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ON CERTAIN RELATIONS FOR C-CLOSURE OPERATIONS ON AN ORDERED SEMIGROUP

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Abstract: In this paper, a relation for C-closure operations on an ordered semigroup is introduced, using this relation regular and simple ordered semigroups are characterized.

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1. Preliminaries

It is known that a semigroup S is regular if and only if it satisfies:

$$A \cap B = AB$$

for all right ideals A and for all left ideals B of S. Using this property, Ponděĺíček [2] introduced a relation for C-closure operations on S, and studied some types of semigroups using the relation. The purpose of this paper is to extend Ponděĺíček's results to ordered semigroups. In fact, we define a relation for C-closure operations on an ordered semigroup, and characterize regular and simple ordered semigroups using the relation. Firstly, let us recall some certain definitions and results which are in [2].

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Let S be a nonempty set. A mapping $\mathbf{U}$:Su$(S) \to$ Su(S) (The symbol Su(S) stands for the set of all subsets of S) is called a *C-closure operation* on S if, for any A, B in Su(S), it satisfies:

(i) $\mathbf{U}(\emptyset) = \emptyset$;

(ii) $A \subseteq B \Rightarrow \mathbf{U}(A) \subseteq \mathbf{U}(B)$;

(iii) $A \subseteq \mathbf{U}(A)$;

(iv) $\mathbf{U}(\mathbf{U}(A)) = \mathbf{U}(A)$.

For an element x in S we write $\mathbf{U}(x)$ instead of $\mathbf{U}(\{x\})$. A subset A of S is said to be $\mathbf{U}$-*closed* if $\mathbf{U}(A) = A$; and A is said to be $\mathbf{U}$-*open* if $S \setminus A$ is $\mathbf{U}$-closed. Let $\mathcal{F}(\mathbf{U})$ denote the set of all $\mathbf{U}$-closed subsets of S, and let $\mathcal{O}(\mathbf{U})$ denote the set of all $\mathbf{U}$-open subsets of S.

Define a relation $\leq$ on $\mathcal{C}(S)$, the set of all C-closure operations on a nonempty set S, by

$$\mathbf{U} \leq \mathbf{V} \text{ if and only if } \mathbf{U}(A) \subseteq \mathbf{V}(A) \text{ for any } A \text{ in Su}(S).$$

A C-closure operation $\mathbf{I}$ on S is defined by $\mathbf{I}(\emptyset) = \emptyset$, and $\mathbf{I}(A) = S$ for any nonempty subset A of S. A C-closure operation $\mathbf{O}$ on S is defined by $\mathbf{O}(A) = A$ for all subsets A of S. For any $\mathbf{U}$ and $\mathbf{V}$ in $\mathcal{C}(S)$ it is known that:

(1) $\mathbf{O} \leq \mathbf{U} \leq \mathbf{I}$.

(2) $\mathbf{U} \leq \mathbf{V}$ if and only if $\mathcal{F}(\mathbf{V}) \subseteq \mathcal{F}(\mathbf{U})$.

(3) $\mathbf{U} \vee \mathbf{V}, \mathbf{U} \wedge \mathbf{V}$ exist, and

(3.1) $\mathcal{F}(\mathbf{U} \vee \mathbf{V}) = \mathcal{F}(\mathbf{U}) \cap \mathcal{F}(\mathbf{V})$,
(3.2) $\mathcal{F}(\mathbf{U} \wedge \mathbf{V}) = \{A \cap B \mid A \in \mathcal{F}(\mathbf{U}), B \in \mathcal{F}(\mathbf{V})\}$.

A C-closure operation $\mathbf{U}$ on a nonempty set S is said to be a *D-closure operation* if, for any indexed family $\{A_i \mid i \in I\}$ of subsets of S, it satisfies:

$$\mathbf{U}(\bigcup A_{i \in I}) = \bigcup_{i \in I} \mathbf{U}(A_i).$$

Note that if $\mathbf{U}$ and $\mathbf{V}$ are D-closure operations on S, then $\mathbf{U} \vee \mathbf{V}$ is a D-closure operation on S.

For each C-closure operation $\mathbf{U}$ on a nonempty set S, a D-closure operation $\mathbf{U}^*$ is defined on S by

$$\mathbf{U}^*(A) = \{x \mid \mathbf{U}(x) \cap A \neq \emptyset\}$$

for any A in $\mathrm{Su}(S)$.

