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ON A PERTURBED SYSTEM OF CHEMOTAXIS, II

Masaki Kurokiba[1 §], Takashi Suzuki[2], Ryo Takahashi[3]

[1]Mathematical Science Rearch Unit
Muroran Institute of Technology
27-1, Mizumoto Muroran, Hokkaido, 050-8585, JAPAN

[2,3]Division of Mathematical Science
Graduate School of Engineering Science
Osaka University
Toyonaka, 560-8531, JAPAN

Abstract: We study a parabolic-elliptic system with nonlinear boundary condition, describing the chemotactic aggregation of cellular slime molds. We show the well-posedness locally in time, blowup criterion of the solution, and finiteness of the blowup points.

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1. Introduction

This paper is concerned with the parabolic-elliptic system describing chemo-

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§Correspondence author

tactic aggregation, that is,

$$\begin{cases} u_t = \nabla\cdot(\nabla u - \chi u \nabla v) & \text{in } \Omega\times(0,T) \\ 0 = \Delta v - \gamma v - f(v) + \alpha u & \text{in } \Omega\times(0,T) \\ \frac{\partial u}{\partial \nu} - \chi u \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega\times(0,T) \\ \frac{\partial v}{\partial \nu} + g(v) = 0 & \text{on } \partial\Omega\times(0,T) \\ u|_{t=0} = u_0 & \text{on } \overline{\Omega}, \end{cases} \tag{GCZ}$$

where Ω is a bounded domain in $\boldsymbol{R}^n$ with smooth boundary $\partial\Omega$, χ, α, γ are positive constants, $f = f(s)$, $g = g(s)$ are smooth functions, and ν is the unit outward normal vector on $\partial\Omega$. The unknown functions $u = u(x,t)$ and $v = v(x,t)$ stand for the cell density and the concentration of the chemical substance at $(x,t) \in \overline{\Omega}\times[0,T)$, respectively, and $F = -\nabla u + \chi u \nabla v$ is the flux of u so that the effect of diffusion $-\nabla\cdot\nabla u$ and that of chemotaxis compete for u to vary. Another description is the movement from the gravitational equilibrium of polytropic fluid, see [4], [6].

For the problem (GCZ) with $f(v) = g(v) = 0$, Nagai [16] showed that the conjecture of Childress and Percus [9] concerning $n = 2$ is true for radially symmetric solutions, that is, the chemotaxis collapse can occur if the total cell number on $\Omega \subset \boldsymbol{R}^2$ is larger than $8\pi/\alpha\chi$ and can not occur in the other case, and then, Nagai, Senba and Yosida [17] and Senba and Suzuki [22] corrected this value to $4\pi/\alpha\chi$ in the general case, see also Biler [5] and Gajewski and Zacharias [11]. Then, Senba and Suzuki [21] and Suzuki [23] showed the formation of collapses with the quantized mass for the blowup solution in finite time, refining the conjecture of Nanjundiah [19].

For the problem (GCZ) with $f(v) = \beta|v|^{p-1}v$ and $g(v) = 0$, Chen and Zhong [8] studied the existence and the non-existence of the solution global in time, while Kurokiba and Suzuki [13] showed the finiteness of the blowup points of the blowup solution in finite time.

In this paper we impose the following conditions on (GCZ):

$$u_0 \geq 0,\ \ u_0 \not\equiv 0 \text{ on } \Omega \tag{1.1}$$

$$u_0 \in C^3(\overline{\Omega}) \tag{1.2}$$

$$f, g \text{ is smooth, say } C^2 \tag{1.3}$$

$$f'(s), g'(s) \geq 0 \tag{1.4}$$

$$f(0), g(0) \leq 0 \tag{1.5}$$

$$|f'(s)| \leq C(1 + |s|^{p_1 - 1}) \tag{1.6}$$

$$|g'(s)| \leq C(1 + |s|^{p_2 - 1}) \tag{1.7}$$

for all $s \in \boldsymbol{R}$, where $C > 0$, $p_1 \in (1, p^*)$ and $p_2 \in (1, p^{**})$ with

$$p^* = \begin{cases} \infty & \text{if } n = 2 \\ \frac{n+2}{n-2} & \text{if } n \geq 3 \end{cases}$$

$$p^{**} = \begin{cases} \infty & \text{if } n = 2 \\ \frac{n}{n-2} & \text{if } n \geq 3. \end{cases} \tag{1.8}$$

We assume the compatibility condition to the parabolic part:

$$\frac{\partial u_0}{\partial \nu} - u_0 \frac{\partial v_0}{\partial \nu} = 0, \tag{1.9}$$

where v_0 denotes the solution to (2.1) for $u = u_0$, see the following section. Then, we obtain the classical solution local in time:

Theorem 1. If (1.1)-(1.9) hold, then there exists a unique non-negative classical solution $(u, v) \in C^{2+\theta, 1+\theta/2}(\overline{Q_T}) \times C([0, T]; C^{2+\theta}(\overline{\Omega}))$ to (GCZ) with $\theta \in (0, 1)$ provided that T is sufficiently small.

Henceforth $T_{\max} \in (0, +\infty]$ and $\mathcal{B}$ stand for the maximal existence time of the solution and the blowup set of u, respectively:

$$T_{\max} = \sup \{T > 0 \mid \text{the solution } (u(\cdot, t), v(\cdot, t)) \text{ exists for } t \in [0, T)\},$$
$$\mathcal{B} = \big\{x_0 \in \overline{\Omega} \mid \text{there exist } t_k \uparrow T_{\max} \text{ and } x_k \to x_0 \text{ such that } u(x_k, t_k) \to +\infty \text{ as } k \to \infty\},$$

and call each $x_0 \in \mathcal{B}$ the blowup point. The following theorem assures that $T_{\max} < +\infty$ implies $\mathcal{B} \neq \emptyset$:

Theorem 2. If (1.1)-(1.9), $n = 2$ and $T_{\max} < +\infty$ hold, then it holds that

$$\lim_{t \uparrow T_{\max}} \int_{\Omega} u \log u dx = \infty.$$

Now, we state the main theorem.

Theorem 3. If (1.1)-(1.9), $n = 2$ and $T_{\max} < +\infty$ hold, then it holds that $\sharp(\mathcal{B} \cap \Omega) < +\infty$.

Concerning the finiteness of $\mathcal{B} \cap \partial\Omega$, see the final remark of this paper.

This paper is composed of four sections. Theorems 1, 2, and 3 are proven in sections 2, 3, and 4, respectively. Henceforth, we put $\chi = \alpha = \gamma = 1$ without loss of generality.

2. Proof of Theorem 1

To begin with, we study the elliptic equation

$$\begin{cases} -\Delta v + v + f(v) = u & \text{in } \Omega \\ \frac{\partial v}{\partial \nu} + g(v) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Lemma 4. If (1.3)-(1.7) hold and $u = u(x) \geq 0$ is in $C^1(\overline{\Omega})$, then (2.1) has a unique non- negative classical solution $v = v(x) \in C^{2+\theta}(\overline{\Omega})$.

Proof. To confirm the uniqueness of v, let v_1 and v_2 be the classical solutions to (2.1) and set $w := v_1 - v_2$. Then, it holds that

$$\begin{cases} -\Delta w + w = -f(v_1) + f(v_2) & \text{in } \Omega \\ \frac{\partial w}{\partial \nu} = -g(v_1) + g(v_2) & \text{on } \partial\Omega \end{cases}$$

and therefore,

$$\begin{aligned} \int_\Omega |\nabla w|^2 + w^2 dx &= -\int_{\partial\Omega} (v_1 - v_2)(g(v_1) - g(v_2)) dS \\ &\quad - \int_\Omega (v_1 - v_2)(f(v_1) - f(v_2)) dx \leq 0 \end{aligned}$$

by (1.4). This implies $v_1 = v_2$.

To show the non-negativity $v \geq 0$, we take $v_- = \min(v, 0) \leq 0$. Since

$$\int_\Omega |\nabla v_-|^2 + v_-^2 + f(v) v_- dx + \int_{\partial\Omega} g(v) v_- dS = \int_\Omega u v_- dx \leq 0$$

holds by (2.1), it follows that $v_- = 0$ from (1.4)-(1.5).

The existence of v is obtained by the variational method. We set

$$I[v] := \frac{1}{2}\int_\Omega |\nabla v|^2 + v^2 dx - \int_\Omega uv dx + \int_\Omega \tilde{f}(v_+) dx + \int_{\partial\Omega} \tilde{g}(v_+) dS \tag{2.2}$$

for $v_+ = \max(v, 0) \geq 0$ and

$$\tilde{f}(v) := \int_0^v f(s) ds, \qquad \tilde{g}(v) := \int_0^v g(s) ds. \tag{2.3}$$

and show that I attains the minimum in $H^1 = H^1(\Omega)$, using (1.6)-(1.7). Since the assumption (1.6)-(1.7) guarantees the sub-criticalness of the nonlinearity, the minimizer will actually be in $C^{2+\theta}(\overline{\Omega})$ by the bootstrap argument.

