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EXISTENCE OF WEAK SOLUTIONS FOR $p(x)$-KIRCHHOFF-TYPE EQUATION

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Abstract: This paper is considerned with the existence of solutions for $p(x)$-Kirchhoff-type problem under with Dirichlet boundary condition. By direct variational method and the Mountain Pass theorem, we establish some conditions that ensure the existence nontrivial weak solutions for the problem.

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Key Words: variational method, $p(x)$-Kirchhoff-type equation, Mountain Pass Theorem

1. Introduction

In this paper we are concerned with the following problem

$$\begin{cases} -M\left(\int\limits_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx\right) \Delta_{p(x)} u = \lambda m\left(x\right) \left|u\right|^{r(x)-2} u & \text{in } \Omega\ , \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a smooth bounded domain, $p, r, s \in C\left(\overline{\Omega}\right)$ for any $x \in \overline{\Omega}$; m is a non-negative measurable real function.

We assume that M and m are satisfy the following conditions specief conditions.

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(M_1) $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that

$$m_1 t^{\alpha(x)-1} \leq M(t) \leq m_2 t^{\alpha(x)-1}$$

for all $t > 0$ and m_1, m_2, α real numbers such that $0 < m_1 \leq m_2$ and $\alpha(x) \geq 1$;

(**A1**) $m \in L^{\beta(x)}(\Omega)$, $M(x) > 0$ and $\beta \in C_+(\overline{\Omega})$ such that $\frac{1}{\beta(x)} + \frac{1}{\beta_0(x)} = 1$, $p(x) < \frac{\beta(x)-1}{\beta(x)} p^*(x)$ and $1 < r(x) < \frac{\beta(x)}{\beta(x)-1}, \forall x \in \overline{\Omega}$;

(**A2**) $1 < r^- \leq r^+ < \alpha^- p^- < \alpha^+ p^+$ and $\alpha^+ p^+ < (p^*)^-$,where $(p^*)^- = \frac{Np^-}{N-p^-}$.

Problem (**P**) is related to the stationary version of a model, the so-called Kirchhoff equation, introduced by Kirchhoff [16]. To be more precise, Kirchhoff established a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

where ρ, P_0, h, E, L are constants, which extends the classical D'Alambert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations.

In recent years, elliptic problems involving $p-$Kirchhoff and $p(x)-$Kirchhoff type operators have been studied in many papers, we refer to [2, 5, 7, 8, 9, 10, 18, 23, 25].

The $p(x)-$Laplacian operator $\Delta_{p(x)} u = \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right)$ is a generalization of $p-$Laplace operator $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$ where $p > 1$ is a real constant. The $p(x)-$Laplacian possesses more complicated structure than the $p-$Laplacian; for example, it is not homogeneous.This fact implies some diffculties; for example, we can not use the theory of Sobolev spaces in many problems involving this operator. Some of the nonlinear problems involving $p(x)$-growth conditions are extremely attractive because those problems can be used to model dynamical phenomenons that arise from the study of electrorheological fluids or elastic mechanics [1, 11, 15, 22, 26]. Moreover, problems with variable exponent growth conditions also appear in the mathematical modelling of stationary thermo-rheological viscous flows of non-Newtonian fluids, in the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium and image processing [3, 4, 6]. The detailed application backgrounds of the $p(x)-$Laplace operator can be found in [13, 14, 20, 21, 17].

In the present paper, by help of the Mountain Pass theorem, we obtain the existence at least one nontrivial weak solution of problem (**P**). This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces . In Section 3, we show the existence of weak solutions of problem (**P**).

2. Preliminaries

We state some definitions and basic properties of variable exponent Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$, $W_0^{1,p(x)}(\Omega)$ and $L_{c(x)}^{p(x)}(\Omega)$ (for details, see [13, 17, 19]).

