in $M_n(F)$ is called idempotent if $A^2 = A$ or $\|A^2\| = det[A]$, where $A = [a_{ij}]$.

Example 1. If $A = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix}$, then

$$A = [0.5] \begin{bmatrix} 0.1 & 0.4 \\ 0.2 & 0.5 \end{bmatrix} + 0.4 \begin{bmatrix} 0.3 & 0.4 \\ 0.5 & 0.5 \end{bmatrix} + 0.6 \begin{bmatrix} 0.3 & 0.1 \\ 0.5 & 0.2 \end{bmatrix},$$

$A = 0.5[0.1 + 0.2] + 0.4[0.3 + 0.4] + 0.6[0.2 + 0.1] = 0.2 + 0.4 + 0.2,\ A = 0.4.$

If $A^2 = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.4 & 0.3 & 0.4 \\ 0.5 & 0.4 & 0.5 \end{bmatrix}$, then

$$A^2 = [0.5] \begin{bmatrix} 0.3 & 0.4 \\ 0.4 & 0.5 \end{bmatrix} + 0.4 \begin{bmatrix} 0.4 & 0.4 \\ 0.5 & 0.5 \end{bmatrix} + 0.5 \begin{bmatrix} 0.4 & 0.3 \\ 0.5 & 0.4 \end{bmatrix}$$

$A^2 = 0.5[0.3 + 0.4] + 0.4[0.4 + 0.4] + 0.5[0.4 + 0.3] = 0.4 + 0.4 + 0.4,\ A^2 = 0.4.$

Therefore $A^2 = A = 0.4$.

Definition 5. For all A in $M_n(F)$ define:

$$\begin{aligned} A\{1\} &= \{x \in M_n(F)/\|x\| > \|A\|\}, \\ A\{2\} &= \{x \in M_n(F)/\|x\| < \|A\|\}, \\ A\{3\} &= \{x \in M_n(F)/\|x\| = \|A\|\}, \\ A\{4\} &= \{x \in M_n(F)/AXA = A\}, \\ A\{5\} &= \{x \in M_n(F)/XAX = X\}. \end{aligned}$$

Clearly $M_n(F) = A\{1\} \cup A\{2\} \cup A\{3\}$. The set $A\{1\}$ is called as det-superior to A and $A\{2\}$ det-inferior to A. Clearly $A\{3\}$ is det-equivalent to A. $A\{4\}$ and $A\{5\}$ are known as the sets of inner and outer inverses of A.

Example 2. If $A = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix}$ and $x = \begin{bmatrix} 0.7 & 0.4 & 0.5 \\ 0.4 & 0.8 & 0.2 \\ 0.1 & 0.6 & 0.7 \end{bmatrix}$, then

$$\|A\| = [0.5] \begin{bmatrix} 0.1 & 0.4 \\ 0.2 & 0.5 \end{bmatrix} + 0.4 \begin{bmatrix} 0.3 & 0.4 \\ 0.5 & 0.5 \end{bmatrix} + 0.6 \begin{bmatrix} 0.3 & 0.1 \\ 0.5 & 0.2 \end{bmatrix}$$

$\|A\| = 0.5[0.1+0.2]+0.4[0.3+0.4]+0.6[0.2+0.1] = 0.2+0.4+0.2,\ \|A\| = 0.4,$

$$\|x\| = [0.7]\begin{bmatrix}0.8 & 0.2\\0.6 & 0.7\end{bmatrix} + 0.4\begin{bmatrix}0.4 & 0.2\\0.1 & 0.7\end{bmatrix} + 0.5\begin{bmatrix}0.4 & 0.8\\0.1 & 0.6\end{bmatrix}$$

$\|x\| = 0.7[0.7+0.2]+0.4[0.4+0.1]+0.5[0.4+0.1] = 0.7+0.4+0.4,\ \|x\| = 0.7.$
Therefore $\|x\| > \|A\|$;

$$\|x\| = \begin{bmatrix}0.3 & 0.5 & 0.1\\0.4 & 0.7 & 0.2\\0.4 & 0.5 & 0.3\end{bmatrix},$$

$\|x\| = 0.3[0.3+0.2]+0.5[0.3+0.2]+0.1[0.4+0.4] = 0.3+0.3+0.1,\ \|x\| = 0.3.$
Therefore $\|x\| < \|A\|$,

$$\|x\| = \begin{bmatrix}0.6 & 0.4 & 0.2\\0.4 & 0.5 & 0.3\\0.5 & 0.3 & 0.4\end{bmatrix},$$

$\|x\| = 0.6[0.4+0.3]+0.4[0.4+0.3]+0.2[0.3+0.5] = 0.4+0.4+0.2,\ \|x\| = 0.4.$
Therefore $\|x\| = \|A\|$.