First, we confirm the coercivity of the functional I. In fact, given $0 < \delta \ll 1$, we take $L = L(\delta) > 0$, using (1.4), such that

$$\frac{f(s)}{s} \geq \frac{f(0)}{s} \geq -\delta, \quad \frac{g(s)}{s} \geq \frac{g(0)}{s} \geq -\delta \qquad \text{for all } s \geq L.$$

It holds that

$$\begin{aligned} \tilde{f}(v) &= \int_0^L f(s)ds + \int_L^v \frac{f(s)}{s} \cdot s ds \\ &\geq \int_0^L f(s)ds + \frac{\delta}{2}L^2 - \frac{\delta}{2}v^2 \geq -\delta v^2 \end{aligned}$$

for $v \geq l_1$, where $l_1 = \max\{L, \sqrt{2\delta^{-1}|\int_0^L f(s)ds + \delta L^2/2|}\}$, and similarly,

$$\tilde{g}(v) \geq -\delta v^2$$

for $v \geq l_2$, where $l_2 = \max\{L, \sqrt{2\delta^{-1}|\int_0^L g(s)ds + \delta L^2/2|}\}$. Therefore,

$$\begin{aligned} &\int_\Omega \tilde{f}(v_+)dx + \int_{\partial\Omega} \tilde{g}(v_+)dS = \int_{\{0\leq v<l_1\}} \tilde{f}(v)dx + \int_{\{v \geq l_1\}} \tilde{f}(v)dx \\ &\quad + \int_{\{0 \leq v < l_2\}} \tilde{g}(v)dS + \int_{\{v \geq l_2\}} \tilde{g}(v)dS \\ &\geq -\delta \int_\Omega v^2 dx - \delta \int_{\partial\Omega} v^2 dS - C(\Omega, f, g), \end{aligned} \tag{2.4}$$

where $C(\Omega, f, g) = |\Omega| \min_{|s|\leq l_1} \tilde{f}(s) + |\partial\Omega| \min_{|s|\leq l_2} \tilde{g}(s)$.

Since

$$\int_\Omega |\nabla v|^2 dx + \int_\Omega v^2 dx \geq \sigma \int_{\partial\Omega} v^2 dS \tag{2.5}$$

holds for some constant $\sigma = \sigma(n, \Omega) > 0$ by the continuity of the trace operator $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$, we obtain

$$\begin{aligned} I[v] &\geq \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \left(\frac{1}{2} - \varepsilon - \delta\right)\int_\Omega v^2 dx - \delta \int_{\partial\Omega} v^2 dS \\ &\quad -\frac{1}{4\varepsilon}\int_\Omega u^2 dx - C(\Omega, f, g) \\ &\geq \left(\frac{1}{2} - \sigma^{-1}\delta\right)\int_\Omega |\nabla v|^2 dx + \left(\frac{1}{2} - \varepsilon - \delta - \sigma^{-1}\delta\right)\int_\Omega v^2 dx \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{4\varepsilon}\int_\Omega u^2 dx - C(\Omega, f, g) \\
\geq \ & \left(\frac{1}{2} - \varepsilon - \delta - \sigma^{-1}\delta\right)\|v\|_{H^1}^2 - \frac{1}{4\varepsilon}\int_\Omega u^2 dx - C(\Omega, f, g)
\end{aligned}
$$

by (2.2) and (2.4)-(2.5), where $\varepsilon > 0$ is arbitrary. Taking $\delta, \varepsilon > 0$ such that $(\frac{1}{2} - \varepsilon - \delta - \sigma^{-1}\delta) > 0$, we thus obtain the coerciveness of I on $H^1(\Omega)$:

$$
I[v] \geq \nu\|v\|_{H^1}^2 - C, \qquad v \in H^1(\Omega), \tag{2.6}
$$

where $\nu > 0$ and $C > 0$ are constants.

To confirm the weak lower semi-continuity of I, let $\{v_k\} \subset H^1(\Omega)$ be a minimizing sequence:

$$
I[v_k] \to \inf_{w \in H^1} I[w] > -\infty.
$$

Since $\{v_k\}$ is bounded from below by (2.6), we can extract a subsequence, denoted by the same symbol, such that

$$
v_k \rightharpoonup v \ \text{ weakly in } H^1(\Omega)
$$

for some v, and then it follows that

$$
\|v\|_{H^1} \leq \liminf_k \|v_k\|_{H^1}. \tag{2.7}
$$

Meanwhile, we have the compact imbedding

$$
H^1(\Omega) \hookrightarrow W^{s,q}(\Omega)
$$

and the continuous imbedding

$$
W^{s,q}(\Omega) \hookrightarrow W^{s-\frac{1}{q},q}(\partial\Omega)
$$

for $0 < s < 1$ and $1 \leq q < q^*$, $\frac{1}{q^*} = \frac{1}{2} - \frac{1-s}{n}$, see [1], which implies the compact imbeddings

$$
H^1(\Omega) \hookrightarrow L^{p_1+1}(\Omega), \ L^{p_2+1}(\partial\Omega)
$$

for p_1, p_2 prescribed by (1.8). From this compactness, passing if necessary to yet another subsequence, we see that there exist $h_1 \in L^{p_1+1}(\Omega)$ and $h_2 \in L^{p_2+1}(\partial\Omega)$ such that

$$
\begin{aligned}
&v_k(x) \to v(x) \text{ for a.e. } x \in \Omega, \quad v_k(\xi) \to v(\xi) \text{ for a.e. } \xi \in \partial\Omega, \\
&|v_k(x)| \leq h_1(x) \text{ for a.e. } x \in \Omega, \quad |v_k(\xi)| \leq h_2(\xi) \text{ for a.e. } \xi \in \partial\Omega.
\end{aligned}
$$

Since $\tilde{f}$ and $\tilde{g}$ are continuous

$$\tilde{f}(v_k(x)) \to \tilde{f}(v_k(x)) \text{ for a.e. } x \in \Omega, \quad \tilde{g}(v_k(\xi)) \to \tilde{g}(v_k(\xi)) \text{ for a.e. } \xi \in \Omega,$$

and it follows from (1.6)-(1.7) that

$$|\tilde{f}(v_k(x))| \leq C(1 + h_1^{p_1+1}(x)) \text{ for a.e. } x \in \Omega,$$
$$|\tilde{g}(v_k(\xi))| \leq C(1 + h_2^{p_2+1}(\xi)) \text{ for a.e. } \xi \in \Omega.$$

Then we have

$$\begin{aligned} & -\int_\Omega u v_k dx + \int_\Omega \tilde{f}(v_k) dx + \int_{\partial\Omega} \tilde{g}(v_k) dS \\ & \longrightarrow -\int_\Omega u v dx + \int_\Omega \tilde{f}(v) dx + \int_{\partial\Omega} \tilde{g}(v) dS \end{aligned} \tag{2.8}$$

by the Lebesgue dominated convergence theorem together with the above facts, and consequently

$$I[v] \leq \liminf_k I[v_k].$$

We obtain $I[v] = \inf_{w \in H^1} I[w]$ as desired.

Finally, we show the reguality of v. We shall do it only in the case $n \geq 3$, for it is done more easily in the case $n = 2$. Set

$$r = \frac{2n}{p_1(n-2)} \in \left(\frac{2n}{n+2}, \frac{2n}{n-2}\right), \quad s = \frac{2n}{n + (n-2)(p_2 - 1)} \in \left(\frac{2n}{n+2}, 2\right).$$

We use (1.6)-(1.7) to get

$$\|u - f(v)\|_r \leq \|u\|_r + C(1 + \|v\|_{rp_1}^{p_1}) \leq \|u\|_r + C(1 + \|v\|_{H^1}^{p_1})$$

and

$$\begin{aligned} \|g(v)\|_{W^{1-1/s,s}(\partial\Omega)} &\leq C(\|g(v)\|_s + \|g'(v)|\nabla v|\|_s) \\ &\leq C(1 + \|v\|_{p_2 s}^{p_2} + \|g'(v)\|_{\frac{2s}{2-s}} \|\nabla v\|_2) \\ &\leq C(1 + \|v\|_{H^1}^{p_2} + \|\nabla v\|_2 + \|v\|_{\frac{2s(p_2-1)}{2-s}}^{p_2-1} \|\nabla v\|_2) \\ &= C(1 + \|v\|_{H^1}^{p_2} + \|\nabla v\|_2 + \|v\|_{\frac{2n}{n-2}}^{p_2-1} \|\nabla v\|_2) \\ &\leq C(1 + \|v\|_{H^1} + \|v\|_{H^1}^{p_2}). \end{aligned}$$

From these inequalities and the L^p estimates (see [2]), it follows that

$$v \in W^{2,\tilde{q_0}}(\Omega) \quad \text{with } \tilde{q_0} = \min(r, s) \in \left(\frac{2n}{n+2}, 2\right).$$

If $2n/(n+2) < \tilde{q}_{k-1} \le n$, then we define q_k, r_k, s_k, $\tilde{q}_k$ recursively as follows:

$$q_k = \begin{cases} \frac{n\tilde{q}_{k-1}}{n-\tilde{q}_{k-1}} & (\frac{2n}{n+2} < \tilde{q}_{k-1} < n) \\ \text{any larger real than } n & (\tilde{q}_{k-1} = n), \end{cases}$$

$$r_k = \begin{cases} \frac{q_k^*}{p_1} & (\frac{2n}{n+2} < \tilde{q}_{k-1} < \frac{n}{2}) \\ \text{any larger real than } n & (\frac{n}{2} \le \tilde{q}_{k-1} \le n), \end{cases}$$

$$s_k = \begin{cases} \frac{q_k q_k^*}{q_k^* + q_k(p_2 - 1)} & (\frac{2n}{n+2} < \tilde{q}_{k-1} < \frac{n}{2}) \\ \text{any real in } (\frac{n}{2}, n) & (\tilde{q}_{k-1} = \frac{n}{2}) \\ \text{any real in } (n, q_k) & (\frac{n}{2} < \tilde{q}_{k-1} \le n), \end{cases} \qquad \tilde{q}_k = \min(r_k, s_k),$$

where $q_k^* = nq_k/(n - q_k)$ for $\tilde{q}_{k-1} \in (2n/(n+2), n/2)$. For the above r_k and s_k, we similarly deduce

$$\|u - f(v)\|_{r_k} \le \|u\|_{r_k} + C(1 + \|v\|_{W^{1,q_k}}^{p_1}),$$
$$\|g(v)\|_{W^{1-1/s_k, s_k}(\partial\Omega)} \le C(1 + \|v\|_{W^{1,q_k}} + \|v\|_{W^{1,q_k}}^{p_2}).$$