It is known that:

(1) $\mathbf{I}^* = \mathbf{I}, \mathbf{O}^* = \mathbf{O}$.

(2) For any $\mathbf{U}, \mathbf{V}$ in $\mathcal{C}(S)$, $\mathbf{U} \leq \mathbf{V}$ implies $\mathbf{U}^* \leq \mathbf{V}^*$.

(3) For any $\mathbf{U}$ in $\mathcal{C}(S)$, the following conditions are equivalent:

(3.1) $\mathbf{U}$ is a D-closure operation;

(3.2) $\mathbf{U} = \mathbf{U}^{**}$;

(3.3) $\mathcal{F}(\mathbf{U}) = \mathcal{O}(\mathbf{U}^*)$;

(3.4) $\mathcal{O}(\mathbf{U}) = \mathcal{F}(\mathbf{U}^*)$.

(4) For a D-closure operation $\mathbf{U}$ on S, $\mathbf{U} \leq \mathbf{U}^*$ or $\mathbf{U}^* \leq \mathbf{U}$ implies $\mathbf{U} = \mathbf{U}^*$.

2. Main Results

An *ordered semigroup* [1] is defined to be a semigroup $(S, \cdot)$ together with a partial order $\leq$ that is *compatible* with the semigroup operation, meaning that for $x, y, z \in S$,

$$x \leq y \Rightarrow zx \leq zy,\ xz \leq yz.$$

Let $(S, \cdot, \leq)$ be an ordered semigroup. If A, B are nonempty subsets of S, we write AB for the set of all elements xy in S such that x in A and y in B, and write

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

For an element x in S, we write Ax and xA for $A\{x\}$ and $\{x\}A$, respectively. In [4], the following conditions hold:

(1) $A \subseteq (A]$;

(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $(A](B] \subseteq (AB]$;

(4) $((A](B]] = (AB]$;

(5) $(A \cup B] = (A] \cup (B]$.

The following concepts can be found in [3]. Let $(S, \cdot, \leq)$ be an ordered semigroup. A nonempty subset A of S is called a *left* (respectively, *right*) *ideal* [3] of S if it satisfies:

(i) $SA \subseteq A$ (respectively, $AS \subseteq A$);

(ii) $A = (A]$, that is, for $x \in A$ and $y \in S$, $y \leq x$ implies $y \in A$.

If A is both a left and a right ideal of S, then A is called an *ideal* of S.

Let $(S, \cdot, \leq)$ be an ordered semigroup. If A is a nonempty subset of S, then $(A \cup SA]$ (respectively, $(A \cup AS]$, $(A \cup SA \cup AS \cup SAS]$) are left (respectively, right, two-sided) ideals of S.

Definition 1. Let $(S, \cdot, \leq)$ be an ordered semigroup. Define a relation ϱ on $\mathcal{C}(S)$ by

$$\mathbf{U}\varrho\mathbf{V} \text{ if and only if } A \cap B = (AB]$$

for all nonempty subsets A in $\mathcal{F}(\mathbf{U})$ and for all nonempty subsets B in $\mathcal{F}(\mathbf{V})$.

Lemma 2. *Let $(S, \cdot, \leq)$ be an ordered semigroup, and let $\mathbf{U}, \mathbf{U}', \mathbf{V}, \mathbf{V}'$ be C-closure operations on S such that $\mathbf{U}\varrho\mathbf{V}$. If $\mathbf{U} \leq \mathbf{U}'$ and $\mathbf{V} \leq \mathbf{V}'$, then $\mathbf{U}'\varrho\mathbf{V}'$.*

Proof. This follows directly from the definition of ϱ. □

Let $(S, \cdot, \leq)$ be an ordered semigroup. Define a mapping $\mathbf{L}$ on $\mathrm{Su}(S)$ by $\mathbf{L}(\emptyset) = \emptyset$, and

$$\mathbf{L}(A) = (A \cup SA]$$

for any nonempty subset A of S. It is easy to verify that $\mathbf{L}$ is a C-closure operation on S. Note that $\mathcal{F}(\mathbf{L})$ is the set of all left ideals of S (including empty set). Indeed, if L is a left ideal of S, then

$$L \subseteq \mathbf{L}(L) = (L \cup SL] \subseteq (L] = L;$$

hence $L \in \mathcal{F}(\mathbf{L})$. Conversely, if $L \in \mathcal{F}(\mathbf{L})$, then $L = \mathbf{L}(L) = (L \cup SL]$, and thus L is a left ideal of S. Similarly, we define a C-closure operation on S by $\mathbf{R}(\emptyset) = \emptyset$, and

$$\mathbf{R}(A) = (A \cup AS]$$

for any nonempty subset A of S. $\mathcal{F}(\mathbf{R})$ is the set of all right ideals of S (including empty set).