Theorem 1. *For each A in $M_n(F)$ the following results hold true:*

(i) If $X \in A\{i\}$ then X^T is also in $A\{i\}$ for $i = 1, 2, 3$ where X^T is the transpose of X.

(ii) If $A_1 \in A\{1\}$, $A_2 \in A\{2\}$, $A_3 \in A\{3\}$ then $\|A_1 + A_2 + A_3\| = det[A_1]$.

(iii)$\|A_1A_2A_3\| = det[A_2]$.

(iv) $A^T \in A\{3\}$ for all A in $M_n(F)$.

Proof. (i) $\|X\| = \|X^T\|$, since $\|X\| = det[A]$ for all X in $A\{3\}$.

(ii)$\|A_1\| > \|A\|, \|A_2\| < \|A\|, \|A_3\| = \|A\|$. Therefore $\|A_1 + A_2 + A_3\| = det[A_1 + A_2 + A_3] = det[A_1] + det[A_2] + det[A_3] = det[A_3] = A_3$.

(iii) $\|A_1A_2A_3\| = det[A_1A_2A_3] = det[A_1] + det[A_2] + det[A_3] = det[A_2] = \|A_2\|$.

(iv) $\|A\| = \|A^T\|$ or $det[A] = det[A^T]$. Therefore for all A in $M_n(F), A^T \in A\{3\}$. □

Example 3. (i) $\|X\| = \|X^T\|$ for all X in $A\{i\}$, where $i = 1, 2, 3$.

Case (i): $A\{1\}$, $x = \begin{bmatrix} 0.7 & 0.4 & 0.5 \\ 0.4 & 0.8 & 0.2 \\ 0.1 & 0.6 & 0.7 \end{bmatrix}$, $\|x\| = 0.7$, $x^T = \begin{bmatrix} 0.7 & 0.4 & 0.1 \\ 0.4 & 0.8 & 0.6 \\ 0.5 & 0.2 & 0.7 \end{bmatrix}$, $\|x^T\| = 0.7[0.7+0.2]+0.4[0.4+0.5]+0.1[0.2+0.5] = 0.7+0.4+0.1$, $\|x^T\| = 0.7$.

Therefore $\|x\| = \|x^T\| = 0.7$.

Case (i): $A\{2\}$, $\|x\| = \begin{bmatrix} 0.3 & 0.5 & 0.1 \\ 0.4 & 0.7 & 0.2 \\ 0.4 & 0.5 & 0.3 \end{bmatrix}$, $\|x\| = 0.3$, $x^T = \begin{bmatrix} 0.3 & 0.4 & 0.4 \\ 0.5 & 0.7 & 0.5 \\ 0.1 & 0.2 & 0.3 \end{bmatrix}$, $\|x^T\| = 0.3[0.3+0.2]+0.4[0.3+0.1]+0.4[0.2+0.1] = 0.3+0.3+0.2$, $\|x^T\| = 0.3$.

Therefore $\|x\| = \|x^T\| = 0.3$.

Case (i): $A\{3\}$, $\|x\| = \begin{bmatrix} 0.6 & 0.4 & 0.2 \\ 0.4 & 0.5 & 0.3 \\ 0.5 & 0.3 & 0.4 \end{bmatrix}$, $\|x\| = 0.4$, $x^T = \begin{bmatrix} 0.6 & 0.4 & 0.5 \\ 0.4 & 0.5 & 0.3 \\ 0.2 & 0.3 & 0.4 \end{bmatrix}$, $\|x^T\| = 0.6[0.4+0.3]+0.4[0.4+0.2]+0.5[0.3+0.2] = 0.4+0.4+0.3$, $\|x^T\| = 0.4$.

Therefore $\|x\| = \|x^T\| = 0.4$.

