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ON TWO DIOPHANTINE EQUATIONS $7^x + 19^y = z^2$ and $7^x + 91^y = z^2$

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Abstract: In this paper, we prove that both two Diophantine equations $7^x + 19^y = z^2$ and $7^x + 91^y = z^2$ have no non-negative integer solution where x, y and z are non-negative integers.

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1. Introduction

In 2013, Sroysang [9] showed that the Diophantine equation $5^x + 7^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers. Recently, Rabago [5] proved that the two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$ have exactly two solutions (x, y, z) where x, y and z are non-negative integers. The solutions are in $\{(1, 0, 2), (4, 1, 10)\}$ and $\{(1, 0, 2), (2, 1, 10)\}$, respectively. In this paper, we prove that both two Diophantine equations $7^x+19^y = z^2$ and $7^x+91^y = z^2$ have no non-negative integer solution where x, y and z are non-negative integers. For related equations, we list them as follows.

In 2007, Acu [1] proved that $(3, 0, 3)$ and $(2, 1, 3)$ are only two solutions (x, y, z) for the Diophantine equation $2^x + 5^y = z^2$ where x, y and z are non-negative integers. In 2011, Suvarnamani, Singta and Chotchaisthit [13] proved

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that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution where x, y and z are non-negative integers. In 2012, Sroysang [11] showed that the Diophantine equation $31^x + 32^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers. Moreover, he [8] showed that $(1, 0, 2)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $3^x + 5^y = z^2$ where x, y and z are non-negative integers. In 2013, Chotchaisthit [2] proved that $(3, 0, 3)$ is a unique non-negative integer solution (x, y, z) for the Diophantine equation $2^x + 11^y = z^2$ where x, y and z are non-negative integers. Moreover, he [3] proved that $(7, 0, 1, 3)$ and $(3, 2, 2, 5)$ are only two solutions (p, x, y, z) for the Diophantine equation $p^x + (p+1)^y = z^2$ where x, y, z are non-negative integers and p is a Mersenne prime. In the same year, Sroysang [6] showed that $(0, 1, 2)$, $(3, 0, 3)$ and $(4, 2, 5)$ are only three solutions (x, y, z) for the Diophantine equation $2^x + 3^y = z^2$ where x, y and z are non-negative integers.

2. Preliminaries

Proposition 2.1. (see [4], Catalan's Conjecture) $(3, 2, 2, 3)$ *is a unique solution* (a, b, x, y) *for the Diophantine equation* $a^x - b^y = 1$ *where* a, b, x *and* y *are integers such that* $\min\{a, b, x, y\} > 1$.

Lemma 2.2. [9, 10] *The Diophantine equation* $7^x + 1 = z^2$ *has no non-negative integer solution where* x *and* z *are non-negative integers.*

Lemma 2.3. [7] *The Diophantine equation* $1 + 19^y = z^2$ *has no non-negative integer solution where* y *and* z *are non-negative integers.*

Lemma 2.4. [12] *The Diophantine equation* $1 + 91^y = z^2$ *has no non-negative integer solution where* y *and* z *are non-negative integers.*

3. Main Results

Theorem 3.1. *The Diophantine equation* $7^x + 19^y = z^2$ *has no non-negative integer solution where* x, y *and* z *are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and z such that $7^x + 19^y = z^2$. By Lemma 2.2 and 2.3, it follows that $x \geq 1$ and $y \geq 1$. Then z is even. Thus, $z^2 \equiv 0 \pmod 3$ or $z^2 \equiv 1 \pmod 3$. Since $7^x \equiv 1 \pmod 3$ and $19^x \equiv 1 \pmod 3$, we obtain that $z^2 \equiv 2 \pmod 3$. This is a contradiction. □

Corollary 3.2. *Let k be a positive integer. Then the Diophantine equation $7^x + 19^y = w^{2k+2}$ has no non-negative integer solution where x, y and w are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and w such that $7^x + 19^y = w^{2k+2}$. Let $z = w^{k+1}$. This implies that $7^x + 19^y = z^2$. This is a contradiction with Theorem 3.1. □

Theorem 3.3. *The Diophantine equation $7^x + 91^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and z such that $7^x + 91^y = z^2$. By Lemma 2.2 and 2.4, it follows that $x \geq 1$ and $y \geq 1$. Then z is even. Thus, $z^2 \equiv 0 \pmod 3$ or $z^2 \equiv 1 \pmod 3$. Since $7^x \equiv 1 \pmod 3$ and $91^x \equiv 1 \pmod 3$, we obtain that $z^2 \equiv 2 \pmod 3$. This is a contradiction. □

Corollary 3.4. *Let k be a positive integer. Then the Diophantine equation $7^x + 91^y = w^{2k+2}$ has no non-negative integer solution where x, y and w are non-negative integers.*

Proof. Suppose that there are non-negative integers x, y and w such that $7^x + 91^y = w^{2k+2}$. Let $z = w^{k+1}$. This implies that $7^x + 91^y = z^2$. This is a contradiction with Theorem 3.3. □

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