Lemma 3. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then* $\mathbf{L}$ *and* $\mathbf{R}$ *are D-closure operations on S.*

Proof. Let $\{A_i \mid i \in I\}$ be an indexed family of subsets of S. We have

$$\begin{aligned}
\mathbf{L}\left(\bigcup_{i\in I} A_i\right) &= \left(\left(\bigcup_{i\in I} A_i\right) \bigcup S\left(\bigcup_{i\in I} A_i\right)\right] \\
&= \left(\left(\bigcup_{i\in I} A_i\right) \bigcup \left(\bigcup_{i\in I} (SA_i)\right)\right] \\
&= \left(\bigcup_{i\in I} (A_i \cup SA_i)\right] \\
&= \bigcup_{i\in I} (A_i \cup SA_i] \\
&= \bigcup_{i\in I} \mathbf{L}(A_i)
\end{aligned}$$

Then $\mathbf{L}$ is a D-closure operation on S. Similarly, $\mathbf{R}$ is a D-closure operation on S. □

Theorem 4. *Let $(S, \cdot, \leq)$ be an ordered semigroup, and let $\mathbf{U}, \mathbf{V}$ be C-closure operations on S. Then $\mathbf{U}\varrho\mathbf{V}$ if and only if $\mathbf{R} \leq \mathbf{U}, \mathbf{L} \leq \mathbf{V}$ and $x \in (\mathbf{U}(x)\mathbf{V}(x)]$ for all x in S.*

Proof. Assume $\mathbf{U}\varrho\mathbf{V}$. Clearly, $S \in \mathcal{F}(\mathbf{V})$. If $A \in \mathcal{F}(\mathbf{U})$, then $A = A \cap S = (AS]$; hence

$$\mathbf{R}(A) = (A \cup AS] = ((AS] \cup AS] = ((AS]] = (AS] = A.$$

This shows that $A \in \mathcal{F}(\mathbf{R})$; thus $\mathbf{R} \leq \mathbf{U}$. Similarly, $\mathbf{L} \leq \mathbf{V}$. Let x be an element of S. Since $\mathbf{U}(x) \in \mathcal{F}(\mathbf{U})$ and $\mathbf{V}(x) \in \mathcal{F}(\mathbf{V})$, we have $x \in \mathbf{U}(x) \cap \mathbf{V}(x) \subseteq (\mathbf{U}(x)\mathbf{V}(x)]$ as required.

Conversely, assume that $\mathbf{R} \leq \mathbf{U}, \mathbf{L} \leq \mathbf{V}$ and $x \in (\mathbf{U}(x)\mathbf{V}(x)]$ for all x in S. To show that $\mathbf{U}\varrho\mathbf{V}$, let $A \in \mathcal{F}(\mathbf{U})$ and $B \in \mathcal{F}(\mathbf{V})$ be nonempty. Then $A \in \mathcal{F}(\mathbf{R})$ and $B \in \mathcal{F}(\mathbf{L})$. We have

$$(AB] \subseteq (AS] \subseteq (A] = A \text{ and } (AB] \subseteq (SB] \subseteq (B] = B.$$

This shows that $(AB] \subseteq A \cap B$. For the reverse inclusion, let x be an element of $A \cap B$. Then $\mathbf{U}(x) \subseteq \mathbf{U}(A) = A$. Similarly, $\mathbf{V}(x) \subseteq B$. By

$$x \in \mathbf{U}(x)\mathbf{V}(x) \subseteq AB \subseteq (AB]$$

follows $A \cap B \subseteq (AB]$. $\square$

An ordered semigroup $(S, \cdot, \leq)$ is said to be *left regular* if $x \in (Sx^2]$ for every x in S, or equivalently, $x \in (x^2 \cup Sx^2]$ for every x in S. A *right regular ordered semigroup* is defined dually. S is said to be *regular* if $x \in (xSx]$ for every x in S, or equivalently, $x \in (x^2 \cup xSx]$ for every x in S. These concepts can be found in [4].