Then we use the L^p estimates, again, so that $v \in W^{2,\tilde{q}_k}(\Omega)$. Repeating this finite times, we obtain

$$v \in W^{2,l}(\Omega) \hookrightarrow C^{2-n/l}(\overline{\Omega}) \quad \text{for some } l > n,$$

and conclude the desired regularity $v \in C^{2+\theta}(\overline{\Omega})$ by the Schauder estimate, see [2]. The proof is complete. □

Proof of Theorem 1. We define

$$u^{(0)} \equiv u_0 \qquad \text{on } \overline{\Omega},$$

and $v^{(0)}, u^{(1)}, v^{(1)}, \cdots$ inductively as follows:

$$\begin{cases} -\Delta v^{(k)} + v^{(k)} = -f(v^{(k)}) + u^{(k)} & \text{in } Q_T \\ \frac{\partial v^{(k)}}{\partial \nu} = -g(v^{(k)}) & \text{on } \Gamma_T \end{cases} \tag{2.9}$$

and

$$\begin{cases} u_t^{(k+1)} - \Delta u^{(k+1)} + \nabla \cdot (u^{(k+1)} \nabla v_0) = -\nabla \cdot (u^{(k)} \nabla (v^{(k)} - v_0)) & \text{in } Q_T \\ \frac{\partial u^{(k+1)}}{\partial \nu} - \frac{\partial v_0}{\partial \nu} u^{(k+1)} = \left(\frac{\partial v^{(k)}}{\partial \nu} - \frac{\partial v_0}{\partial \nu} \right) u^{(k)} & \text{in } \Gamma_T \\ u^{(k+1)}\big|_{t=0} = u_0 & \text{on } \overline{\Omega}, \end{cases} \tag{2.10}$$

where $Q_T := \Omega \times (0,T)$ and $\Gamma_T := \partial\Omega \times (0,T)$. We take

$$q > n+2 \tag{2.11}$$

and obtain the continuous imbedding

$$W_q^{2,1}(Q_T) \hookrightarrow C^{2-(n+2)/q,1-(n+2)/2q}(\overline{Q_T}), \tag{2.12}$$

where $C^{l,l/2}(\overline{Q_T})$ $(0 < l \notin \boldsymbol{N})$ is the standard Hölder space and its norm is denoted by $[\,\cdot\,]_{l,Q_T}$, see [14].

Set

$$X_{M,T} := \{u \in W_q^{2,1}(Q_T) \mid u \geq 0,\ \|u\|_{W_q^{2,1}(Q_T)} \leq M, \quad u|_{t=0} = u_0\} \tag{2.13}$$

for $M > 0$ and $0 < T \ll 1$ to be decided later. From (2.11), (2.12) and Lemma 4, we can define $v^{(k)}$ as the solution to (2.9) with non-negativity if $u^{(k)} \in X_{M,T}$. Henceforth, C_i^* $(i = 1, 2, \cdots)$ denotes the positive constant which is monotone increasing on T and depends on M and given data, but not on $u^{(k)}$ and $v^{(k)}$. Using the mean value theorem and standard elliptic regularity, we can find the inequality

$$\sup_{\substack{x\in\overline{\Omega} \\ t,s\in[0,T],\ t\neq s}} \frac{|v^{(k)}(x,t) - v^{(k)}(x,s)| + |\nabla v^{(k)}(x,t) - \nabla v^{(k)}(x,s)|}{|t-s|^{1-(n+2)/2q}} \leq C_1^*, \tag{2.14}$$

where C_1^* is independent of T.

If $u^{(k)} \in X_{M,T}$ and $v^{(k)}$ is a solution to (2.9), then there exists a unique solution $u^{(k+1)} \in W_q^{2,1}(Q_T)$ to (2.10) such that

$$\begin{aligned} \|u^{(k+1)}\|_{W_q^{2,1}(Q_T)} \leq C_0^*(T)\Big\{ &\|u_0\|_{W^{2,q}(\Omega)} + \|\nabla\cdot(u^{(k)}\nabla(v^{(k)} - v_0))\|_{L^q(Q_T)} \\ &+ \left\| \left(\frac{\partial v^{(k)}}{\partial \nu} - \frac{\partial v_0}{\partial \nu} \right) u^{(k)} \right\|_{W_q^{1-1/q,1/2-1/2q}(\Gamma_T)} \Big\}, \end{aligned} \tag{2.15}$$

by the $W_q^{2,1}$-theory, see [14]. Note that $C_0^*(T)$ is independent of M and monotone increasing on T. We compute

$$\begin{aligned} &\|\nabla\cdot(u^{(k)}\nabla(v^{(k)} - v_0))\|_{L^q(Q_T)} \\ \leq\ & M(\|v^{(k)} - v_0\|_{L^\infty(Q_T)} + \|\nabla(v^{(k)} - v_0)\|_{L^\infty(Q_T)} \\ &\quad + \|f(v^{(k)}) - f(v_0)\|_{L^\infty(Q_T)} + \|u^{(k)} - u_0\|_{L^\infty(Q_T)}). \end{aligned} \tag{2.16}$$

It holds that

$$\|v^{(k)}-v_0\|_{C(\overline{Q_T})}+\|\nabla(v^{(k)}-v_0)\|_{C(\overline{Q_T})}\le T^{1-(n+2)/2q}C_1^* \tag{2.17}$$

by (2.14). Since f is smooth, we have

$$\begin{aligned}\|f(v^{(k)})-f(v_0)\|_{L^\infty(Q_T)} &\le C_2^*\|v^{(k)}-v_0\|_{L^\infty(Q_T)}\\ &\le T^{1-(n+2)/2q}C_1^*C_2^*.\end{aligned} \tag{2.18}$$

We remark the following fact (see [14]): If (2.11) holds and $0<T\ll 1$ then

$$[u]_{1+\theta,Q_T}\le C_3^*(T^{\theta/2}\|u\|_{W_q^{2,1}(Q_T)}+T^{-1+\theta/2}\|u\|_{L^q(Q_T)}), \tag{2.19}$$

where $\theta=1-(n+2)/q$ and the constant C_3^* is independent of $0<T\ll 1$. Using Hölder's inequality, we have

$$\|u^{(k)}-u_0\|^q_{L^q(Q_T)}\le \frac{1}{q}T^q\|u_t^{(k)}\|^q_{L^q(Q_T)},$$

and then

$$[u^{(k)}-u_0]_{1+\theta,Q_T}\le T^{\theta/2}\cdot 3C_3^*M \tag{2.20}$$

by (2.19). Combining (2.16)-(2.18) and (2.20), we obtain

$$\|\nabla\cdot(u^{(k)}\nabla(v^{(k)}-v_0))\|_{L^q(Q_T)}\le C_4^*T^{1/2-(n+2)/2q}. \tag{2.21}$$

Next, we estimate the boundary integral term as

$$\begin{aligned}&\left\|\left(\frac{\partial v^{(k)}}{\partial\nu}-\frac{\partial v_0}{\partial\nu}\right)u^{(k)}\right\|_{W_q^{1-1/q,1/2-1/2q}(\Gamma_T)}\\ \le\ &\|g(v^{(k)})-g(v_0)\|_{L^\infty(\Gamma_T)}\cdot\|u^{(k)}\|_{W_q^{1-1/q,1/2-1/2q}(\Gamma_T)}\\ &+\|g(v^{(k)})-g(v_0)\|_{W_q^{1-1/q,1/2-1/2q}(\Gamma_T)}\cdot\|u^{(k)}\|_{L^\infty(\Gamma_T)}\\ \le\ &CM(\|g(v^{(k)})-g(v_0)\|_{L^\infty(\Gamma_T)}+\|g(v^{(k)})-g(v_0)\|_{W_q^{1-1/q,1/2-1/2q}(\Gamma_T)}),\end{aligned} \tag{2.22}$$

where C is independent of M and T. Since g is smooth and (2.14) holds, we have

$$\|g(v^{(k)})-g(v_0)\|_{L^\infty(\Gamma_T)}\le T^{1-(n+2)/2q}C_5^* \tag{2.23}$$

Following the notations of [14], we take the canonical set $B\subset \boldsymbol{R}^{n-1}$, for instance the unit cube in $\boldsymbol{R}^{n-1}$, and the finite family of the mappings $\Phi_j : B\to$

$V_j \subset \partial\Omega$ $(j \in \mathcal{J})$ each of which is one-to-one, onto and smooth, and satisfies $\cup_{j\in\mathcal{J}}\Phi_j B = \partial\Omega$. Let $\{\rho_j\}_{j\in\mathcal{J}}$, $0 \le \rho_j \le 1$, be a smooth partition of unity subject to $\{\Phi_j(B)\}_{j\in\mathcal{J}}$. Set

$$\|u\|_{W_q^{\lambda,\lambda/2}(\Gamma_T)} := \|u\|_{L^q(\Gamma_T)} + < u >_{L_q^{\lambda,\lambda/2}(\Gamma_T)}$$

for $0 < \lambda < 1$, where

$$\begin{aligned}
\|u\|_{L^q(\Gamma_T)} &:= \sum_{j\in\mathcal{J}} \|\Phi_j^*(u\rho_j)\|_{L^q(B\times(0,T))}, \\
< u >_{L_q^{\lambda,\lambda/2}(\Gamma_T)} &:= \sum_{j\in\mathcal{J}} < \Phi_j^*(u\rho_j) >_{L_q^{\lambda,\lambda/2}(B\times(0,T))}, \\
< w >_{L_q^{\lambda,\lambda/2}(B\times(0,T))} &:= \left\{\int_0^T dt \int\int_{B\times B} \frac{|w(x,t)-w(y,t)|^q}{|x-y|^{n-1+\lambda q}} dxdy\right\}^{1/q} \\
&\quad + \left\{\int_B dx \int_0^T \int_0^T \frac{|w(x,t)-w(x,s)|^q}{|t-s|^{1+\lambda q/2}} dtds\right\}^{1/q} \\
&=: < w >_{L_q^{\lambda,0}(B\times(0,T))} + < w >_{L_q^{0,\lambda/2}(B\times(0,T))},
\end{aligned}$$

and Φ_j^* is the pull back induced by Φ_j.

