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ON BASES AND MAXIMAL IDEALS IN AN ORDERED SEMIGROUP

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Abstract: In this paper the concepts of left base, right base and two-sided base of an ordered semigroup are introduced. A sufficient condition for an ordered semigroup contains right bases is given.

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1. Preliminaries

Tamura [9] introduced and studied one-sided bases and two-sided bases of semigroups (without order). Fabrici [2] gave a sufficient condition for a semigroup without order contains a right base, and showed that every finite semigroup contains both one-sided bases and two-sided bases. In this paper we introduce the notions of right base, left base and two-sided base of an ordered semigroup, and extend Fabrici's results to ordered semigroups.

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In [1], an *ordered semigroup* $(S, \cdot, \leq)$ is a semigroup $(S, \cdot)$ together with a partial order $\leq$ that is *compatible* with the semigroup operation, meaning that, for any x, y, z in S,

$$x \leq y \text{ implies } zx \leq zy \text{ and } xz \leq yz.$$

If A, B are nonempty subsets of S, we write the set product AB of A and B for the set of all elements xy of S with $x \in A$ and $y \in B$, and write $(A]$ for the set of all elements x of S such that $x \leq a$ for some a in A, i.e.,

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

In particular, we write Ax for $A\{x\}$. It was shown in [6] that $(A \cup B] = (A] \cup (B]$.

A nonempty subset A of an ordered semigroup $(S, \cdot, \leq)$ is called a *left ideal* [3] of S if it satisfies the following conditions:

(i) $SA \subseteq A$;

(ii) for any $x \in A$ and $y \in S$, $y \leq x$ implies $y \in A$.

For a in S, the principal left ideal generated by a is $L(a) := (a \cup Sa]$. It is known that the union of two left ideals of S is a left ideal of S.

A proper left ideal A of an ordered semigroup $(S, \cdot, \leq)$ is called a *maximal left ideal* if there is no a proper left ideal L' of S such that $L \subset L'$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. The equivalence relation $\mathcal{L}$ is defined by:

$$a\mathcal{L}b \text{ if and only if } L(a) = L(b)$$

for any a, b in S. The $\mathcal{L}$-class containing a in S will be written L_a. Define a preorder $\preceq$ on the set of all $\mathcal{L}$-classes by:

$$L_a \preceq L_b \text{ if and only if } L(a) \subseteq L(b).$$

The symbol $L_a \prec L_b$ stands for $L_a \preceq L_b$, but $L_a \neq L_b$. The symbol $a < b$ stands for $a \leq b$, but $a \neq b$. Note that $a \leq b$ implies $L_a \preceq L_b$. In particular, $a < b$ implies $L_a \prec L_b$.

2. Ordered Semigroups Containing One-Sided and Two-Sided Bases

We define one-sided bases and two-sided bases of an ordered semigroup by:

Definition 1. Let $(S, \cdot, \leq)$ be an ordered semigroup. A subset A of S is called a *right base* of S if it satisfies the following conditions:

(i) $S = (A \cup SA]$;

(ii) if B is a subset of A such that $S = (B \cup SB]$, then $A = B$.

Dually, one can define for A to be a *left base* of S.

By a *two-sided base* of S we mean a subset A of S such that

(i) $S = (A \cup SA \cup AS \cup SAS]$;

(ii) if B is a subset of A such that $S = (B \cup SB \cup BS \cup SBS]$, then $A = B$.

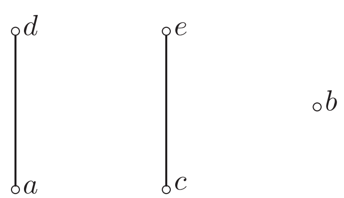
Example 2. ([7]) Let $(S, \cdot, \leq)$ be an ordered semigroup such that the multiplication and the order relation are defined by:

$\cdot$	a	b	c	d	e
a	a	e	c	d	e
b	a	b	c	d	e
c	a	e	c	d	e
d	a	e	c	d	e
e	a	e	c	d	e

$$\leq \, = \{(a,a), (b,b), (c,c), (d,d), (e,e), (a,d), (c,e)\}.$$

The covering relation and the figure of S are given by:

$$< \, = \{(a,d), (c,e)\}$$



The left bases of S are $\{a\}, \{b\}, \{c\}, \{d\}$ and $\{e\}$. The right base of S is $\{b, d\}$. S has only one two-sided base: $\{b\}$.

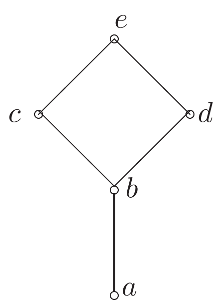
Example 3. ([4]) Let $(S, \cdot, \leq)$ be an ordered semigroup such that the multiplication and the order relation are defined by:

$\cdot$	a	b	c	d	e
a	a	a	c	a	c
b	a	a	c	a	c
c	a	a	c	a	c
d	d	d	e	d	e
e	d	d	e	d	e

$$\leq\ = \{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(b,c),\\ (b,d),(b,e),(c,c),(c,e),(d,d),(d,e),(e,e)\}.$$

The covering relation and the figure of S are given by:

$$<\ = \{(a,b),(a,c),(a,d),(a,e),(b,c),(b,d),(b,e),(c,e),(d,e)\}$$



The right bases of S are $\{e\}$ and $\{c\}$. The left bases of S are $\{d\}$ and $\{e\}$. The two-sided bases of S are $\{c\}$, $\{d\}$ and $\{e\}$.