Example 4. Let $A = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0.2 & 0.3 & 0.7 \\ 0.7 & 0.5 & 0.6 \\ 0.8 & 0.6 & 0.7 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.3 & 0.6 & 0.7 \\ 0.1 & 0.5 & 0.2 \\ 0.2 & 0.6 & 0.3 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 0.4 & 0.1 & 0.3 \\ 0.2 & 0.6 & 0.4 \\ 0.3 & 0.5 & 0.4 \end{bmatrix}$.

$$\begin{aligned}
\|A\| &= 0.4 \\
\|A_1\| &= 0.2[0.5 + 0.6] + 0.3[0.7 + 0.6] + 0.6[0.6 + 0.5] \\
&= 0.2 + 0.3 + 0.6 \\
\|A_1\| &= 0.6 \\
\|A_2\| &= 0.3[0.3 + 0.2] + 0.6[0.1 + 0.2] + 0.7[0.1 + 0.2] \\
&= 0.3 + 0.2 + 0.2 \\
\|A_2\| &= 0.3 \\
\|A_3\| &= 0.4[0.4 + 0.4] + 0.1[0.2 + 0.3] + 0.3[0.2 + 0.3] \\
&= 0.4 + 0.1 + 0.3 \\
\|A_3\| &= 0.4
\end{aligned}$$

$$A_1 + A_2 + A_3 = \begin{bmatrix} 0.4 & 0.6 & 0.7 \\ 0.7 & 0.6 & 0.6 \\ 0.8 & 0.6 & 0.7 \end{bmatrix}$$

$$\begin{aligned} \|A_1 + A_2 + A_3\| &= 0.4[0.6 + 0.6] + 0.6[0.7 + 0.6] + 0.7[0.6 + 0.6] \\ &= 0.4 + 0.6 + 0.6 \\ \|A_1 + A_2 + A_3\| &= 0.6 \\ \|A_1 + A_2 + A_3\| &= det[A_1] + det[A_2] + det[A_3] \\ &= 0.6 + 0.3 + 0.4 = 0.6 = \|A_3\| \end{aligned}$$

$$A_1A_2A_3 = \begin{bmatrix} 0.2 & 0.6 & 0.4 \\ 0.3 & 0.6 & 0.4 \\ 0.3 & 0.6 & 0.4 \end{bmatrix}$$

$$\begin{aligned} \|A_1A_2A_3\| &= 0.2[0.4 + 0.4] + 0.6[0.3 + 0.3] + 0.4[0.3 + 0.3] \\ &= 0.2 + 0.3 + 0.3 \\ \|A_1A_2A_3\| &= 0.3 \\ \|A_1A_2A_3\| &= det[A_1]det[A_2]det[A_3] \\ &= [0.6][0.3][0.4] = 0.3 = \|A_2\| \end{aligned}$$

(iv) $\|A\| = \|A_T\|$.

Therefore $\|A\| = \|A_T\| = 0.4$ or $det[A] = det[A^T]$ for all$A \in M_n(F)$, $A^T \in A(3)$.

Theorem 2. *(i) For all $X \in A\{4\}$, $\|A\| \leq \|X\|$.*

(ii) For all $X \in A\{5\}$, $\|X\| \leq \|A\|$.
Further for all X in $A\{4\} \cap A\{5\}$ the matrices AX and XA are idempotent.

Proof. If $X \in A\{4\}$, then $AXA = A$. Therefore

$$\|AXA\| = det[A] \ \Rightarrow \ det[A]det[X]det[A] = det[A] = \|A\| \ \Rightarrow \ \|A\| \leq \|X.$$

(i) If $XinA\{5\}$, then $XAX = X$. Therefore

$$\|XAX\| = det[X] \Rightarrow det[X]det[A]det[X] = det[X] = \|X\| \Rightarrow \|X\| \leq \|A.$$

(ii) If X in $A\{4\} \cap A\{5\}$, then

$$AXA = A, \tag{1}$$

$$XAX = X, \tag{2}$$

$XAXA = XA \Rightarrow (XA)^2 = XA$ from (1) and $AXAX = Ax \Rightarrow (AX)^2 = AX$ from (2).

