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BLOCK NUMERICAL INTEGRATOR FOR THE SOLUTION OF $y''' = f(x, y, y', y'')$

Fasasi M. Kolawole[1 §], A. Olaide Adesanya[2], S.O. Adee[3]
[1,2,3]Department of Mathematics
Modibbo Adama University of Technology
Yola, Adamawa State, NIGERIA

Abstract: A block method for solution of third order initial value problems of ordinary differential equations is presented in this paper. The scheme was derived using collocation and interpolation approach to generate a continuous linear multistep method which was solved for the independent solution to give a continuous block method. The result was evaluated at selected grid points to give a discrete block which gave simultaneous solutions at both grid and off grid points. The stability and convergence of the method were investigated, and found to be A -stable with good region of absolute stability. The accuracy of the block method was established numerically after comparing it with the existing method.

AMS Subject Classification: 65L05, 65L06, 65D30
Key Words: collocation, hybrid points, independent solution, interpolation, zero stable, consistent, convergent

1. Introduction

We consider the solution to general third order initial value problem of the form

$$y''' = f(x, y, y', y''), \quad y^k(x_0) = y_0^k, \quad k = 0, 1, 2 \quad y \epsilon \mathbb{R} \tag{1}$$

where x_0 is the initial point and f is continuous within the interval of integra-

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§Correspondence author

tion and satisfies the existence and uniqueness condition. Conventionally (1) is solved by reduction to an equivalent system of first order ordinary differential equation of the form $y^{'} = f(x, y)$, $y(a) = \mu$, $a \leq x \leq b$; $x, y \epsilon \mathbb{R}^n$ and $f \ \epsilon \ C^{} [a, b]$ before any appropriate numerical method could be used to solve the resultant problem. The method of reduction however has some setbacks such as complication in writing computer program, inability to utilize additional information associated with specific ordinary differential equation and most importantly a given system of equations to be solved may not be solved explicitly with respect to the derivatives of the highiest order. These setbacks were reported by and Awoyemi [1] and Adesanya et al [2].

Other improved methods have been proposed in the literature for solving (1) directly without first reducing it to first order systems. For instance, Onumanyi, *et al.* [3], Awoyemi [4], Jator [5] , Awoyemi and Kayode [6], developed Linear Multistep Methods with continuous coefficient for higher order initial value problems based on collocation method using power series polynomial as the basis function. The method guarrantee easy approximation of solution at all the interior points of integration interval but the procedure is more costly to implement when the method is to be implemented in predictor-corrector mode.

In order to avoid the limitations mentioned above, scholars developed block Method in which approximation are simultaneously generated at different grid points in the interval of integration and is less expensive in terms of the number of function evaluation compare to linear multistep methods.This assertion has been reported by (see Jator [7], Adesanya *et al* [8], Anake *et al* [9] and Awoyemi et al. [10]).

In this paper, we developed a one step with four off-grid points implemented in block method. It should be noted that this work is an improvement on the work of Adesanya *et al.* [11] where one step and three off steps were considered to develop a numerical method for the solution of general third order initial value problems of ordinary differential equations. Moreover, the points considered in this paper were carefully selected to ensure zero stability of the developed method.

2. Derivation of the Method

we define the general power series approximate solution in the form:

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j, \tag{2}$$

where r and s are the numbers of interpolation and collocation points respectively. The third derivative of (2) gives

$$y'''(x) = \sum_{j=2}^{r+s-1} j\,(j-1)\,(j-2)\,a_j x^{j-3}. \tag{3}$$

Substituting (3) into (1) gives

$$f\left(x, y, y', y''\right) = \sum_{j=2}^{r+s-1} j\,(j-1)\,(j-2)\,a_j x^{j-3}. \tag{4}$$

we interpolate (2) at $x_{n+r}, r = 0, \frac{1}{5}, \frac{2}{5}$ and collocate (3) at $x_{n+s}, s = 0(\frac{1}{5})1$ this results into a system of non linear equation

$$AX = U \tag{5}$$

where

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \end{bmatrix}^T$$

$$U = \begin{bmatrix} y_n & y_{n+\frac{1}{5}} & y_{n+\frac{2}{5}} & f_n & f_{n+\frac{1}{5}} & f_{n+\frac{2}{5}} & f_{n+\frac{3}{5}} & f_{n+\frac{4}{5}} & f_{n+1} \end{bmatrix}^T,$$

