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FEKETE-SZEGÖ FUNCTIONAL FOR SOME SUBCLASS OF ANALYTIC FUNCTIONS

Xiaoxue Jiang[1], Lifeng Guo[2] §

[1]College of Petroleum Engineering
Northeast Petroleum University
Daqing, 163318, P.R. CHINA

[2]School of Mathematica Science and Technology
Northeast Petroleum University
Daqing, 163318, P.R. CHINA

Abstract: In this paper we obtain sharp upper bound of $|a_2|$ and of the Fekete-Szegö functional $|a_3 - \kappa a_2^2|$ for the generalized class of non-Bazilevič functions.

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1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + a_2 z^2 + a_2 z^2 + \cdots,\ z \in \mathbb{U}. \tag{1}$$

Let $\mathcal{P}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = 1 + c_1 z + c_2 z^2 + \cdots,\ z \in \mathbb{U}. \tag{2}$$

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§Correspondence author

A function $f \in \mathcal{A}$ is said to be starlike of order α $(0 \leq \alpha < p)$ in $\mathbb{U}$, that is, $f \in \mathcal{S}^*(\alpha)$, if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ 0 \leq \alpha < 1, z \in \mathbb{U} \tag{3}$$

with $\mathcal{S}_1^*(0) := \mathcal{S}^*$.

For $0 < \mu < 1$, a function $f(z) \in N(\mu)$ if and only if $f(z) \in \mathcal{A}$ and

$$\mathbf{Re}\left\{\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\mu}\right\} > 0,\ z \in \mathbb{U}. \tag{4}$$

$N(\mu)$ was introduced by M.Obradović [1] recently, and he called this class of functions to be non-Bazilevič type. Tuneski and Darus [2] obtained Fekete-Szegö inequality for the non-Bazilevic class of functions. Using this non-Bazilevič class, Bao et al. [3] and Sahoo [4] studied some starlikeness criterions for the class

$$\left|(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu} + \alpha\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| < M. \tag{5}$$

Let

$$\mathcal{N}(\alpha, \mu, f; z) = (1-\alpha)\left(\frac{z^p}{f(z)}\right)^{\mu} + \alpha\frac{zf'(z)}{f(z)}\left(\frac{z^p}{f(z)}\right)^{\mu} \tag{6}$$

where $z \in \mathbb{U}, 0 < \mu < 1, 0 < \alpha \leq 1$ and $f \in \mathcal{A}$.

The main object of the present sequel to the aforementioned works is to sharp upper bound of $|a_2|$ and of the Fekete-Szegö functional $|a_3 - \kappa a_2^2|$ for the generalized class of non-Bazilevič functions.. We improved results of N. Tuneski and M. Darus (see [2]).

In that purpose we will need the following lemma.

Lemma 1. (see [5]) *Let* $p(z) \in \mathcal{P}$ *is an analytic function in* U, *be given by* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *and* $\mathbf{Re}\,p(z) > 0$ *for* $z \in \mathbb{U}$. *Then*

$$\left|c_2 - \frac{1}{2}c_1^2\right| \leq 2 - \frac{|c_1|^2}{2} \tag{7}$$

and $|c_n| \leq 2$ *for all* $n \in \mathbb{N}$. *If* $p(z) = \frac{1+z^2}{1-z^2}$ *or* $p(z) = \frac{1+z}{1-z}$, *the result is sharp.*

2. Main Results

By using Lemma 1, we first prove the following Theorem.

Theorem 2. *Let* $0 < \mu < 1, 0 < \alpha \leq 1, \mu \neq \alpha, \mu \neq 2\alpha, 0 \leq \gamma < 1$ *and* $f \in \mathcal{A}$. *If*

$$\mathbf{Re}\big(\mathcal{N}(\alpha, \mu, f; z)\big) > \gamma, z \in \mathbb{U}, \tag{8}$$

then $|a_2| \leq 2(1-\gamma)/|\alpha - \mu|$ *and for all* $\kappa \in \mathbb{C}$ *the following bound is sharp.*

$$|a_3 - \kappa a_2^2| \leq \frac{2(1-\gamma)}{|2\alpha - \mu|} \max\left\{1, \left|1 + \frac{(1-\gamma)(2\alpha-\mu)(1+\mu-2\kappa)}{(\alpha-\mu)^2}\right|\right\}. \tag{9}$$