Therefore XA and AX are idempotent. □

Example 5. (i)If X in $A\{4\}$, then $\|AXA\| = \|A\| \leq \|X\|$

$A = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix}$ and $x = \begin{bmatrix} 0.7 & 0.4 & 0.5 \\ 0.4 & 0.8 & 0.2 \\ 0.1 & 0.6 & 0.7 \end{bmatrix}$

$\|AXA\| = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.4 & 0.5 \\ 0.4 & 0.8 & 0.2 \\ 0.1 & 0.6 & 0.7 \end{bmatrix} \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix}$

$\|AXA\| = 0.4$

$\Rightarrow \|A\| \leq \|X$

If X in $A5$, then $\|XAX\| = \|X\| \leq \|A\|$

$A = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix}$ and $\|x\| = \begin{bmatrix} 0.3 & 0.5 & 0.1 \\ 0.4 & 0.7 & 0.2 \\ 0.4 & 0.5 & 0.3 \end{bmatrix}$

$\|XAX\| = \begin{bmatrix} 0.3 & 0.5 & 0.1 \\ 0.4 & 0.7 & 0.2 \\ 0.4 & 0.5 & 0.3 \end{bmatrix} \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.3 & 0.5 & 0.1 \\ 0.4 & 0.7 & 0.2 \\ 0.4 & 0.5 & 0.3 \end{bmatrix}$

$\|XAX\| = 0.4 \Rightarrow \|X\| \leq \|A\|$.

(ii) If X in $A\{5\}$ $XAXA = XA \Rightarrow (XA)^2 = XA$

$XAXA = \begin{bmatrix} 0.3 & 0.5 & 0.1 \\ 0.4 & 0.7 & 0.2 \\ 0.4 & 0.5 & 0.3 \end{bmatrix} \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.3 & 0.5 & 0.1 \\ 0.4 & 0.7 & 0.2 \\ 0.4 & 0.5 & 0.3 \end{bmatrix} \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix}$

$XAXA = 0.4\ XA = \begin{bmatrix} 0.3 & 0.5 & 0.1 \\ 0.4 & 0.7 & 0.2 \\ 0.4 & 0.5 & 0.3 \end{bmatrix} \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix}$

$XA = 0.4$, $(XA)^2 = XA = 0.4$.

If $X \in A\{4\}$

$AXAX = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.4 & 0.5 \\ 0.4 & 0.8 & 0.2 \\ 0.1 & 0.6 & 0.7 \end{bmatrix} \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.4 & 0.5 \\ 0.4 & 0.8 & 0.2 \\ 0.1 & 0.6 & 0.7 \end{bmatrix}$

$AXAX = 0.4$

$AX = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.4 & 0.5 \\ 0.4 & 0.8 & 0.2 \\ 0.1 & 0.6 & 0.7 \end{bmatrix}$

$(AX)^2 = AX = 0.4$.

Therefore XA and AX are idempotent.

4. Properties of Det-Norm Ordering with Fuzzy Matrices

Definition 6. The det-norm ordering $A \leq B$ in $M_n(F)$ is defined as $A \leq B \Leftrightarrow \|A\| \leq \|B\|$ Or $A \leq B \Leftrightarrow det[A] \leq det[B]$

Example 6. $A = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix}$ and $B = \begin{bmatrix} 0.7 & 0.6 & 0.8 \\ 0.4 & 0.5 & 0.6 \\ 0.8 & 0.6 & 0.7 \end{bmatrix}$, $\|A\| = 0.4$, $\|B\| = 0.7[0.5+0.6]+0.6[0.4+0.6]+0.8[0.4+0.6] = 0.6+0.6+0.6$, $\|B\| = 0.6$.

Therefore $A \leq B \Leftrightarrow \|A\| \leq \|B\|$.

Theorem 3. *The det-ordering is not a partial ordering.*

Proof. (i) $det[A] \leq det[B]$ for all $A \in M_n(F)$. Hence $A \leq B$.

Therefore reflexivity is true.

(ii) $A \leq B \Rightarrow \|A\| \leq \|B\|$, $B \leq A \Rightarrow \|B\| \leq \|A\|$, $A \leq B$ and $B \leq A \Rightarrow \|A\| = \|B\|$.

But $\|A\| = \|B\|$ does not imply $A = B$.

Therefore anti symmetry is not true.