Theorem 5. *An ordered semigroup $(S, \cdot, \leq)$ is regular if and only if* $\mathbf{R}\varrho\mathbf{L}$.

Proof. Assume that S is regular. Then for any x in S we have

$$x \in (xSx] = ((x](Sx]] \subseteq (\mathbf{R}(x)\mathbf{L}(x)];$$

hence $\mathbf{R}\varrho\mathbf{L}$ by Theorem 4.

Conversely, $\mathbf{R}\varrho\mathbf{L}$ implies S is regular since, for any x in S, we have

$$\begin{aligned} x \in (\mathbf{R}(x)\mathbf{L}(x)] &= ((x \cup xS](x \cup Sx]] \\ &= ((x \cup xS)(x \cup Sx)] \\ &\subseteq (x^2 \cup xSx]. \end{aligned}$$

$\square$

Let $(S, \cdot, \leq)$ be an ordered semigroup. We denote the D-closure operation $\mathbf{R} \vee \mathbf{L}$ on S by $\mathbf{M}$. Note that $\mathcal{F}(\mathbf{M})$ is the set of all ideals of S (including empty set).

Theorem 6. *The following statements are equivalent on an ordered semigroup $(S, \cdot, \leq)$:*

(1) $\mathbf{L}\varrho\mathbf{L}$;

(2) $\mathbf{L}\varrho\mathbf{M}$;

(3) *S is left regular and* $\mathbf{R} \leq \mathbf{L}$.

Proof. (1) $\Rightarrow$ (2). Since $\mathbf{L} \leq \mathbf{M}$, it follows by Lemma 2 that $\mathbf{L}\varrho\mathbf{M}$.

(2) $\Rightarrow$ (3). Assume $\mathbf{L}\varrho\mathbf{M}$. Then $\mathbf{R} \leq \mathbf{L}$ by Theorem 4; hence $\mathbf{M} = \mathbf{L}$. For any x in S, we have

$$\begin{aligned} x \in (\mathbf{L}(x)\mathbf{M}(x)] &= (\mathbf{L}(x)\mathbf{L}(x)] \\ &= ((x \cup Sx](x \cup Sx]] \\ &= ((x \cup Sx)(x \cup Sx)] \\ &\subseteq (x^2 \cup xSx \cup Sx^2 \cup SxSx] \end{aligned}$$

$$\begin{aligned} &= ((x \cup xS)x \cup S(x \cup xS)x] \\ &\subseteq (\mathbf{R}(x)x \cup S\mathbf{R}(x)x] \\ &\subseteq (\mathbf{L}(x)x \cup S\mathbf{L}(x)x] \\ &\subseteq (x^2 \cup Sx^2]. \end{aligned}$$

This shows that S is left regular.

(3) $\Rightarrow$ (1). Assume that S is left regular and $\mathbf{R} \leq \mathbf{L}$. Then, for any x in S, we have

$$x \in (Sx^2] = ((Sx](x]] \subseteq (\mathbf{L}(x)\mathbf{L}(x)].$$

By Theorem 4, $\mathbf{L}\varrho\mathbf{L}$. □

Theorem 7. *The following statements are equivalent on an ordered semigroup* $(S, \cdot, \leq)$:

(1) $\mathbf{R}\varrho\mathbf{R}$;

(2) $\mathbf{M}\varrho\mathbf{R}$;

(3) *S is right regular and* $\mathbf{L} \leq \mathbf{R}$.

Proof. The proof is left-right dual of Theorem 6. □

An ordered semigroup $(S, \cdot, \leq)$ is said to be *left simple* (*right simple*) if it contains no proper left (right) ideal, and S is said to be *simple* if it contains no proper ideal.

Lemma 8. *Let* $(S, \cdot, \leq)$ *be an ordered semigroup. Then*:

(1) *S is left simple if and only if* $\mathbf{L} = \mathbf{I}$;

(2) *S is right simple if and only if* $\mathbf{R} = \mathbf{I}$;

(3) *S is simple if and only if* $\mathbf{M} = \mathbf{I}$.