Lemma 4. *Let A be a subset of an ordered semigroup $(S,\cdot,\leq)$. Then A is a right base of S if and only if it satisfies the following conditions*:

(i) *for any x in S there exists a in A such that $L_x \preceq L_a$;*

(ii) *if $a,b \in A$ such that $a \neq b$, then neither $L_a \preceq L_b$ nor $L_b \preceq L_a$;*

Proof. Assume that A is a right base of S. Let $x \in S$. Since $S = (A \cup SA]$, so $x \in (A]$ or $x \in (SA]$. If $x \in (A]$, then $x \leq a$ for some a in A; hence $L_x \preceq L_a$. If $x \in (SA]$, then $x \leq sa'$ for some s in S and a' in A; hence $L_x \preceq L_{a'}$. This proves that (i) holds. Let $a,b \in A$ such that $a \neq b$. Suppose $L_a \preceq L_b$. We set $B = A \setminus \{a\}$. Let $x \in S$. By (i), there exists c in A such that $L_x \preceq L_c$. If $c \neq a$, then $c \in B$; thus

$$x \in L(x) \subseteq L(c) \subseteq (B \cup SB].$$

If $c = a$, then, by $b \in B$, we have

$$x \in L(x) \subseteq L(b) \subseteq (B \cup SB].$$

Thus $S = (B \cup SB]$. This is a contradiction. The case $L_b \preceq L_a$ is proved similarly. Thus (ii) holds true.

Conversely, assume that the conditions (i) and (ii) hold. By (i), $S = (A \cup SA]$. Suppose that $S = (B \cup SB]$ for some a proper subset B of A. Let $a \in A \backslash B$. If $a \leq b$ for some b in B, then $L_a \preceq L_{a'}$. This contradicts to (ii). Similarly, if $a \leq sb'$ for some s in S and b' in B, then $L_a \preceq L_{b'}$. This is a contradiction. Hence A is a right base of S.

Theorem 5. *Assume that an ordered semigroup $(S, \cdot, \leq)$ contains a left ideal. Then L is a maximal left ideal of S if and only if $S \setminus L$ is a maximal $\mathcal{L}$-class.*

Proof. Assume that L is a maximal left ideal of S. Let $x, y \in S \setminus L$. Since $L \subset L \cup L(x) \subseteq S$, we have $L \cup L(x) = S$, and so $y \in L(x)$. Similarly, $x \in L(y)$. Thus $L(x) = L(y)$. This proves that $S \setminus L$ is an $\mathcal{L}$-class. If $S \setminus L \prec L_a$ for some a in S, then $S \setminus L \subseteq L(a) \subseteq L$. This is a contradiction. Therefore, $S \setminus L$ is a maximal $\mathcal{L}$-class.

Conversely, assume that $S \setminus L$ is a maximal $\mathcal{L}$-class such that $S \setminus L = L_a$ for some a in S. If $x \in SL \setminus L$, then $x \in L_a$; hence $L(x) = L(a)$. By $x \in SL$, $x = sb$ for some s in S and b in L. We have $L_a \prec L_b$. This is a contradiction. Hence $SL \subseteq L$. Let $y \in L$ and $c \in S$ be such that $c \leq y$. Suppose that $c \in L_a$. Then $c < y$, and so $L_a \prec L_y$. This is a contradiction. Thus $c \in L$. This shows that L is a left ideal of S. Let L' be a left ideal of S such that $L \subset L' \subset S$. Then there is $z \in S \setminus L'$. We have $L_a = L_z$. Similarly, there exists $w \in L' \setminus L$ such that $L_w = L_a$. Then

$$z \in L(z) = L(a) = L(w) \subseteq L'.$$

This is a contradiction. Therefore, L is a maximal left ideal of S.

Corollary 6. *If an ordered semigroup $(S, \cdot, \leq)$ contains a right base, then S contains a maximal left ideal.*

Proof. This is a consequence of Lemma 4 and Theorem 5.

Definition 7. A proper left ideal L of an ordered semigroup $(S, \cdot, \leq)$ is called a *covered left ideal* if $L \subseteq (S(S \setminus L)]$.