We set $w_j^{(k)} = \Phi_j^*(v^{(k)}\rho_j)$ and may take the unit cube as the above B. We calculate according to the above definition and get

$$\|v^{(k)}\|_{L^q(\Gamma_T)} + < v^{(k)} >_{L_q^{1-1/q,0}(\Gamma_T)} \le T^{1/q} C \sup_{0<t<T} \|v^{(k)}(\cdot,t)\|_{W^{1,q}(\Omega)}, \tag{2.24}$$

$$< w_j^{(k)} >_{L_q^{0,(1-1/q)/2}(B\times(0,T))} \le T^{(q+1-n)/2q} C_1^*. \tag{2.25}$$

From the elliptic regularity and (2.20) we have

$$\begin{aligned}
\|v^{(k)}(t) - v_0\|_{W^{1,q}(\Omega)} &\le C_6^* \|u^{(k)}(t) - u_0\|_{C(\overline{\Omega})} \\
&\le T^{\frac{1}{2}(1-(n+2)/q)} \cdot 3C_3^* C_6^* M,
\end{aligned} \tag{2.26}$$

and then

$$\sup_{0<t<T} \|v^{(k)}(\cdot,t)\|_{W^{1,q}(\Omega)} \le \|v_0\|_{W^{1,q}} + T^{\frac{1}{2}(1-(n+2)/q)} \cdot 2C_3^* C_6^* M. \tag{2.27}$$

Combining (2.24), (2.25) and (2.27), we get

$$\|v^{(k)}\|_{W_q^{1-1/q,1/2-1/2q}(\Gamma_T)} \le T^{1/q} C_7^*. \tag{2.28}$$

One can see that if there exist constants A_i $(i=1,2)$ such that

$$\|v^{(k)}\|_{L^\infty(\Gamma_T)} + \|v_0\|_{L^\infty(\Gamma_T)} \leq A_1,$$
$$< v^{(k)} >_{L_q^{1-1/q,1/2-1/2q}(\Gamma_T)} + < v_0 >_{L_q^{1-1/q,1/2-1/2q}(\Gamma_T)} \leq A_2,$$

then there exists a constant C_8^* depending only on q, A_1, A_2, g and Ω such that

$$\begin{aligned} &\|g(v^{(k)}) - g(v_0)\|_{W_q^{1-1/q,1/2-1/2q}(\Gamma_T)} \\ &\leq C_8^*(\|v^{(k)} - v_0\|_{W_q^{1-1/q,1/2-1/2q}(\Gamma_T)} + \|v^{(k)} - v_0\|_{L^\infty(\Gamma_T)}). \end{aligned} \tag{2.29}$$

It holds that

$$\begin{aligned} &\|v^{(k)} - v_0\|_{L^q(\Gamma_T)} + < v^{(k)} - v_0 >_{L_q^{1-1/q,0}(\Gamma_T)} \\ \leq\ & T^{1/q} C \sup_{0<t<T} \|v^{(k)}(\cdot,t) - v_0\|_{W^{1,q}(\Omega)} \\ \leq\ & T^{\frac{1}{2}(1-n/q)} C_9^* \end{aligned}$$

by (2.26) and the trace embedding, and that

$$\begin{aligned} < v^{(k)} - v_0 >_{L_q^{0,1/2-1/2q}(\Gamma_T)} &= < v^{(k)} >_{L_q^{0,1/2-1/2q}(\Gamma_T)} \\ &\leq T^{(q+1-n)/2q} C_1^* \sharp \mathcal{J} \end{aligned}$$

by (2.25). Hence, we get

$$\|v^{(k)} - v_0\|_{W_q^{1-1/q,1/2-1/2q}(\Gamma_T)} \leq T^{(q-n)/2q} C_{10}^*. \tag{2.30}$$

Combining (2.22), (2.23), (2.17) and (2.29)-(2.30), we obtain

$$\left\| \left(\frac{\partial v^k}{\partial \nu} - \frac{\partial v_0}{\partial \nu} \right) u^{(k)} \right\|_{W_q^{1-1/q,1/2-1/2q}(\Gamma_T)} \leq T^{(q-n)/2q} C_{11}^*. \tag{2.31}$$

From (2.15), (2.21) and (2.31) we arrive at the following inequality:

$$\|u^{(k+1)}\|_{W_q^{2,1}(Q_T)} \leq C_0^*(T)(\|u_0\|_{W^{2,q}(\Omega)} + T^{(q-n)/2q} C_{12}^*). \tag{2.32}$$

We fix $0 < \delta \ll 1$. Noting that C_0^* and C_{12}^* are monotone increasing on T, we can take M and T in (2.13) such that

$$M = 2\|u_0\|_{W^{2,q}(\Omega)} \max\{1, C_0^*(\delta)\}, \quad T^{(q-n)/2q} C_{12}^*(\delta) \leq \|u_0\|_{W^{2,q}(\Omega)}. \tag{2.33}$$

Then we can define a mapping

$$\Psi :\ X_{M,T} \ni u^{(k)} \mapsto u^{(k+1)} \in X_{M,T}.$$

To prove the existence of the solution in local-time, it suffices to prove that Ψ is a contraction mapping on $X_{M,T}$. But we omit the proof since the calculations are similar to the above ones which assure that Ψ is a mapping on $X_{M,T}$. We may need to replace T with $\tilde{T}$ which is sufficiently smaller than the previous T. Fix $M > 0$ and $0 < T \ll 1$ such that Ψ is a contraction mapping on $X_{M,T}$, and let (u, v) be a solution to (GCZ) satisfying $u \in X_{M,T}$. The uniqueness of the solution is reduced to that of u by Lemma 4. Let (u_1, v_1) be another solution to (GCZ). By the above argument, we see that there exist M_1 and T_1 such that Ψ is a contraction mapping on X_{M_1,T_1}. If we set

$$M' := \max\{M, M_1\}, \qquad T' := \min\{T, T_1\},$$

then both u and u_1 are in $X_{M',T'}$. Since Ψ is also a contraction mapping on $X_{M',T'}$, the fixed point is unique in $X_{M',T'}$, and therefore

$$u_1 = u_2 \qquad \text{in } Q_{T'}.$$

The uniqueness, $u = u_1$ in Q_T, is clear if $T' = T$. Otherwise, we take

$$T_0 := \inf\{\tau \in (0, T) \mid u \neq u_1 \text{ in } Q_\tau\} \in [T_1, T).$$

It holds that $\sup_{t\in[0,T_0)} \|u_1(t)\|_{W_q^{2,1}(Q_T)} \leq C$ for some $C > 0$. In fact, smoothness of the solution in $\overline{Q_{T_0}}$ follows from (2.11), (2.12), (2.14), Lemma 4 and the Schauder theory with (1.9), see [14]. Therefore, there exists $0 < \delta \ll 1$ such that $u = u_1$ in $Q_{T_0+\delta}$, which is a contradiction by the definition of T_0. Thus the uniqueness is shown. Smoothness of the solution is similar to the above. The proof of Theorem 1 is complete. □

3. Proof of Theorem 2

We confirm several facts used later (see [21], [23]). The Gagliardo-Nirenberg inequality in two space dimentions is described by

$$\|w\|_2^2 \leq K^2(\|\nabla w\|_1^2 + \|w\|_1^2) \qquad \text{for all } w \in W^{1,1}(\Omega), \tag{3.1}$$

where K is a constant determined by Ω. We put $B_R(x_0) = \{x \in \boldsymbol{R}^2 \mid |x - x_0| < R\}$. To derive local in space estimates, we take the smooth cut-off function φ satisfying

$$0 \leq \varphi \leq 1 \quad \text{in } \boldsymbol{R}^2, \qquad \frac{\partial \varphi}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

First, given $x_0 \in \Omega$ and $0 < R' < R$ with $B_{2R}(x_0) \subset \Omega$, we take such φ by

$$\varphi(x) = \begin{cases} 1 & (x \in B_{R'}(x_0)) \\ 0 & (x \in \boldsymbol{R}^2 \setminus B_R(x_0)). \end{cases}$$

Given $x_0 \in \partial\Omega$, next, we take a smooth conformal mapping $X : B_{2R} \cap \overline{\Omega} \to \boldsymbol{R}^2$ satisfying $x_0 \mapsto 0$ and

$$\begin{aligned}
&X(B_{2R}(x_0) \cap \Omega) \subset \{(x_1, x_2) \mid x_2 > 0\}, \\
&X(B_{2R}(x_0) \cap \partial\Omega) \subset \{(x_1, x_2) \mid x_2 = 0\}, \\
&X(B_{R'}(x_0) \cap \Omega) \subset B_{1/2}(0), \qquad X(\Omega \setminus B_R(x_0)) \subset \boldsymbol{R}^2 \setminus B_1(0)
\end{aligned}$$

for $0 < R' < R \ll 1$. Then set $\varphi(x) := \eta(X(x))$, where $\eta \in C_0^\infty(\boldsymbol{R}^2)$, $0 \leq \eta \leq 1$, $\eta = 1$ in $B_{1/2}(0)$, and $\eta = 0$ in $\boldsymbol{R}^2 \setminus B_1(0)$. It holds that

$$\frac{\partial}{\partial \nu}(\eta \circ X) = \frac{\partial X}{\partial \nu} \cdot (\nabla \eta \circ X) = 0 \qquad \text{on } \partial\Omega$$

because $(\partial X)/(\partial \nu)$ is proportional to $(0, 1)$.