and

$$\begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\
1 & x_{n+\frac{1}{5}} & x_{n+\frac{1}{5}}^2 & x_{n+\frac{1}{5}}^3 & x_{n+\frac{1}{5}}^4 & x_{n+\frac{1}{5}}^5 & x_{n+\frac{1}{5}}^6 & x_{n+\frac{1}{5}}^7 & x_{n+\frac{1}{5}}^8 \\
1 & x_{n+\frac{2}{5}} & x_{n+\frac{2}{5}}^2 & x_{n+\frac{2}{5}}^3 & x_{n+\frac{2}{5}}^4 & x_{n+\frac{2}{5}}^5 & x_{n+\frac{2}{5}}^6 & x_{n+\frac{2}{5}}^7 & x_{n+\frac{2}{5}}^8 \\
0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 & 336x_n^5 \\
0 & 0 & 0 & 6 & 24x_{n+\frac{1}{5}} & 60x_{n+\frac{1}{5}}^2 & 120x_{n+\frac{1}{5}}^3 & 210x_{n+\frac{1}{5}}^4 & 336x_{n+\frac{1}{5}}^5 \\
0 & 0 & 0 & 6 & 24x_{n+\frac{2}{5}} & 60x_{n+\frac{2}{5}}^2 & 120x_{n+\frac{2}{5}}^3 & 210x_{n+\frac{2}{5}}^4 & 336x_{n+\frac{2}{5}}^5 \\
0 & 0 & 0 & 6 & 24x_{n+\frac{3}{5}} & 60x_{n+\frac{3}{5}}^2 & 120x_{n+\frac{3}{5}}^3 & 210x_{n+\frac{3}{5}}^4 & 336x_{n+\frac{3}{5}}^5 \\
0 & 0 & 0 & 6 & 24x_{n+\frac{4}{5}} & 60x_{n+\frac{4}{5}}^2 & 120x_{n+\frac{4}{5}}^3 & 210x_{n+\frac{4}{5}}^4 & 336x_{n+\frac{4}{5}}^5 \\
0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 & 336x_{n+1}^5
\end{bmatrix}.$$

Solving (5) for the unknown constant a_j's using Guassian elimination method and substituting back into (2) gives a continuous hybrid linear multistep method in the form

$$y(x) = \alpha_0 y_n + \alpha_{\frac{1}{5}} y_{n+\frac{1}{5}} + \alpha_{\frac{2}{5}} y_{n+\frac{2}{5}} + h^3 \left[\sum_{j=0}^{1} \beta_j\ f_{n+j} + \beta_v\ f_{n+v}\right], \tag{6}$$

$$v = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5},$$

where

$$\alpha_0 = \frac{25}{2}t^2 - \frac{15}{2}t + 1, \quad \alpha_{\frac{1}{5}} = 10t - 25t^2, \quad \alpha_{\frac{2}{5}} = \frac{25}{2}t^2 - \frac{5}{2}t,$$

$$\beta_0 = -\frac{1}{1008000}(78125t^8 - 375000t^7 + 743750t^6 - 787500t^5 + 479500t^4 - 168000t^3 + 31075t^2 - 2286t),$$

$$\beta_{\frac{1}{5}} = \frac{1}{1008000}(390625t^8 - 1750000t^7 + 3106250t^6 - 2695000t^5 + 1050000t^4 - 92565t^2 + 13538t),$$

$$\beta_{\frac{1}{5}} = -\frac{1}{504000}(390625t^8 - 1625000t^7 + 2581250t^6 - 1872500t^5 + 525000t^4 - 20745t^2 + 2218t),$$

$$\beta_{\frac{3}{5}} = \frac{1}{504000}(390625t^8 - 1500000t^7 + 2143750t^6 - 1365000t^5 + 350000t^4 - 13465t^2 + 1482t),$$

$$\beta_{\frac{4}{5}} = -\frac{1}{1008000}(390625t^8 - 1375000t^7 + 1793750t^6 - 1067500t^5 + 262500t^4 - 9825t^2 + 1082t),$$

$$\beta_1 = \frac{1}{201600}(15625t^8 - 50000t^7 + 61250t^6 - 35000t^5 + 8400t^4 - 309t^2 + 34t).$$