Set

$$C_+\left(\overline{\Omega}\right) = \left\{p;\ p \in C\left(\overline{\Omega}\right),\ p(x) > 1, \text{ for any } x \in \overline{\Omega}\right\}.$$

For any $p(x) \in C_+\left(\overline{\Omega}\right)$, we denote

$$1 < p^- := \min_{x\in\overline{\Omega}} p(x) \le p(x) \le p^+ := \max_{x\in\overline{\Omega}} p(x) < \infty.$$

For any $p(x) \in C_+\left(\overline{\Omega}\right)$, we define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u \,|u : \Omega \to \mathbb{R} \text{ is measurable, } \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

The modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx,\ \forall u \in L^{p(x)}(\Omega).$$

Proposition 2.1. (see [13, 17]) *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\left|\int_\Omega uv dx\right| \le (\frac{1}{p^-} + \frac{1}{(p^-)'}) \left|u\right|_{p(x)} \left|v\right|_{p'(x)}.$$

Proposition 2.2. (see [13, 17]) *If $u, u_n \in L^{p(x)}(\Omega)$, $n = 1, 2, ...$ we have*

$$(i)\ |u|_{p(x)} > 1 \implies |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+};$$

(ii) $|u|_{p(x)} < 1 \implies |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}$;

(iii) $\lim_{n\to\infty} |u_n|_{p(x),\Omega} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(x)}(u_n) = 0$;

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is denined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \mid \ |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \forall u \in W^{1,p(x)}(\Omega).$$

The space $W_0^{1,p(x)}(\Omega)$ is denoted by the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. We can define an equivalent norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}.$$

Proposition 2.3. (see [12]) *Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^\infty(\Omega)$, and $1 \leq p(x)q(x) \leq \infty$, for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Then*

$$|u|_{p(x)q(x)} \leq 1 \implies |u|_{p(x)q(x)}^{p^+} \leq \left| |u|^{p(x)} \right|_{q(x)} \leq |u|_{p(x)q(x)}^{p^-};$$

$$|u|_{p(x)q(x)} \geq 1 \implies |u|_{p(x)q(x)}^{p^-} \leq \left| |u|^{p(x)} \right|_{q(x)} \leq |u|_{p(x)q(x)}^{p^+}.$$

In particular, if $p(x) = p$ is constant, then $||u|^p|_{q(x)} = |u|_{pq(x)}^p$.

We also consider the weighted variable exponent Lebesgue space $L_{c(x)}^{p(x)}(\Omega)$. Let $c : \Omega \to \mathbb{R}$ be a measurable real function such that $c(x) > 0$ a.e. $x \in \Omega$. We define

$$L_{c(x)}^{p(x)}(\Omega) = \left\{ u \, | u : \Omega \to \mathbb{R} \text{ is measurable, } \int_\Omega c(x) \, |u(x)|^{p(x)} \, dx < \infty; \ c(x) > 0 \right\},$$

The then $L_{c(x)}^{p(x)}(\Omega)$ is a Banach space which has similar properties with the usual variable exponent Lebesgue spaces. The modular of this space is $\rho_{(c(x),p(x))} : L_{c(x)}^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{(c(x),p(x))}(u) = \int_\Omega c(x) \, |u(x)|^{p(x)} \, dx.$$

Proposition 2.4. (see [17]) *If* $p^+ < \infty$ *and* $u, u_n \in L^{p(x)}_{c(x)}(\Omega)$, $n = 1, 2, ...$ *we have*

$$(i)\ |u|_{(c(x),p(x))} > 1 \Longrightarrow |u|^{p^-}_{(c(x),p(x))} \leq \rho_{(c(x),p(x))}(u) \leq |u|^{p^+}_{(c(x),p(x))},$$
$$(ii)\ |u|_{(c(x),p(x))} < 1 \Longrightarrow |u|^{p^+}_{(c(x),p(x))} \leq \rho_{(c(x),p(x))}(u) \leq |u|^{p^-}_{(c(x),p(x))},$$

$$(iii) \lim_{n\to\infty} |u_n|_{(c(x),p(x))} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{(c(x),p(x))}(u_n) = 0.$$

Proposition 2.5. (see [13, 17]) *(i) If* $1 < p^- \leq p^+ < \infty$ *then, the spaces* $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ *and* $W_0^{1,p(x)}(\Omega)$ *are separable and reflexive Banach spaces;*