The above φ is sometimes written as $\varphi_{x_0,R',R}$. Then, $\psi := (\varphi_{x_0,R',R})^6$ satisfies

$$\begin{aligned}
&\psi(x) = \begin{cases} 1 & (x \in B_{R'}(x_0)) \\ 0 & (x \in \boldsymbol{R}^2 \setminus B_R(x_0)), \end{cases} \\
&0 \leq \psi \leq 1 \quad \text{in } \boldsymbol{R}^2, \qquad \frac{\partial \psi}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \\
&|\nabla \psi| \leq A\psi^{5/6}, \qquad |\Delta \psi| \leq B\psi^{2/3} \qquad \text{in } \boldsymbol{R}^2,
\end{aligned}$$

where A and B are positive constants determined by R' and R. We use the following estimates derived from (3.1), see [21, 23] for the proof.

Lemma 5. The following inequalities hold for any $s > 1$, where $C > 0$ is a constant:

$$\begin{aligned}
\int_\Omega u^2 \psi dx &\leq 2K^2 \int_{B_R(x_0)\cap\Omega} u dx \cdot \int_\Omega u^{-1} |\nabla u|^2 \psi dx + K^2 (\frac{A^2}{2} + 1) \|u\|_1^2, \\
\int_\Omega u^2 dx &\leq \frac{2K^2}{\log s} \int_\Omega (u \log u + e^{-1}) dx \cdot \int_\Omega u^{-1} |\nabla u|^2 dx + 2K^2 \|u\|_1^2 + 3s^2 |\Omega|, \\
\int_\Omega u^3 \psi dx &\leq \frac{72K^2}{\log s} \int_{B_R(x_0)\cap\Omega} (u \log u + e^{-1}) dx \cdot \int_\Omega |\nabla u|^2 \psi dx \\
&\quad + C \|u\|_{L^1(B_R(x_0)\cap\Omega)}^3 + 10|\Omega| s^3.
\end{aligned}$$

Lemma 5 is irrelevant to (GCZ) and the above u is not necessarily to be the solution. Here, we emphasize the mass conservation, i.e.

$$\|u(t)\|_1 = \|u_0\|_1 = \text{constant} \qquad \text{for } 0 \le t < T_{\max}. \tag{3.2}$$

This property is derived from the first boundary condition of (GCZ) and the non-negativity of u. The second equation of (GCZ) is written as

$$-\Delta v + v = -f(v) + u \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \nu} = -g(v) \quad \text{on } \partial\Omega.$$

Since $|f(v)| = -f(v) \le -f(0) = |f(0)|$ if $f(v) \le 0$ by (1.4)-(1.5), we obtain

$$|f(v)| \le |f(0)| + f(v)$$

and hence

$$\begin{aligned}\int_\Omega |f(v)|dx &\le |\Omega| \cdot |f(0)| + \int_\Omega f(v)dx \\ &= |\Omega| \cdot |f(0)| - \int_{\partial\Omega} g(v)dS - \int_\Omega vdx + \int_\Omega udx \\ &\le |\Omega| \cdot |f(0)| + |\partial\Omega| \cdot |g(0)| + \|u\|_1 \le \text{constant}.\end{aligned}$$

It also holds that

$$\begin{aligned}\int_{\partial\Omega} |g(v)|dS &\le |\partial\Omega| \cdot |g(0)| + \int_{\partial\Omega} g(v)dS \\ &= |\partial\Omega| \cdot |g(0)| + \int_\Omega -v - f(v) + u \; dx \\ &\le |\partial\Omega| \cdot |g(0)| + \int_\Omega |f(v)|dx + \|u\|_1 \le \text{constant}.\end{aligned}$$

Then, we apply the L^1 estimate of [7] to obtain

$$\sup_{0\le t<T_{\max}} \|v(t)\|_{W^{1,q}} \le C_q \qquad \text{for } q \in [1,2), \tag{3.3}$$

$$\sup_{0\le t<T_{\max}} \|v(t)\|_q \le C_q \qquad \text{for } q \in [1,\infty), \tag{3.4}$$

$$\sup_{0\le t<T_{\max}} \|v(t)\|_{L^q(\partial\Omega)} \le C_q \qquad \text{for } q \in [1,\infty), \tag{3.5}$$

by Sobolev's and the trace embedding theorems, see [1].

Henceforth, we assume $T_{\max} < \infty$.

Lemma 6. We obtain $x_0 \in \mathcal{B}$ if and only if

$$\limsup_{t\uparrow T_{\max}} \int_{B_R(x_0)\cap\Omega} u \log u dx = \infty \tag{3.6}$$

for any $0 < R \ll 1$.

Proof. The 'if' part is obvious by the definition of $\mathcal{B}$. To show the converse, we assume

$$\limsup_{t\uparrow T_{\max}} \int_{B_R(x_0)\cap\Omega} u \log u dx < \infty \tag{3.7}$$

for some $0 < R \ll 1$ and show $x_0 \notin \mathcal{B}$. In fact, this implies

$$\limsup_{t\uparrow T_{\max}} \int_{\Omega} (u \log u)\psi dx < \infty$$

for $\psi = (\varphi_{x_0,R',R})^6$ $(0 < R' < R)$. Multiplying the fist equation of (GCZ) by $u\psi$ and integrating over Ω, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\psi dx + \int_\Omega |\nabla u|^2 dx + \int_\Omega u\nabla u \cdot \nabla\psi dx \\ = \; & \int_\Omega u\psi\nabla u \cdot \nabla v dx + \int_\Omega u^2 \nabla v \cdot \nabla \psi dx. \end{aligned} \tag{3.8}$$

The first term of the right hand of (3.8) is equal to

$$\begin{aligned} \int_\Omega u\psi\nabla u \cdot \nabla v dx &= \frac{1}{2}\int_\Omega \psi\nabla u^2 \cdot \nabla v dx \\ &= I - \frac{1}{2}\int_\Omega u^2\psi\Delta v dx - \frac{1}{2}\int_\Omega u^2\nabla v \cdot \nabla\psi dx, \end{aligned} \tag{3.9}$$

where

$$I := -\frac{1}{2}\int_{\partial\Omega} \psi u^2 g(v) dS. \tag{3.10}$$

Then, using the second equation of (GCZ) and (1.4)-(1.5), we obtain

$$\begin{aligned} &\int_\Omega u\psi\nabla u \cdot \nabla v dx \\ = \; & I - \frac{1}{2}\int_\Omega u^2(v + f(v))\psi dx + \frac{1}{2}\int_\Omega u^3\psi dx - \frac{1}{2}\int_\Omega u^2\nabla v \cdot \nabla\psi dx \\ \leq \; & I + \frac{1}{2}|f(0)|\int_\Omega u^2\psi dx + \frac{1}{2}\int_\Omega u^3\psi dx - \frac{1}{2}\int_\Omega u^2\nabla v \cdot \nabla\psi dx \end{aligned}$$

$$\leq \quad I + \frac{1}{2}|f(0)| \int_\Omega u^2 \psi dx + \frac{1}{2} \int_\Omega u^3 \psi dx$$
$$+\frac{1}{2} \int_\Omega v \nabla u^2 \cdot \nabla \psi dx + \frac{1}{2} \int_\Omega u^2 v \Delta \psi dx. \tag{3.11}$$

From (3.8)-(3.11) it follows that

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt} \int_\Omega u^2 \psi dx + \int_\Omega |\nabla u|^2 \psi dx + \int_\Omega u \nabla u \cdot \nabla \psi dx \\ \leq \quad & I + \frac{1}{2}|f(0)| \int_\Omega u^2 \psi dx + \frac{1}{2} \int_\Omega u^3 \psi dx \\ & -\frac{1}{2} \int_\Omega v \nabla u^2 \cdot \nabla \psi dx - \frac{1}{2} \int_\Omega u^2 v \Delta \psi dx. \end{aligned} \tag{3.12}$$

Using Young's inequality, we get the following inequalities:

$$\begin{aligned} \frac{1}{2} \left| \int_\Omega u^2 v \Delta \psi dx \right| & \leq \frac{1}{2} \int_\Omega u^2 \psi^{2/3} \cdot B v dx \\ & \leq \frac{1}{3} \int_\Omega u^3 \psi dx + \frac{B^3}{6} \|v\|_3^3, \end{aligned} \tag{3.13}$$

$$\begin{aligned} \frac{1}{2} \left| \int_\Omega v \nabla u^2 \cdot \nabla \psi dx \right| & \leq \int_\Omega |\nabla u| \psi^{1/2} \cdot u \psi^{1/3} \cdot A v dx \\ & \leq \frac{1}{4} \int_\Omega |\nabla u|^2 \psi dx + \int_\Omega u^2 \psi^{2/3} \cdot A^2 v^2 dx \\ & \leq \frac{1}{4} \int_\Omega |\nabla u|^2 \psi dx + \frac{1}{3} \int_\Omega u^3 \psi dx + \frac{4A^6}{3} \|v\|_6^6, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \left| \int_\Omega u \nabla u \cdot \nabla \psi dx \right| & \leq \int_\Omega |\nabla u| \psi^{1/2} \cdot u \psi^{1/3} \cdot A dx \\ & \leq \frac{1}{4} \int_\Omega |\nabla u|^2 \psi dx + \frac{1}{3} \int_\Omega u^3 \psi dx + \frac{4A^6}{3} |\Omega|. \end{aligned} \tag{3.15}$$

Combining (3.12)-(3.15), we obtain

$$\frac{1}{2}\frac{d}{dt} \int_\Omega u^2 \psi dx + \frac{1}{2} \int_\Omega |\nabla u|^2 \psi dx \leq I + \frac{1}{2}|f(0)| \int_\Omega u^2 \psi dx + \frac{3}{2} \int_\Omega u^3 \psi dx + C_1, \tag{3.16}$$

where $C_1 > 0$ is an absolute constant induced by (3.4).