Solving (6) for the independent solution at the selected grid points give a continuous block method of the form

$$y_{n+k}^{(m)} = \sum_{m=1}^{0} \frac{(kh)^m}{m!} y_n^m + h^3 \left[\sum_{j=1}^{0} \sigma_j \ f_{n+j} + \sigma_v \ f_{n+v} \right], \quad v = \frac{1}{5}(\frac{1}{5})\frac{4}{5}, \qquad (7)$$

$$\sigma_0 = -\frac{1}{8064}(625t^8 - 3000t^7 + 5950t^6 - 6300t^5 + 3836t^4 - 1344t^3),$$
$$\sigma_{\frac{1}{5}} = \frac{1}{8064}(3125t^8 - 14000t^7 + 24850t^6 - 21560t^5 + 8400t^4),$$
$$\sigma_{\frac{2}{5}} = -\frac{1}{4032}(3125t^8 - 13000t^7 + 20650t^6 - 14980t^5 + 4200t^4),$$
$$\sigma_{\frac{3}{5}} = \frac{1}{4032}(3125t^8 - 12000t^7 + 17150t^6 - 10920t^5 + 2800t^4),$$
$$\sigma_{\frac{4}{5}} = -\frac{1}{8064}(3125t^8 - 11000t^7 + 14350t^6 - 8540t^5 + 2100t^4),$$
$$\sigma_1 = \frac{1}{8064}(625t^8 - 2000t^7 + 2450t^6 - 1400t^5 + 336t^4).$$

Evaluating (7) at $t = \frac{1}{5}(\frac{1}{5})1$ gives a discrete block method of the form

$$A^0 Y_m^{(i)} = \sum_{j=0}^{2} h^1 e_i y_n^{(i)} + h^{(3-i)}\left[d_i\ f(y_n) + b_i F(Y_m)\right], \tag{8}$$

$$Y_m^{(i)} = \left[y_{n+\frac{1}{5}}^{(i)},\ y_{n+\frac{2}{5}}^{(i)},\ y_{n+\frac{3}{5}}^{(i)},\ y_{n+\frac{4}{5}}^{(i)},\ y_{n+1}^{(i)}\right]^T,$$
$$F(Y_m) = \left[f_{n+\frac{1}{5}},\ f_{n+\frac{2}{5}},\ f_{n+\frac{3}{5}},\ f_{n+\frac{4}{5}},\ f_{n+1}\right],$$
$$y_n^{(i)} = \left[y_{n-\frac{1}{5}}^{(i)},\ y_{n-\frac{2}{5}}^{(i)},\ y_{n-\frac{3}{5}}^{(i)},\ y_{n-\frac{4}{5}}^{(i)},\ y_n^{(i)}\right],$$

$A^0 = 5 \times 5 -$ identity matrix, when $i = 0$,

$$e_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & \frac{2}{5} \\ 0 & 0 & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & \frac{4}{5} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{50} \\ 0 & 0 & 0 & 0 & \frac{2}{25} \\ 0 & 0 & 0 & 0 & \frac{9}{50} \\ 0 & 0 & 0 & 0 & \frac{8}{25} \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}, \qquad d_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{3929}{5040000} \\ 0 & 0 & 0 & 0 & \frac{317}{78750} \\ 0 & 0 & 0 & 0 & \frac{783}{80000} \\ 0 & 0 & 0 & 0 & \frac{712}{39375} \\ 0 & 0 & 0 & 0 & \frac{233}{8064} \end{bmatrix},$$

when $i = 1$:

$$e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & \frac{2}{5} \\ 0 & 0 & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & \frac{4}{5} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$d_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1231}{126000} \\ 0 & 0 & 0 & 0 & \frac{71}{3150} \\ 0 & 0 & 0 & 0 & \frac{123}{3500} \\ 0 & 0 & 0 & 0 & \frac{376}{7875} \\ 0 & 0 & 0 & 0 & \frac{61}{1008} \end{bmatrix}, \quad b_1 = \begin{bmatrix} \frac{863}{50400} & \frac{-761}{63000} & \frac{941}{126000} & \frac{-341}{126000} & \frac{107}{252000} \\ \frac{544}{7875} & \frac{-37}{1575} & \frac{136}{7875} & \frac{-101}{15750} & \frac{8}{7875} \\ \frac{3501}{28000} & \frac{-9}{3500} & \frac{87}{2800} & \frac{-9}{875} & \frac{9}{5600} \\ \frac{1424}{7875} & \frac{176}{7875} & \frac{608}{7875} & \frac{-16}{1575} & \frac{16}{7875} \\ \frac{475}{2016} & \frac{25}{504} & \frac{125}{1008} & \frac{25}{1008} & \frac{11}{2016} \end{bmatrix},$$

when $i = 2$:

$$e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad d_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{19}{288} \\ 0 & 0 & 0 & 0 & \frac{14}{225} \\ 0 & 0 & 0 & 0 & \frac{51}{800} \\ 0 & 0 & 0 & 0 & \frac{14}{225} \\ 0 & 0 & 0 & 0 & \frac{19}{288} \end{bmatrix},$$