Proof. Condition (8) is equivalent to

$$1 - \alpha + \alpha\frac{zf'(z)}{f(z)} = \left(\frac{z^p}{f(z)}\right)^{\mu}\Big((1-\gamma)p(z) + \gamma\Big), z \in \mathbb{U}, \tag{10}$$

for some $p(z) \in \mathcal{P}$. Equating coeffcients we obtain

$$a_2 = \frac{1-\gamma}{\alpha-\mu}c_1 \tag{11}$$

and

$$a_3 = \frac{1-\gamma}{2\alpha-\mu}c_2 + \frac{(1-\gamma)^2(1+\mu)}{2(\alpha-\mu)^2}c_1^2, \tag{12}$$

and further

$$\begin{aligned} & a_3 - \kappa a_2^2 \\ = \; & \tfrac{1-\gamma}{2\alpha-\mu}\big(c_2 - \tfrac{1}{2}c_1\big) + \tfrac{(1-\gamma)^2(2\alpha-\mu)(1+\mu-2\kappa)+(1-\gamma)(\alpha-\mu)^2}{2(\alpha-\mu)^2(2\alpha-\mu)}c_1^2. \end{aligned} \tag{13}$$

Now, using Lemma 1 we receive

$$|a_3 - \kappa a_2^2| \leq H(x) = A + \frac{AB}{4}x^2 \tag{14}$$

where

$$x = |c_1| \leq 2, A = \frac{2(1-\gamma)}{|\alpha-\mu|} > 0 \tag{15}$$

$$B = \frac{|C| - (\alpha-\mu)^2}{(\alpha-\mu)^2} \tag{16}$$

where

$$C=(1-\gamma)(2\alpha-\mu)(1+\mu-2\kappa)+(\alpha-\mu)^2. \tag{17}$$

Thus, we have

$$|a_3-\kappa a_2^2|\le\begin{cases} H(0)=A, & if\ |C|\le(\alpha-\mu)^2,\\ H(2)=\frac{A|C|}{(\alpha-\mu)^2}, & if\ |C|>(\alpha-\mu)^2,\end{cases} \tag{18}$$

Equality is attained for functions given by

$$\mathcal{N}(\alpha,\mu,f;z)=\frac{1+(1-2\gamma)z^2}{1-z^2} \tag{19}$$

and

$$\mathcal{N}(\alpha,\mu,f;z)=\frac{1+(1-2\gamma)z}{1-z} \tag{20}$$

respectively.

Taking $\gamma=0$ in Theorem 2, we have the following corollary.

Corollary 3. *Let $0<\mu<1, 0<\alpha\le1, \mu\ne\alpha, \mu\ne2\alpha$ and $f\in\mathcal{A}$. If*

$$\mathbf{Re}\big(\mathcal{N}(\alpha,\mu,f;z)\big)>0, z\in\mathbb{U}, \tag{21}$$

then $|a_2|\le 2/|\alpha-\mu|$ and for all $\kappa\in\mathbb{C}$ the following bound is sharp.

$$|a_3-\kappa a_2^2|\le\frac{2}{|2\alpha-\mu|}\max\left\{1,\left|1+\frac{(2\alpha-\mu)(1+\mu-2\kappa)}{(\alpha-\mu)^2}\right|\right\}. \tag{22}$$

Taking $\alpha=1$ in Theorem 2, we have the following corollary (see [2]).

Corollary 4. *Let $0<\mu<1, 0\le\gamma<1$ and $f\in\mathcal{A}$. If*

$$\mathbf{Re}\big(\mathcal{N}(1,\mu,f;z)\big)>\gamma, z\in\mathbb{U}, \tag{23}$$

then $|a_2|\le 2(1-\gamma)/(1-\mu)$ and for all $\kappa\in\mathbb{C}$ the following bound is sharp.

$$|a_3-\kappa a_2^2|\le\frac{2(1-\gamma)}{1-\mu}\max\left\{1,\left|1+\frac{(1-\gamma)(2-\mu)(1+\mu-2\kappa)}{(1-\mu)^2}\right|\right\}. \tag{24}$$

Theorem 5. *Let $0<\mu<1, 0<\alpha\le1, \mu\ne\alpha, \mu\ne2\alpha, 0<\gamma\le1$ and $f\in\mathcal{A}$. If*

$$\big|\mathcal{N}(\alpha,\mu,f;z)-1\big|<\gamma, z\in\mathbb{U}, \tag{25}$$

then $|a_2| \le \gamma/|\alpha-\mu|$ *and for all* $\kappa \in \mathbb{C}$ *the following bound is sharp.*

$$|a_3 - \kappa a_2^2| \le \frac{\gamma}{|2\alpha-\mu|} \max\left\{1, \frac{\gamma|2\alpha-\mu|}{(\alpha-\mu)^2}\left|\frac{1+\mu}{2} - \kappa\right|\right\}. \tag{26}$$