Now, we estimate I defined by (3.10). We see that there exists an absolute constant M_1 (only depending on $g(0)$) such that

$$I \leq M_1 \int_{\partial\Omega} u^2 \psi dS \tag{3.17}$$

by (1.3)-(1.5). We take a constant L such that

$$\|w\|_{L^1(\partial\Omega)} \leq M_2 \|w\|_{W^{1,1}(\Omega)} \qquad \text{for all } w \in W^{1,1}(\Omega). \tag{3.18}$$

Fix $0 < \varepsilon_i \ll 1$ ($i = 1, 2$). It holds that

$$\|u^2\psi\|_{W^{1,1}} = \int_\Omega u^2 \psi dx + \int_\Omega |\nabla(u^2\psi)| dx, \tag{3.19}$$

$$\int_\Omega |\nabla(u^2\psi)| dx \leq 2 \int_\Omega u|\nabla u|\psi dx + \int_\Omega u^2 |\nabla \psi| dx, \tag{3.20}$$

$$\begin{aligned} 2\int_\Omega u|\nabla u|\psi dx &= 2\int_\Omega u\psi^{1/2} \cdot |\nabla u|\psi^{1/2} dx \\ &\leq 2\varepsilon_1 \int_\Omega |\nabla u|^2 \psi dx + 2\varepsilon_1^{-1} \int_\Omega u^2 \psi dx, \end{aligned} \tag{3.21}$$

and

$$\int_\Omega u^2 |\nabla\psi| dx \leq \int_\Omega u^2 \psi^{2/3} \cdot A\psi^{1/6} dx \leq \varepsilon_2 \int_\Omega u^3 \psi dx + \frac{4}{27}\varepsilon_2^{-2} A^6 |\Omega|. \tag{3.22}$$

Combining (3.17)-(3.22), we get

$$\begin{aligned} I \leq M_1 M_2 \Big\{ &(1 + 2\varepsilon_1^{-1}) \int_\Omega u^2 \psi dx + 2\varepsilon_1 \int_\Omega |\nabla u|^2 \psi dx \\ &+ \varepsilon_2 \int_\Omega u^3 \psi dx + \frac{4}{27}\varepsilon_2^{-2} A^6 |\Omega| \Big\}. \end{aligned} \tag{3.23}$$

From (3.12) and (3.23), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\psi dx + \left(\frac{1}{2} - 2\varepsilon_1 M_1 M_2\right) \int_\Omega |\nabla u|^2 \psi dx \\ \leq\; & C_1 + \frac{4}{27}\varepsilon_2^{-2} M_1 M_2 A^6 |\Omega| + \left(\frac{3}{2} + \varepsilon_2 M_1 M_2\right) \int_\Omega u^3 \psi dx \\ &\qquad + \left\{ (1 + 2\varepsilon_1^{-1}) M_1 M_2 + \frac{1}{2}|f(0)| \right\} \int_\Omega u^2 \psi dx. \end{aligned}$$

Taking $\varepsilon_1 = (8M_1M_2)^{-1}$, $\varepsilon_2 = (2M_1M_2)^{-1}$, we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u^2\psi dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx \\ \le\ & C_2 + 4\int_\Omega u^3\psi dx + \{2(1+16M_1M_2) + |f(0)|\}\int_\Omega u^2\psi dx. \end{aligned} \tag{3.24}$$

Now we use (3.7) and prescribe $s \gg 1$ by

$$\frac{72K^2}{\log s}\int_{B_R(x_0)\cap\Omega}(u\log u + e^{-1})dx \le \frac{1}{8}.$$

Then it follows that

$$\frac{d}{dt}\int_\Omega u^2\psi dx \le C_3 + \{2(1+16M_1M_2) + |f(0)|\}\int_\Omega u^2\psi dx$$

by Lemma 5. We use Gronwall's inequality and our assumption $T = T_{\max} < \infty$ to get

$$\sup_{0\le t<T}\int_\Omega u^2\psi dx < \infty. \tag{3.25}$$

We continue the process, multiplying the first equation of (GCZ) by $u^2\psi$ and integrating by parts. We have

$$\begin{aligned} &\frac{1}{3}\frac{d}{dt}\int_\Omega u^3\psi + 2\int_\Omega u|\nabla u|^2 + \int_\Omega u^2\nabla u\cdot\nabla\psi dx \\ &= 2\int_\Omega u^2(\nabla u\cdot\nabla v)\psi dx + \int_\Omega u^3\nabla v\cdot\nabla\psi dx, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\frac{d}{dt}\int_\Omega w^2\psi dx + \frac{8}{3}\int_\Omega |\nabla w|^2\psi dx + 2\int_\Omega w\nabla w\cdot\nabla\psi dx \\ =\ & 4\int_\Omega w(\nabla w\cdot\nabla v)\psi dx + 3\int_\Omega w^2\nabla v\cdot\nabla\psi dx, \end{aligned}$$

where $w := u^{3/2}$. Using the second equation of (GCZ), we have

$$\begin{aligned} &4\int_\Omega w(\nabla w\cdot\nabla v)\psi dx = 2\int_\Omega(\nabla w^2\cdot\nabla v)\psi dx \\ =\ & II - 2\int_\Omega w^2\psi\Delta v dx - 2\int_\Omega w^2\nabla v\cdot\nabla\psi dx \end{aligned}$$

$$
\begin{aligned}
&= II - 2\int_\Omega w^2(v+f(v))\psi dx + 2\int_\Omega w^{8/3}\psi dx - 2\int_\Omega w^2\nabla v\cdot\nabla\psi dx \\
&\le II + 2|f(0)| + 2\int_\Omega w^2\psi dx + 2\int_\Omega w^{8/3}\psi dx - 2\int_\Omega w^2\nabla v\cdot\nabla\psi dx,
\end{aligned}
$$

where

$$
II := -2\int_{\partial\Omega}\psi w^2 g(v)dS. \tag{3.26}
$$

Then, it follows that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega w^2\psi dx + \frac{8}{3}|\nabla w|^2\psi dx + 2\int_\Omega w\nabla w\cdot\nabla\psi dx \\
\le\ & II + 2|f(0)|\int_\Omega w^2\psi dx + 2\int_\Omega w^{8/3}\psi dx \\
&- \int_\Omega v\nabla w^2\cdot\nabla\psi dx - \int_\Omega vw^2\Delta\psi dx.
\end{aligned}\tag{3.27}
$$

By the calculation analogous to (3.13)-(3.15) and (3.23), we get the folowing inequalities:

$$
2\left|\int_\Omega w\nabla w\cdot\nabla\psi dx\right| \le \frac{1}{3}\int_\Omega |\nabla w|^2\psi dx + 3\int_\Omega w^3\psi dx + \frac{4}{9}A^6|\Omega|
$$

$$
2\int_\Omega w^{8/3}\psi dx \le 2\int_\Omega w^3\psi dx + \left(\frac{8}{9}\right)^8\frac{2}{9}|\Omega| \tag{3.28}
$$

$$
\left|\int_\Omega v\nabla w^2\cdot\nabla\psi dx\right| \le \frac{1}{3}\int_\Omega |\nabla w|^2\psi dx + 3\int_\Omega w^3\psi dx + \frac{4}{9}A^6\|v\|_6^6
$$

$$
\left|\int_\Omega vw^2\Delta\psi dx\right| \le \int_\Omega w^3\psi dx + \frac{4}{27}B^3\|v\|_3^3 \tag{3.29}
$$

$$
\begin{aligned}
II \le 4M_1M_2\Bigg\{&(1+2\varepsilon_1^{-1})\int_\Omega u^2\psi dx + 2\varepsilon_1\int_\Omega |\nabla u|^2\psi dx \\
&+\varepsilon_2\int_\Omega u^3\psi dx + \frac{4}{27}\varepsilon_2^{-2}A^6|\Omega|\Bigg\},
\end{aligned}\tag{3.30}
$$

where ε_1 and ε_2 are arbitrary positive constants. Inequalities (3.27)-(3.30) are summarized by

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega w^2\psi dx + (2-8\varepsilon_1M_1M_2)\int_\Omega |\nabla w|^2\psi dx \\
\le\ & C_4 + \frac{16}{27}\varepsilon_2^{-2}M_1M_2A^6|\Omega| + (9+4\varepsilon_2M_1M_2)\int_\Omega w^3\psi dx
\end{aligned}
$$

$$+\left\{4M_1M_2(1+2\varepsilon_1^{-1})+2|f(0)|\right\}\int_\Omega w^2\psi dx,$$

where $C_4 > 0$ is an absolute constant induced by (3.4). If we take $\varepsilon_1 = (16M_1M_2)^{-1}$ and $\varepsilon_2 = (4M_1M_2)^{-1}$, then it holds that

$$\frac{d}{dt}\int_\Omega w^2\psi dx + \frac{3}{2}\int_\Omega |\nabla w|^2\psi dx \le C_5 + 10\int_\Omega w^3\psi dx + \{4M_1M_2(1+32M_1M_2)+2|f(0)|\}\int_\Omega w^2\psi dx. \quad (3.31)$$