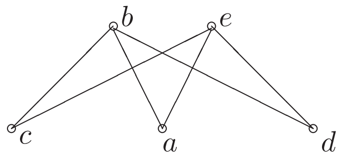
Example 8. ([8]) Let $(S, \cdot, \leq)$ be an ordered semigroup such that $S = \{a, b, c, d, e\}$ and

$\cdot$	a	b	c	d	e
a	a	b	a	a	a
b	a	b	a	a	a
c	a	b	a	a	a
d	a	b	a	a	a
e	a	b	a	a	e

$$\leq\,=\{(a,a),(a,b),(a,e),(b,b),(c,b),(c,c),(c,e),(d,d),(d,b),(d,e),(e,e)\}.$$

The covering relation and the figure of S are given by:

$$\preceq\,=\{(a,b),(a,e),(c,b),(c,e),(d,b),(d,e)\}$$



The left ideals of S are $\{a\}$, $\{a,c\}$, $\{a,d\}$, $\{a,c,d\}$, $\{a,b,c,d\}$, $\{a,c,d,e\}$ and S. The covered left ideals of S are $\{a\}$, $\{a,c\}$, $\{a,d\}$, $\{a,c,d\}$.

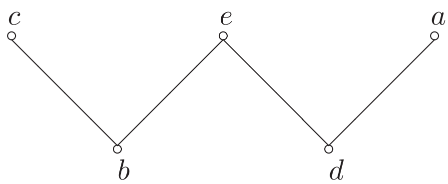
Example 9. ([5]) Let $(S,\cdot,\leq)$ be an ordered semigroup such that $S=\{a,b,c,d,f\}$ and

$\cdot$	a	b	c	d	e
a	b	d	a	b	e
b	d	b	b	d	e
c	d	b	c	d	e
d	b	d	d	b	e
e	e	e	e	e	e

$$\leq\,=\{(a,a),(b,b),(b,c),(b,e),(c,c),(d,a),(d,d),(d,e),(e,e)\}.$$

The covering relation and the figure of S are given by:

$$\preceq\,=\{(b,c),(b,e),(d,a),(d,e)\}$$



The left ideals of S are $\{b, d, e\}$, $\{a, b, d, e\}$ and S. The covered left ideal of S is $\{b, d, e\}$.

Corollary 10. *Let $(S, \cdot, \leq)$ be an ordered semigroup containing a right base. Then the following statements hold.*

(1) *S contains maximal left ideals.*

(2) *Every maximal left ideal L_i of S is $L_i = S \setminus L_{a_i}$ for some a_i in S.*

Proof. It is a consequence of Theorem 4 and 5.

Theorem 11. *Let $(S, \cdot, \leq)$ be an ordered semigroup containing maximal left ideals. If the intersection of all maximal left ideals of S is empty or a covered left ideal, then S contains a right base.*

Proof. Let $\{L_i \mid i \in I\}$ be the set of all maximal left ideals of S. By Theorem 5, for each $i \in I$, $S \setminus L_i$ is a maximal $\mathcal{L}$-class. Setting

$$S \setminus L_i := L_{a_i} \text{ for each } i \text{ in } I$$

then

$$L := \bigcap_{i \in I} L_i = \bigcap_{i \in I}(S \setminus L_{a_i}) = S \setminus \bigcup_{i \in I} L_{a_i}.$$

Let A denote the set of all elements a_i. We assert that A is a right base of S, and hence the theorem is proved. We consider two cases:

Case 1: $L = \emptyset$. Then $S = \bigcup_{i \in I} L_{a_i}$. If $x \in S$, then $x \in L_{a_i}$ for some i in I, and so $L(x) = L(a_i)$. Thus $L_x \preceq L_{a_i}$. Since L_{a_i} is a maximal $\mathcal{L}$-class for all i in I, it follows that, for any different i, j in I, neither $L_{a_i} \preceq L_{a_j}$ nor $L_{a_j} \preceq L_{a_i}$. By Lemma 4, A is a right base of S.

Case 2: L is a covered left ideal of S. That is, $L \subseteq (S(S \setminus L)]$. If $x \in S \setminus L$, then $x \in \bigcup_{i \in I} L_{a_i}$, and so $x \in L_{a_{i_0}}$ for some i_0 in I. Since

$$L(x) = L(a_{i_0}) \subseteq (A \cup SA],$$

we have $x \in (A \cup SA]$. This proves that

$$S \setminus L \subseteq (A \cup SA].$$

By

$$L \subseteq (S(S \setminus L)] \subseteq (S(A \cup SA)] \subseteq (SA \cup SSA] \subseteq (A \cup SA]$$

it follows that

$$S = L \cup (S \setminus L) \subseteq (A \cup SA].$$

This implies that if $x \in S$ then there exists $a_i \in A$ such that $L_x \preceq L_{a_i}$. It follows by Lemma 4 that A is a right base of S.

It is not true in general that an ordered semigroup contains one-sided bases implies the semigroup contains two-sided bases. This was shown by Example 2 in [2]. However, it is easy to see that this statement holds true for any finite ordered semigroups.

Theorem 12. *If an ordered semigroup $(S, \cdot, \leq)$ contains a left or a right base which is finite, then S contains a two-sided base.*

Corollary 13. *Every finite ordered semigroup contains both one-sided bases and two-sided bases.*

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