Proof. Assume that S is a left simple. If A is a nonempty subset of S, then

$$\mathbf{L}(A) = (A \cup SA] = S = \mathbf{I}(A);$$

hence $\mathbf{L} = \mathbf{I}$. Conversely, if A is a left ideal of S, then

$$S = \mathbf{I}(A) = \mathbf{L}(A) = (A \cup SA] \subseteq A \subseteq S,$$

i.e., $A = S$. This proves (1). The statements (2) and (3) are proved similarly. □

Lemma 9. $\mathbf{L}\vee\mathbf{R}^*=\mathbf{I}=\mathbf{L}^*\vee\mathbf{R}$.

Proof. Let A be an element of $\mathcal{F}(\mathbf{L}\vee\mathbf{R}^*)$; then $A\in\mathcal{F}(\mathbf{L})$ and $S\setminus A\in\mathcal{F}(\mathbf{R})$. If A is a proper subset of S, then $(S\setminus A)A\subseteq A\cap(S\setminus A)$. This is a contradiction. Thus $A=S$, and hence $\mathbf{L}\vee\mathbf{R}^*=\mathbf{I}$. Similarly, $\mathbf{L}^*\vee\mathbf{R}=\mathbf{I}$. □

Theorem 10. *The following statements are equivalent on an ordered semigroup* $(S,\cdot,\leq)$:

(1) *S is simple*;

(2) $\mathbf{L}\leq\mathbf{M}^*$;

(3) $\mathbf{R}\leq\mathbf{M}^*$;

(4) $\mathbf{M}^*\varrho\mathbf{I}$;

(5) $\mathbf{I}\varrho\mathbf{M}^*$.

Proof. If S is simple, then $\mathbf{M}=\mathbf{I}$; hence

$$\mathbf{L}\leq\mathbf{I}=\mathbf{I}^*=\mathbf{M}^*.$$

Conversely, assume that $\mathbf{L}\leq\mathbf{M}^*$. Since $\mathbf{R}\leq\mathbf{M}$, we have $\mathbf{R}^*\leq\mathbf{M}^*$. By Lemma 9,

$$\mathbf{I}=\mathbf{L}\vee\mathbf{R}^*\leq\mathbf{M}^*.$$

Thus $\mathbf{M}^*=\mathbf{I}$, $\mathbf{M}=\mathbf{I}$. This proves (1) $\Leftrightarrow$ (2). For (1) $\Leftrightarrow$ (3) can be proved similarly. By Theorem 4, (5) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1). If S is simple, then $\mathbf{M}=\mathbf{I}=\mathbf{I}^*$. Since $(S^2]=S$, it follows that $\mathbf{I}\varrho\mathbf{I}$. Thus (4) and (5) hold. □

Lemma 11. *An ordered semigroup $(S,\cdot,\leq)$ is left simple if and only if* $\mathbf{R}\leq\mathbf{L}^*$.

Proof. If S is left simple, then $\mathbf{L}=\mathbf{I}$; hence

$$\mathbf{R}\leq\mathbf{I}=\mathbf{I}^*=\mathbf{L}^*.$$

Conversely, assume that $\mathbf{R}\leq\mathbf{L}^*$. By Lemma 9,

$$\mathbf{I}=\mathbf{L}^*\vee\mathbf{R}\leq\mathbf{L}^*\leq\mathbf{I}.$$

Hence $\mathbf{L}^*=\mathbf{I}$, and so $\mathbf{L}=\mathbf{I}$. □

Theorem 12. *The following are equivalent for an ordered semigroup* $(S,\cdot,\leq)$:

(1) *S is left simple*;

(2) $\mathbf{L}^*\varrho\mathbf{I}$;

(3) $\mathbf{L}\varrho\mathbf{L}^*$.

Proof. This can be proved as Theorem 10. □

Lemma 13. *An ordered semigroup $(S, \cdot, \leq)$ is right simple if and only if $\mathbf{L} \leq \mathbf{R}^*$.*

Proof. This can be proved as Lemma 11. □

Theorem 14. *The following are equivalent for an ordered semigroup $(S, \cdot, \leq)$:*

(1) *S is right simple*;

(2) $\mathbf{I}\varrho\mathbf{R}^*$;

(3) $\mathbf{R}^*\varrho\mathbf{R}$.

Proof. This can be proved as Theorem 10. □

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