Since (3.25) implies

$$\sup_{0\le t<T}\int_{B_{R'}(x_0)\cap\Omega} w\log w dx < \infty$$
$$\sup_{0\le t<T}\|w\|_{L^1(B_{R'}(x_0)\cap\Omega)} < \infty$$

for $R' \in (0,R)$, we can repeat the argument of the previous stage. We see

$$\sup_{0\le t<T}\|w\|^{2/3}_{L^2(B_r(x_0)\cap\Omega)} = \sup_{0\le t<T}\|u\|_{L^3(B_r(x_0)\cap\Omega)} < \infty, \quad (3.32)$$

replacing u, R and $\psi = (\varphi_{x_0,R',R})^6$ by w, R', and $\psi = (\varphi_{x_0,R'',R'})^6$ respectively, where $R'' \in (0,R')$ and $r \in (0,R)$. For $h := u - f(v)$, we have

$$\sup_{0\le t<T}\|h\|_{L^3(B_r(x_0)\cap\Omega)} < \infty$$

by (3.4) and (3.32), and then

$$\sup_{0\le t<T}\|v\|_{W^{2,3}(B_{r'}(x_0)\cap\Omega)} < \infty \qquad \text{for } r' \in (0,r),$$

follows from the elliptic regularity. Since $R' \in (0,R)$ and $r' \in (0,R')$ are arbitrary, Sobolev's embedding guarantees

$$\sup_{0\le t<T}\|v\|_{C^1(B_r(x_0)\cap\Omega)} < \infty \qquad \text{for all } r \in (0,R). \quad (3.33)$$

Repeating the arguments once more, we have

$$\sup_{0\le t<T}\|u\|_{L^4(B_r(x_0)\cap\Omega)} < \infty \qquad \text{for all } r \in (0,R). \quad (3.34)$$

From this stage, we use only the first equation of (GCZ). The rest of the proof is thus anagolous to Step 3 of Lemma 5 of [21] (see also [23]). Skipping details, we just mention that estimates (3.33)-(3.34) and the Moser's iteration scheme (see [3]) are used to obtain

$$\sup_{0\le t<T} \|u\psi\|_\infty < \infty.$$

This implies

$$\limsup_{t\uparrow T} \|u\|_{L^\infty(B_{r'}(x_0)\cap\Omega)} < \infty,$$

and therefore $x_0 \notin \mathcal{B}$. □

Proof of Theorem 2. We shall prove that

$$\limsup_{t\uparrow T} \int_\Omega u\log u dx < \infty \tag{3.35}$$

follows from

$$\liminf_{t\uparrow T} \int_\Omega u\log u dx < \infty. \tag{3.36}$$

In fact, the proof of Lemma 6 is valid even to $\varphi \equiv 1$, and hence we can show that (3.35) implies

$$\limsup_{t\uparrow T} \|u\|_\infty < \infty. \tag{3.37}$$

If (3.37) is the case, then the standard theories (see [2], [12] and [14]) and (3.3)-(3.5) guarantee that the solution u to (GCZ) is continued after $t = T$. Thus, $T_{\max} < \infty$ implies (3.36) and hence we obtain the result $\lim_{t\uparrow T}\int_\Omega u\log u dx = \infty$.

Now, multiplying $\log u$ by the first equation of (GCZ), and we have

$$\frac{d}{dt}\int_\Omega u\log u dx + \int_\Omega u^{-1}|\nabla u|^2 + uv + u(f(v) - f(0))dx$$
$$= |f(0)|\cdot\|u\|_1 + \int_\Omega u^2 dx. \tag{3.38}$$

Using Lemma 5 and (3.2), we see

$$\frac{d}{dt}\int_\Omega u\log u dx + \left\{1 - \frac{2K^2}{\log s}\int_\Omega (u\log u + e^{-1})dx\right\}\int_\Omega u^{-1}|\nabla u|^2 dx$$
$$\le C_6\|u_0\|_1^2 + 3s^2|\Omega|,$$

where $s > 0$ is arbitrary. Then, we take

$$s = s(t) = \exp(2K^2\int_\Omega (u\log u + e^{-1})dx) > 1,$$

and obtain

$$\frac{dJ}{dt} \leq C_6\|u_0\|_1^2 + 3|\Omega| \exp(4K^2 J), \quad J(t) := \int_\Omega (u \log u + e^{-1}) dx. \tag{3.39}$$

From this differential inequality, we can conclude that (3.36) ⇒ (3.35). □

4. Proof of Theorem 3

We take the Green's function $G = G(x, x')$ defined by

$$(-\Delta_{x'} + 1)G = \delta(x' - x) \ (x' \in \Omega), \qquad \frac{\partial}{\partial \nu_{x'}} G = 0 \ (x' \in \partial\Omega)$$

for $x \in \Omega$, where δ denotes the Dirac' delta function, and shall use the following lemma (see [21],[23] for the proof):

Lemma 7. It holds that $\rho \in L^\infty(\Omega \times \Omega)$, where

$$\rho(x, x') = \nabla\psi(x) \cdot \nabla_x G(x, x') + \nabla\psi(x') \cdot \nabla_{x'} G(x, x').$$

First, we show the following lemma:

Lemma 8. It holds that

$$\frac{d}{dt} \int_\Omega (u \log u)\psi + \frac{1}{4} u^{-1} |\nabla u|^2 \psi \ dx \leq 5 \int_\Omega u^2 \psi dx + C_8.$$

Proof. Multiplying $\psi \log u$ to the first equation of (GCZ), we have

$$\begin{aligned} \frac{d}{dt} \int_\Omega (u \log u)\psi dx &= -\int_\Omega \nabla u \cdot \nabla\{(\log u + 1)\psi\} dx \\ &\quad + \int_\Omega u \nabla v \cdot \nabla\{(\log u + 1)\psi dx =: -F_1 + F_2. \end{aligned}$$

From the second equation of (GCZ)

$$\begin{aligned} F_2 &= \int_\Omega \nabla v \cdot \nabla u dx + \int_\Omega u(\log u + 1) \nabla v \cdot \nabla\psi dx \\ &= III - \int_\Omega u \nabla \cdot (\psi \nabla v) dx + \int_\Omega u(\log u + 1)\nabla v \cdot \nabla\psi dx \end{aligned}$$

$$= \quad III + \int_\Omega u^2\psi dx - \int_\Omega u\psi(v+f(v))dx + \int_\Omega (u\log u)\nabla v\cdot\nabla\psi dx,$$

where

$$III := \int_{\partial\Omega} u\psi g(v)dS. \tag{4.1}$$

We also have that

$$F_1 = \int_\Omega u^{-1}|\nabla u|^2\psi dx + \int_\Omega (\log u + 1)\nabla u\cdot\nabla\psi dx.$$

Then we obtain the following inequality:

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (u\log u)\psi dx + \int_\Omega u^{-1}|\nabla u|^2\psi dx + \int_\Omega u\psi(v+f(v)-f(0))dx\\ = \quad & III + |f(0)|\int_\Omega u\psi dx - \int_\Omega (\log u+1)\nabla u\cdot\nabla\psi dx + \int_\Omega (u\log u)\nabla v\cdot\nabla\psi dx.\end{aligned} \tag{4.2}$$

Here, we recall the elementary inequality: Let $\alpha > 0$ and $0 < \beta < 2$. Then it holds that

$$(|\log u| + 1)^\alpha u^\beta \le u^2 + d_{\alpha,\beta},$$

where $k_{\alpha,\beta}$ is a positive constant determined by only α and β.

$$\begin{aligned}&\left|\int_\Omega (\log u+1)\nabla u\cdot\nabla\psi dx\right| \le A\int_\Omega (|\log u|+1)u^{1/2}\psi^{1/3}\cdot u^{-1/2}|\nabla u|\psi^{1/2}dx\\ &\le A|\partial\Omega|^{1/6}\left\{\int_\Omega (|\log u|+1)^3u^{3/2}\psi dx\right\}^{1/3}\left\{\int_\Omega u^{-1}|\nabla u|^2\psi dx\right\}^{1/2}\\ &\le A|\partial\Omega|^{1/6}\left\{\int_\Omega u^2\psi dx + d_{3,3/2}|\Omega|\right\}^{1/3}\left\{\int_\Omega u^{-1}|\nabla u|^2\psi dx\right\}^{1/2}\\ &\le \int_\Omega \frac{1}{4}u^{-1}|\nabla u|^2\psi u^2\psi\ dx + d_{3,3/2}|\Omega| + \frac{4}{27}A^6|\Omega|.\end{aligned} \tag{4.3}$$