(ii) Let $q \in C_+(\overline{\Omega})$ *.If* $q(x) < p^*(x)$, *for all* $x \in \overline{\Omega}$, *then the embedding* $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ *is compact and continuous* ($p^*(x) = \frac{Np(x)}{N-p(x)}$ *if* $N > p(x)$ *or* $p^*(x) = \infty$ *if* $N \leq p(x)$), *also there is a constant* $c > 0$ *such that*

$$|u|_{q(x)} \leq c\,\|u\|, \text{ for all } u \in W_0^{1,p(x)}(\Omega).$$

Proposition 2.6. (see [14]) *Let* X *be a Banach space and*

$$\Lambda(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx.$$

The functional $\Lambda : X \to \mathbb{R}$ *is convex. The mapping* $\Lambda' : X \to X^*$ *is a strictly monotone, bounded homeomorphism, and of* (S_+) *type, namely*

$$u_n \rightharpoonup u \text{ (weakly) and } \overline{\lim_{n\to\infty}} \langle \Lambda'(u_n), u_n - u\rangle \leq 0 \text{ implies } u_n \to u \text{ (strongly)},$$

where $X = W_0^{1,p(x)}(\Omega)$.

3. Main Results

We say that $u \in X$ is a weak solution of (**P**) if

$$M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx\right) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx = \lambda \int_\Omega m(x)\,|u|^{r(x)-2}\, u\varphi dx,$$

where $\varphi \in X$.

We associate to the problem (**P**) the energy functional, defined as $I : X \to \mathbb{R}$,

$$I(u) = \widehat{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right) - \lambda \int_\Omega \frac{m(x)}{r(x)} |u|^{r(x)} dx,$$

where $\widehat{M}(t) = \int_0^t M(s)\,ds$. In a standart way, it can be shown that $I \in C^1(X, \mathbb{R})$.Moreover, we have

$$\langle I'(u), v\rangle = M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx \\ - \lambda \int_\Omega m(x) |u|^{r(x)-2} uv dx,$$

for any $u, v \in X$. Hence, we can infer that critical points of functional I are the weak solutions for problem (**P**).

Theorem 3.1. *Assume that the conditions* (M_1), (**A1**) *and* (**A2**) *are satisfied, then there exists* $\lambda^* > 0$ *such that for any* $\lambda \in (0, \lambda^*)$, *problem* (**P**) *has a nontrivial weak solution.*

Definition 3.2. *We say that* I *satisfies Palais-Smale condition in* X ((PS) *condition for short) if every sequence* $\{u_n\} \subset X$ *such that* $|I(u_n)| \le c$ *and* $I'(u_n) \to 0$ *contains a convergent subsequence in the norm of* X.

Lemma 3.3. *Assume that* (M_1), (**A1**) *and* (**A2**) *hold. There exist two positive real numbers* ρ *and* γ *such that for any* $\lambda \in (0, \lambda^*)$, *we have*

$$I(u) \ge \rho > 0, \forall u \in X \text{ with } \|u\| = \gamma.$$

Proof. By using the conditions (**A2**) and (**A1**) ,the embedding from X to $L^{r(x)}_{m(x)}(\Omega)$ is compact (see Theorem 2.8 [19]). Then, we can write

$$\int_\Omega m(x) |u_n|^{r(x)} dx \le C\left(\|u_n\|^{r^+} + \|u_n\|^{r^-}\right), \tag{3.1}$$

for all $u \in X$. We consider $\|u\| < 1$. Then, by (M_1), (3.1) and Proposition 2.2, we have

$$I(u) \quad \ge \quad \frac{m_1}{\alpha^+ (p^+)^{\alpha^+}} \|u\|^{\alpha^+ p^+} - \frac{\lambda}{r^-}\int_\Omega m(x) |u|^{r(x)} dx$$

$$
\begin{aligned}
&\geq \frac{m_1}{\alpha^+ (p^+)^{\alpha^+}} \|u\|^{\alpha^+ p^+} - \frac{\lambda c_2}{r^-} \|u\|^{r^-} \\
&= \left(\frac{m_1}{\alpha^+ (p^+)^{\alpha^+}} \|u\|^{\alpha^+ p^+ - r^-} - \frac{\lambda c_2}{r^-} \right) \|u\|^{r^-} .
\end{aligned}
$$

By the above inequality, if we choose

$$
\lambda^* = \frac{m_1 r^-}{2\alpha^+ (p^+)^{\alpha^+} c_2} \gamma^{\alpha^+ p^+ - r^-},
$$

then, there exist two positive real numbers ρ *and* γ such that $\lambda \in (0, \lambda^*)$, we have

$$
I(u) \geq \rho > 0, \forall u \in W_0^{1,p(x)}(\Omega) \text{ with } \|u\| = \gamma \in (0,1) .
$$