(iii) $A \leq B, B \leq C \Rightarrow A \leq C$ for all $A, B, C \in M_n(F)$. For $A \leq B = \|A\| \leq \|B\|$

$$B \leq C = \|B\| \leq \|C\|,$$

$$A \leq C = \|A\| \leq \|C\|.$$

Therefore transitivity is true.

Thus the det-ordering is not a partial ordering in $M_n(F)$. □

Example 7. $A = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix}$, $B = \begin{bmatrix} 0.7 & 0.6 & 0.8 \\ 0.4 & 0.5 & 0.6 \\ 0.8 & 0.6 & 0.7 \end{bmatrix}$ and $C = \begin{bmatrix} 0.8 & 0.6 & 0.8 \\ 0.5 & 0.9 & 0.7 \\ 0.8 & 0.6 & 0.8 \end{bmatrix}$, $\|A\| = 0.4$, $\|B\| = 0.6$, $\|C\| = 0.8[0.8+0.6]+0.6[0.5+0.7]+0.8[0.5+0.8] = 0.8+0.6+0.8$, $\|C\| = 0.8$.

(i) $\|A\| \leq \|A\|$ for all $A \in M_n(F)$, $A \leq A$.

Therefore reflexivity is true.

(ii) $A \leq B = \|A\| \leq \|B\| = 0.4 \leq 0.6$, $B \leq A = \|B\| \leq \|A\| = 0.6 \nleq 0.4$, $A \leq BandB \leq A \Rightarrow \|A\| = \|B\|$.

But $\|A\| = \|B\|$ does not imply $A = B$.

Therefore anti symmetry is not true.

(iii) $A \leq B = \|A\| \leq \|B\| = 0.4 \leq 0.6$, $B \leq C = \|B\| \leq \|C\| = 0.6 \nleq 0.8$, $A \leq C = \|A\| \leq \|C\| = 0.4 \nleq 0.8$.

Therefore transitivity is true.

Thus the det-ordering is not a partial ordering in $M_n(F)$.

Theorem 4. *If $A \leq B$, then:*

(i) $A^T \leq B^T$;

(ii) $AB^T \leq BB^T$, $B^T A \leq B^T B$;

(iii) $A^T A \leq B^T B$, $AA^T \leq BB^T$, $A^n \leq B^n$ for any positive integer n.

Proof. (i) $\|A\| = det[A^T]$, $\|B\| = det[B^T]$.

Therefore $\|A\| \leq \|B\| \Rightarrow det[A^T] \leq det[B^T]$, i.e. $A \leq B \Rightarrow A^T \leq B^T$.

(ii) $det[AB^T] \leq det[A]det[B^T] = det[A]det[B] = det[A]$. Since $A \leq B$,

$$det[BB^T] = det[B]det[B^T] = det[B]det[B] = det[B],$$

$$A \leq B \Rightarrow det[A] \leq det[B] \Rightarrow det[AB^T] \leq det[BB^T].$$

Similarly $A \leq B \Rightarrow B^T A \leq B^T B$.

(iii) $det[A^T A] = det[A^T]det[A] = det[A]det[A] = det[A]$,

$$det[B^T B] = det[B^T]det[B] = det[B]det[B] = det[B],$$

$$A \leq B \Rightarrow det[A] \leq det[B] \Rightarrow det[A^T A] \leq det[B^T B] \Rightarrow A^T A \leq B^T B.$$

Similarly $A \leq B \Rightarrow A^T A \leq B^T B$.

(iv) $det[A^n] = det[A..ntimes] = det[A]$
$det[A]ntimes = det[A]$
$det[B^n] = det[B..ntimes] = det[B]$
$det[B]ntimes = det[B]$
$A \leq B \Rightarrow det[A] \leq det[B] \Rightarrow det[A^n] \leq det[B^n]$.