$$b_2 = \begin{bmatrix} \frac{1427}{7200} & \frac{-133}{1200} & \frac{241}{3600} & \frac{-173}{7200} & \frac{3}{800} \\ \frac{43}{150} & \frac{7}{225} & \frac{7}{225} & \frac{-1}{75} & \frac{1}{450} \\ \frac{219}{800} & \frac{57}{400} & \frac{57}{400} & \frac{-21}{800} & \frac{3}{800} \\ \frac{64}{225} & \frac{8}{75} & \frac{64}{225} & \frac{14}{225} & 0 \\ \frac{25}{96} & \frac{25}{144} & \frac{25}{144} & \frac{25}{96} & \frac{19}{288} \end{bmatrix}.$$

3. Implementation of the Method

Writing equation [8] in the generalised form

$$Y_m = Ey_n + h^{\mu} DF(y_n) + h^{\mu} BF(Y_m), \tag{9}$$

where $Y_m = [y_{n+1}, y_{n+2,\ldots,} y_{n+k}]^T$, μ is the order of the differential equation, k is the steplength, E, D and B are matrices. We then propose a prediction equation in the form

$$Y_m^{(0)} = Ey_n + \sum_{\lambda=0}^{2} h^{\mu+\lambda} F^{\lambda}(y_n) \tag{10}$$

where $F^{(\lambda)}\ (y_n) = \frac{\delta^\lambda}{\delta x^\lambda} f(x, y, y^{\prime})_{y_n}$. Substituting (9) into (8) gives

$$Y_m = Ey_n + h^\mu DF(y_n) + h^\mu BF(y_m^{(0)}). \tag{11}$$

Equation (11) is our non self starting block method since the prediction equation is not gotten directly from the block formula (Adesanya et al [3])

4. Basic Properties of the Developed Method

4.1. Order of the block

Let the linear operator l $L\{y(x) : h\}$ on (7) as

$$L\{y(x) : h\} = \mathbf{A}^0 y_m^{(i)} - \sum_{i=0}^{2} h^i e_i y_n^{(i)} - h^{3-i}\ [df(y_n) + bF(y_m)]\,. \tag{12}$$

Expanding y_{n+j} and f_{n+j} in Taylor series and comparing the coefficients of h gives

$$L\{y(x) : h\} = C_0 y(x) + C_1 y^{\prime}(x) + ... + C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x) \\ + C_{p+2} h^{p+2} y^{p+2}(x) + ...$$

Definition 1. The linear operator L and associated block method are said to be of order p if $C_0 = C_1 = ... = C_p = C_{p+1} = 0\ C_{p+2} \neq 0.C_{p+2}$ is called the error constant and implies that the truncation error is given by $t_{n+k} = C_{p+2} h^{p+2} y^{p+2}(x) + O\left(h^{p+3}\right)$. Comparing the coefficient of h , the order of the method is five with error constant

$$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = 0,\ c_6 = 0,$$

$$c_7 = \left[\frac{-9807}{7087500000000}, \frac{-491}{55371093750}, \frac{-1917}{87500000000}, \frac{-1136}{27685546875}, \frac{-149}{2268000000}\right]^T.$$

4.2. Consistency

A method is said to be consistent, if it has order greater than one. From the above analysis, it is obvious that our method is consistent.

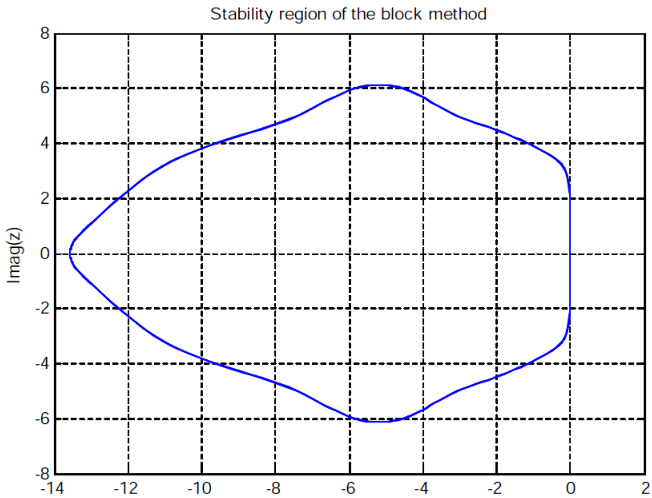

Figure 1: Stability region of the method

4.3. Zero Stability

A block method is said to be zero stable as $h \to 0$ the r_j, $j = 1(1)\ k$ of the first characteristics polynomial $\rho(r) = 0$ that is $\left|\left[\sum A^0 R^{k-1}\right]\right| \leq 1$, for those root with $|R| = 1$ must be simple.