The fifth term of the right-hand side of equality (4.2) is equal to

$$-\int_\Omega v\nabla\cdot(u\log u\nabla\psi)dx = -\int_\Omega v(\log u+1)\nabla u\cdot\nabla\psi dx - \int_\Omega (vu\log u)\Delta\psi dx \tag{4.4}$$

by $\left.\frac{\partial\psi}{\partial\nu}\right|_{\partial\Omega} = 0$. Each term of the right-hand side of equality (4.4) is estimated as follows:

$$\left|\int_\Omega v(\log u+1)\nabla u\cdot\nabla\psi dx\right|$$

$$\le A \int_\Omega v \cdot u^{1/2}(|\log u| + 1)\psi^{1/3} \cdot u^{-1/2}|\nabla u|\psi^{1/2} dx$$
$$\le A\|v\|_6 \left\{u^{3/2}(|\log u| + 1)^3 \psi dx\right\}^{1/3} \left\{u^{-1}|\nabla u|^2 \psi dx\right\}^{1/2}$$
$$\le \frac{1}{4}\int_\Omega u^2 \psi dx + d_{3,3/2}|\Omega| + \frac{4}{27}A^6\|v\|_6^6. \tag{4.5}$$

$$\left|\int_\Omega (vu\log u)\Delta\psi dx\right| \le B\int_\Omega v|u\log u|\psi^{2/3}dx$$
$$\le \quad B\|v\|_3 \left\{\int_\Omega u^{3/2}(|\log u| + 1)^{3/2}\psi dx\right\}^{2/3}$$
$$\le \quad \int_\Omega u^2\psi dx + d_{3/2,3/2}|\Omega| + \frac{4}{27}B^3\|v\|_3^3. \tag{4.6}$$

Finally, III, defined by (4.1), is estimated as follows:

$$III \le M_1 \int_{\partial\Omega} u\psi dS \le M_1 M_2 \left(\int_\Omega u(\psi + |\nabla\psi|)dx + \int_\Omega |\nabla u|\psi dx\right)$$
$$\le M_1 M_2 \left(\|u\|_1 + \int_\Omega u\psi^{1/2} \cdot A\psi^{1/3} dx + \int_\Omega u^{-1/2}|\nabla u| \cdot u^{1/2}\psi dx\right)$$
$$\le \frac{1}{4}\int_\Omega u^{-1}|\nabla u|^2 dx + \int_\Omega u^2\psi dx + C_7, \tag{4.7}$$

where M_1 and M_2 are positive constants determined by (3.17) and (3.18), and C_7 is an constant depending only on $\|u_0\|_1, M_1, M_2$ and A. Combining (4.2)-(4.6), we get the desired estimate. □

We are now in a position to prove the finiteness of blowup points.

Proof of Theorem 3. We take $x_0 \in \mathcal{B}$. From Lemmas 5 and 8, it follows that

$$\frac{d}{dt}\int_\Omega u\log u\psi dx + \frac{1}{4}\left(1 - 40K^2 \int_{B_R(x_0)\cap\Omega} udx\right)\int_\Omega u^{-1}|\nabla u|^2\psi dx \le C_9, \tag{4.8}$$

and therefore, the condition

$$\limsup_{t\uparrow T} \int_{B_R(x_0)\cap\Omega} udx < \varepsilon_0 := \frac{1}{40K^2}$$

implies

$$\limsup_{t\uparrow T} \int_{B'_R(x_0)\cap\Omega} u\log u dx \le \limsup_{t\uparrow T} \int_\Omega (u\log u)\psi dx < \infty$$

for any $0 < R' < R$. This implies $x_0 \notin \mathcal{B}$ by Lemma 6, a contradiction. That is, each $x_0 \in \mathcal{B}$ admits the estimate

$$\limsup_{t\uparrow T} \int_{B_R(x_0)\cap\Omega} udx \ge \varepsilon_0 \qquad \text{for any } 0 < R \ll 1. \tag{4.9}$$

We shall replace this inequality by

$$\liminf_{t\uparrow T} \int_{B_R(x_0)\cap\Omega} udx \ge \varepsilon_0 \qquad \text{for any } 0 < R \ll 1. \tag{4.10}$$

If this is the case, it follows that

$$\sharp\mathcal{B} \le \frac{\|u_0\|_1}{\varepsilon_0} < \infty$$

by virtue of the mass conservation (3.2), and then the proof is complete.

To prove (4.10), it suffices to show that

$$\sup_{0<t<T} \left| \frac{d}{dt}\int_\Omega u\psi dx \right| \le C(x_0) < \infty \tag{4.11}$$

by the assumption $T_{\max} < +\infty$. In fact, if (4.11) is the case, the convergence

$$\lim_{t\uparrow T} \int_\Omega u\psi dx = \int_\Omega u_0(x)\psi(x)dx + \int_0^T \left(\frac{d}{dt}\int_\Omega u(\cdot,t)\psi dx \right) dt$$

exists, and therefore it holds that

$$\liminf_{t\uparrow T} \int_{B_R(x_0)\cap\Omega} udx \ge \lim_{t\uparrow T} \int_\Omega u\psi dx \ge \limsup_{t\uparrow T} \int_{B'_R(x_0)\cap\Omega} udx \ge \varepsilon_0$$

because $0 < R \ll 1$ is arbitrary in (4.9).

Now, we use the first equation of (GCZ) and $\left.\frac{\partial\psi}{\partial\nu}\right|_{\partial\Omega} = 0$, and derive

$$\frac{d}{dt}\int_\Omega u\psi dx = \int_\Omega u\Delta\psi dx + \int_\Omega u\nabla v\cdot\nabla\psi dx =: F_3 + F_4. \tag{4.12}$$

It is clear that

$$|F_3| \le B\|u_0\|_1. \tag{4.13}$$

Also, F_4 is equal to

$$\begin{aligned}
&\int_\Omega u(x,t)\nabla\psi(x)\cdot\nabla_x\left\{\int_\Omega G(x,x')[u(x',t)-f(v(x',t))]dx'\right.\\
&\qquad\qquad\qquad\left.-\int_{\partial\Omega}G(x,\xi)g(v(\xi,t))dS_\xi\right\}dx\\
&=\frac{1}{2}\int_\Omega\int_\Omega\rho(x,x')u(x,t)u(x',t)dxdx'\\
&\quad-\int_\Omega\int_\Omega u(x,t)\nabla\psi(x)\cdot\nabla_x G(x,x')f(v(x',t))dx'dx\\
&\quad-\int_\Omega u(x,t)\nabla\psi(x)\cdot\left\{\int_{\partial\Omega}\nabla_x G(x,\xi)g(v(\xi,t))dS_\xi\right\}dx\\
&=: F_4^{(1)}-F_4^{(2)}-F_4^{(3)}.
\end{aligned}\tag{4.14}$$

Lemma 7 implies that

$$|F_4^{(1)}|\le\|\rho\|_\infty\|u_0\|_1^2,\tag{4.15}$$

while

$$\begin{aligned}
&|F_4^{(2)}|\le A\|u_0\|_1\|(-\Delta+1)^{-1}f(v)\|_{W^{1,\infty}}\le C_{10}\|(-\Delta+1)^{-1}f(v)\|_{W^{2,3}}\\
&\le C_{11}\|f(v)\|_3\le C_{12}(1+\|v\|_{3p_1}^{p_1})\le C_{13}
\end{aligned}\tag{4.16}$$

by the Sobolev embedding, (1.7) and (3.4), where C_{13} is independent of t. Finally, if ψ is supported in $\overline{\Omega}$ it holds that

$$|F_4^{(3)}|\le C_{15}(x_0)\tag{4.17}$$

which is the case of $x_0\in\Omega$. We obtain the finiteness of the inner blowup points and the proof is complete. □

The finiteness of boundary blowup points is an open qustion. In this connection, we recall that this situation is also the case of the Dirichlet boundary condition to v where the non-existence of the boundary blowup point arises in the stationary state, see [18].

The crucial estimate in our argument arises in accordance with

$$H=\int_{\partial\Omega}G(x,\xi)g(v(\xi,t))dS_\xi$$

satisfying

$$(-\Delta+1)H=0\text{ in }\Omega,\qquad\frac{\partial H}{\partial\nu}=g\ \text{ on }\partial\Omega$$

for $g(\xi,t)=g(v(\xi,t))$. An expected sharp estimate is concerned with the weak norm, see [10],

$$\|v\|_{H^1_w(\Omega)}\leq C$$

in (2.1) for the prescribed $\|u\|_1=\lambda$. Then, it will follow that

$$\|g\|_{H^1_w(\Omega)}\leq C$$

if $g=g(v)$ is appropriate. Then, we take $\zeta\in H^2_w(\Omega)$ such that $\frac{\partial\zeta}{\partial\nu}=g$ on $\partial\Omega$. Then we obtain

$$(-\Delta+1)h=(\Delta-1)\zeta \text{ in } \Omega, \qquad \frac{\partial h}{\partial\nu}=0 \text{ on } \partial\Omega$$

for $h=H-\zeta$ with $(\Delta-1)\zeta\in L^2_w(\Omega)$. This relation will imply

$$\|H\|_{H^2_w(\Omega)}\leq C,$$

and consequently,

$$\left|\int_\Omega u\nabla\psi\cdot\nabla H dx\right|\leq C\int_\Omega (u\log u+e^{-1})dx$$

by

$$\left|\int_\Omega u\nabla\psi\cdot\nabla H dx\right|\leq C[u]_{L\log L}\|H\|_{H^2_w(\Omega)}$$

$$[f]_{L\log L}=\int_\Omega |f(x)|\log\left(1+\frac{|f(x)|}{\|f\|_1}\right)dx.$$

Thus we obtain

$$\left|\frac{d}{dt}\int_\Omega u\psi dx\right|\leq C\left(1+\int_\Omega (u\log u+e^{-1})\psi dx\right) \tag{4.18}$$

besides (4.8).

Inequality (4.18) is not appropriate to control the local L^1 norm of u in time, but we can infer the finiteness of boundary blowup points, assuming

$$\iint_{\Omega\times[0,T)} u\log u\ dxdt<+\infty.$$

In this connection, we remind

$$\int_\Omega u\log u\ dx\geq\delta_1\log\frac{1}{T-t}-C$$

derived from (3.39) with $\delta_1>0$.

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