The proof of Lemma 3.3 is complete. □

Lemma 3.4. *Assume that* (M_1), $(\mathbf{A1})$ *and* $(\mathbf{A2})$ *hold. Then, there exists* $\omega \in X$ *such that* $\omega \geq 0$, $\omega \neq 0$ *and* $I(t\omega) < 0$ *for all* $t > 0$ *small enough.*

Proof. Then by (M_1), we have

$$
\begin{aligned}
I(t\omega) &= \widehat{M} \left(\int_\Omega \frac{|\nabla t\omega|^{p(x)}}{p(x)} dx \right) - \lambda \int_\Omega \frac{m(x)}{r(x)} |t\omega|^{r(x)} dx \\
&\leq \frac{m_2 t^{\alpha^- p^-}}{\alpha^- (p^-)^{\alpha^-}} \left(\int_\Omega |\nabla \omega|^{p(x)} dx \right)^{\alpha^-} - \frac{\lambda t^{r+}}{r^+} \int_\Omega m(x) |\omega|^{r(x)} dx,
\end{aligned}
$$

Thus,

$$
I(t\omega) < 0
$$

for all $t < \sigma^{\frac{1}{\alpha^- p^- - r^+}}$ with

$$
0 < \sigma < \min \left\{ 1, \frac{\lambda \alpha^- (p^-)^{\alpha^-} \int_\Omega m(x) |\omega|^{r(x)} dx}{m_2 r^+ \left(\int_\Omega |\nabla \omega|^{p(x)} dx \right)^{\alpha^-}} \right\} .
$$

The proof of Lemma 3.4 is complete. □

Lemma 3.5. *Assume that* (M_1), (**A1**) *and* (**A2**) *hold. Then, I satisfies* (PS) *condition.*

Proof. Let assume that there exists a sequence $\{u_n\} \subset X$ such that

$$|I(u_n)| \leq c \ \text{ and } I'(u_n) \to 0 \ \text{ as } \ n \to \infty. \tag{3.2}$$

Firstly, we prove that $\{u_n\}$ is bounded in X. Arguing by contradiction and passing to a subsequence, we have $\|u_n\| \to \infty$ as $n \to \infty$. Using (M_1), (3.2) and considering $\|u_n\| > 1$, for n large enough, we have

$$\begin{aligned} c \ \geq \ I(u_n) &\geq \frac{m_1}{\alpha^+ (p^+)^{\alpha^+}} \left(\int_\Omega |\nabla u_n|^{p(x)} \, dx \right)^{\alpha(x)} - \frac{\lambda}{r^-} \int_\Omega m(x) |u_n|^{r(x)} \, dx \\ &\geq \frac{m_1}{\alpha^+ (p^+)^{\alpha^+}} \|u_n\|^{\alpha^- p^-} - \frac{\lambda c_1}{r^-} \|u_n\|^{r^+}. \end{aligned}$$

Since $r^+ < \alpha^- p^-$, we obtain that $\{u_n\}$ is bounded in X. Therefore, there exists a subsequence, again denoted by $\{u_n\}$ and $u \in X$ such that $u_n \rightharpoonup u$ in X. By (3.2) ,we have $\langle I'(u_n), u_n - u\rangle \to 0$.

$$\begin{aligned} &\left\langle I'(u_n), u_n - u \right\rangle = \\ &M\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)} dx \right) \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) \\ &-\lambda \int_\Omega m(x) |u_n|^{r(x)-2} u_n (u_n - u) \, dx \to 0. \end{aligned}$$

On the other hand, using Proposition 2.1-2.3-2.7 and the compact embedding $\left(X \hookrightarrow L^{r(x)}_{m(x)}(\Omega)\right)$, we have

$$\int_\Omega m(x) |u_n|^{r(x)-2} u_n (u_n - u) \, dx \to 0.$$

Therefore, we obtain

$$M\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)} dx \right) \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) \to 0.$$

In view of (M_1), we conclude that

$$\int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) \to 0.$$

Thus, from Proposition 2.6, it follows that $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. I satisfies (PS) condition. □

Proof of Theorem 3.1, from Lemma 3.3, Lemma 3.4, Lemma 3.5 and the fact that $I(0) = 0$, I satisfies the Mountain Pass theorem [24]. Therefore, I has at least one nontrivial weak solution. □

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