For our method:

$$\rho(r) = \left| r \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0,$$

$$r^4(r-1) = 0,$$

$r = 0, 0, 0, 0$ or r=1. It can be concluded tha our method is zero stable

4.4. Convergence

Definition 2. The necessary and sufficient conditions for a linear multistep method to be convergent is that it must be consistent and zero stable. Hence our method is convergent.

X	Error in NM	Error in [11]	Error in [8]
0.1	3.330669(−16)	3.330669(−16)	0.0000 + 00
0.2	2.220446(−16)	3.330669(−16)	1.981670(−16)
0.3	0.000000(+00)	3.330669(−16)	6.507412(−15)
0.4	1.110223(−16)	1.110223(−16)	1.559238(−15)
0.5	0.000000(+00)	1.110223(−16)	3.150448(−15)
0.6	2.220446(−16)	4.440892(−16)	5.637458(−15)
0.7	1.110223(−16)	5.551115(−16)	9.616405(−15)
0.8	1.665335(−16)	5.551115(−16)	1.568680(−14)
0.9	1.665335(−16)	7.216450(−16)	2.486977(−14)
1.0	1.387779(−16)	1.054712(−15)	3.879839(−14)

Table 1: Comparison of absolute errors for Problem I

4.5. Stability Region

The method (6) is said to be absolutely stable if for a given h, all roots z_s ofthe characteristics polynomial $\pi(z,h) = \rho(z) + h^3\sigma(z) = 0$, satisfies $|z_s| < 1$, $s = 1, 2, ..., n$. *where* $h = -\lambda^3 h^3$ and $\lambda = \frac{\partial f}{\partial y}$. The boundary locus method is adopted to determine the region of absolute stability. Substituting the test equation $y''' = -\lambda^3 y$, $y'' = -\lambda^2 y$ and $y' = \lambda y$ into the polynomial gives the stability region as shown in Figure 1.

5. Numerical Experiments

5.1. Numerical Examples

Problem 1. $y''' = 3\sin x$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$, $0 \le x \le 1$, $h = 0.05$.

Exact Solution: $y(x) = 3\cos x + \frac{x^2}{2} - 2$.
Source (see Adesanya et al [11]): The result is shown in Table 1

Problem 2. $y''' + y' = 0$, $y(0) = 0$, $y'(0) = 1$, $y''(0) = 2$, $x \in [0, 1]$, $h = 0.05$.

Exact Solution: $y(x) = 2(1 - \cos x) + \sin x$.
Source (see Adesanya et al [11]): The result is shown in Table 2.
Error: = |Exact result – Computed result|, NM= Error in New Method.

X	Error in NM	Error in [11]	Error in [8]
0.1	9.714451(−16)	3.018419(−14)	1.54055(−09)
0.2	0.000000(+00)	6.272760(−15)	9.84550(−09)
0.3	0.000000(+00)	3.398948(−13)	2.36528(−08)
0.4	2.220446(−16)	1.235012(−12)	4.32732(−08)
0.5	3.330669(−16)	2.985279(−12)	3.90181(−08)
0.6	3.330669(−16)	5.907719(−12)	6.97008(−08)
0.7	2.220446(−16)	1.033573(−11)	5.20329(−08)
0.8	2.220446(−16)	1.661249(−11)	1.35224(−07)
0.9	2.220446(−16)	2.508349(−11)	4.74034(−07)
1.0	4.440892(−16)	3.6088915(−11)	1.06936(−06)

Table 2: Comparison of absolute errors for Problem 2

6. Discussion of the Results

Two numerical examples were used to test the efficiency of our developed scheme. Adesanya et al [11] solved Problems 1 and 2, where they developed methods of order five implemented in predictor corrector mode where the predictors were implemented in block method with step size $(h) = 0.01$ and 0.1.respectively. Our block method compete favourably with the existing method and with less computational cost.

7. Conclusion

An improved block method for direct solution of third order ordinary differential equations has been developed and implemented in this paper. The good convergent and stability properties of our method makes it more attractive for numerical integrator of linear and non-linear initial value problems. Its accuracy and effectiveness has also been shown clearly in tables 1 and 2. Our method proves to be a good estimate of the exact solution for the test examples.

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