Proof. Similarly as in the proof of Theorem 1, condition (25) is equivalent to

$$1-\alpha+\alpha\frac{zf'(z)}{f(z)} = \left(\frac{z^p}{f(z)}\right)^{\mu}\left(\frac{2\gamma}{1-p(z)} + 1 - \gamma\right), z \in \mathbb{U}, \tag{27}$$

for some $p(z) \in \mathcal{P}$. Equating coeffcients we obtain

$$a_2 = \frac{\gamma}{2(\alpha-\mu)}c_1 \tag{28}$$

and

$$a_3 = -\frac{\gamma}{2(2\alpha-\mu)}\left(c_2 - \frac{1}{2}c_1\right) + \frac{(1+\mu)\gamma^2}{8(\alpha-\mu)^2}c_1^2, \tag{29}$$

and further

$$a_3 - \kappa a_2^2 = -\tfrac{\gamma}{2(2\alpha-\mu)}\left(c_2 - \tfrac{1}{2}c_1\right) + \tfrac{\gamma^2}{4(\alpha-\mu)^2}\left(\tfrac{1+\mu}{2} - \kappa\right)c_1^2, . \tag{30}$$

Now, using Lemma 1 we receive

$$|a_3 - \kappa a_2^2| \le H(x) = A + \frac{B}{4}x^2 \tag{31}$$

where

$$x = |c_1| \le 2, A = \frac{\gamma}{|2\alpha-\mu|} > 0 \tag{32}$$

$$B = \frac{\gamma^2|C|}{(\alpha-\mu)^2} - \frac{\gamma}{|2\alpha-\mu|}, \tag{33}$$

where

$$C = \frac{1+\mu}{2} - \kappa. \tag{34}$$

Thus, we have

$$|a_3 - \kappa a_2^2| \le \begin{cases} H(0) = A, & if\ |C| \le \frac{\gamma(\alpha-\mu)^2}{\gamma^2|2\alpha-\mu|}, \\ H(2) = \frac{A\gamma|C||2\alpha-\mu|}{(\alpha-\mu)^2}, & if\ |C| > \frac{\gamma(\alpha-\mu)^2}{\gamma^2|2\alpha-\mu|}, \end{cases} \tag{35}$$

Equality is attained for functions given by

$$\mathcal{N}(\alpha, \mu, f; z) = 1 - \gamma z^2 \tag{36}$$

and

$$\mathcal{N}(\alpha, \mu, f; z) = 1 - \gamma z \tag{37}$$

respectively.

Taking $\gamma = 1$ in Theorem 5, we have the following corollary.

Corollary 6. *Let* $0 < \mu < 1, 0 < \alpha \leq 1, \mu \neq \alpha, \mu \neq 2\alpha$ *and* $f \in \mathcal{A}$. *If*

$$\left|\mathcal{N}(\alpha, \mu, f; z) - 1\right| < 1, z \in \mathbb{U}, \tag{38}$$

then $|a_2| \leq 1/|\alpha - \mu|$ *and for all* $\kappa \in \mathbb{C}$ *the following bound is sharp.*

$$|a_3 - \kappa a_2^2| \leq \frac{1}{|2\alpha - \mu|} \max\left\{1, \frac{|2\alpha - \mu|}{(\alpha - \mu)^2}\left|\frac{1+\mu}{2} - \kappa\right|\right\}. \tag{39}$$

Taking $\alpha = 1$ in Theorem 5, we have the following corollary (see [5]).

Corollary 7. *Let* $0 < \mu < 1, 0 < \gamma \leq 1$ *and* $f \in \mathcal{A}$. *If*

$$\left|\mathcal{N}(1, \mu, f; z) - 1\right| < \gamma, z \in \mathbb{U}, \tag{40}$$

then $|a_2| \leq \gamma/(1 - \mu)$ *and for all* $\kappa \in \mathbb{C}$ *the following bound is sharp.*

$$|a_3 - \kappa a_2^2| \leq \frac{\gamma}{(2 - \mu)} \max\left\{1, \frac{\gamma(2 - \mu)}{(1 - \mu)^2}\left|\frac{1+\mu}{2} - \kappa\right|\right\}. \tag{41}$$

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