Therefore $A^n?B^n$ for any positive integer n. □

Example 8. $A = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix}$, and $B = \begin{bmatrix} 0.7 & 0.6 & 0.8 \\ 0.4 & 0.5 & 0.6 \\ 0.8 & 0.6 & 0.7 \end{bmatrix}$

$\|A\| = 0.4$ and $\|B\| = 0.6$

$A^T = \begin{bmatrix} 0.5 & 0.3 & 0.5 \\ 0.4 & 0.1 & 0.2 \\ 0.6 & 0.4 & 0.5 \end{bmatrix}$
$\|A^T\| = 0.5[0.1 + 0.2] + 0.3[0.4 + 0.2] + 0.5[0.4 + 0.1]$
$= 0.2 + 0.3 + 0.4$
$\|A^T\| = 0.4$
$B^T = \begin{bmatrix} 0.7 & 0.4 & 0.8 \\ 0.6 & 0.5 & 0.6 \\ 0.8 & 0.6 & 0.7 \end{bmatrix}$
$\|B^T\| = 0.7[0.5 + 0.6] + 0.4[0.6 + 0.6] + 0.8[0.6 + 0.5]$
$= 0.6 + 0.4 + 0.6$
$\|A^T\| = 0.6$
(i) $\|A\| = det[A^T] = 0.4$, $\|B\| = det[B^T] = 0.6$.

Therefore $\|A\| \leq \|B\| \Rightarrow = det[A^T] \leq = det[B^T] = 0.4 \leq 0.6$
ie., $A \leq B \Rightarrow A^T \leq B^T$
$\|AB^T\| = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.4 & 0.8 \\ 0.6 & 0.5 & 0.6 \\ 0.8 & 0.6 & 0.7 \end{bmatrix}$
$\|AB^T\| = 0.6[0.4 + 0.4] + 0.6[0.4 + 0.4] + 0.6[0.4 + 0.4]$
$= 0.4 + 0.4 + 0.4$
$\|AB^T\| = 0.4$
$\|A\|\|B^T\| = [0.4][0.6] = 0.4$
$\|BB^T\| = \begin{bmatrix} 0.7 & 0.6 & 0.8 \\ 0.4 & 0.5 & 0.6 \\ 0.8 & 0.6 & 0.7 \end{bmatrix} \begin{bmatrix} 0.7 & 0.4 & 0.8 \\ 0.6 & 0.5 & 0.6 \\ 0.8 & 0.6 & 0.7 \end{bmatrix}$
$\|BB^T\| = 0.8[0.6 + 0.6] + 0.6[0.6 + 0.6] + 0.7[0.6 + 0.5]$
$= 0.6 + 0.6 + 0.6$
$\|BB^T\| = 0.6$
$\|A^TA\| = \begin{bmatrix} 0.5 & 0.3 & 0.5 \\ 0.4 & 0.1 & 0.2 \\ 0.6 & 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix}$
$\|A^TA\| = 0.5[0.4 + 0.4] + 0.4[0.4 + 0.4] + 0.5[0.4 + 0.4]$
$= 0.4 + 0.4 + 0.4$
$\|A^TA\| = 0.4$
$\|A^T\|\|A\| = [0.4][0.4] = 0.4$
$\|B^TB\| = \begin{bmatrix} 0.7 & 0.4 & 0.8 \\ 0.6 & 0.5 & 0.6 \\ 0.8 & 0.6 & 0.7 \end{bmatrix} \begin{bmatrix} 0.7 & 0.6 & 0.8 \\ 0.4 & 0.5 & 0.6 \\ 0.8 & 0.6 & 0.7 \end{bmatrix}$
$\|B^TB\| = 0.8[0.6 + 0.6] + 0.6[0.6 + 0.6] + 0.7[0.6 + 0.6]$

$= 0.6 + 0.6 + 0.6$
$\|B^T B\| = 0.6$
$A \leq B \Rightarrow \|A\| \leq \|B\| \Rightarrow \|A^T A\| \leq \|B^T B\| = 0.4 \leq 0.6 \Rightarrow A^T A \leq B^T B$

$$\|AA^T\| = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.3 & 0.5 \\ 0.4 & 0.1 & 0.2 \\ 0.6 & 0.4 & 0.5 \end{bmatrix}$$

$\|AA^T\| = 0.5[0.4 + 0.4] + 0.4[0.4 + 0.4] + 0.5[0.4 + 0.4]$
$= 0.4 + 0.4 + 0.4$
$\|AA^T\| = 0.4$
$\|A\|\|A^T\| = [0.4][0.4] = 0.4$

Therefore $A \leq B \Rightarrow AA^T \leq BB^T$.

5. Conclusion

In this paper, a new definition for the det-norm ordering and its properties are suggested in fuzzy environment. A numerical example is given to clarify the developed theory and the proposed det-norm ordering with fuzzy matrix.

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