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COMPACT SPACES WITH RESPECT TO AN IDEAL

Asha Gupta[1 §], Ramandeep Kaur[2]
[1,2]Department of Applied Sciences
PEC University of Technology
Chandigarh, INDIA

Abstract: The aim of this paper is to study compactness modulo an ideal called I-compact spaces and discuss their properties. Some of the results in compact spaces have been generalized in terms of I-compact spaces.

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1. Introduction and Preliminaries

Ideals in topological spaces have been considered since 1930. These have been studied by Kuratowski[5] in 1933 and Vaidyanathaswamy[14] in 1946. An ideal I on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies:

(i) $A \in I$ and $B \subseteq A$ implies $B \in I(heredity)$
(ii)$A \in I$ and $B \in I$ implies $A \cup B \in I$(finite additivity).

We denote a topological space (X, τ) with an ideal I defined on X by (X, τ, I). An ideal I is said to be condense or a boundary ideal (see [8]) if $\tau \cap I = \{\phi\}$. If $A \subset X$, $cl(A)$ will denote the closure of A in (X, τ).

A subset A of a space (X, τ) is said to be g-closed[7] if $cl(A) \subseteq U$, whenever $A \subseteq U$ and $U \in \tau$. Every closed set is g-closed but converse is not true.

If (X, τ, I) is an ideal space,(Y, σ) is a topological space and $f : (X, \tau, I) \longrightarrow$

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§Correspondence author

(Y, σ) is a function, then $f(I) = \{f(I_1) : I_1 \in I\}$ is an ideal of Y[8]. If I is ideal of subsets of X and Y is subset of X, then $I_Y = \{Y \cap I_1 : I_1 \in I\}$ is an ideal of subsets of Y[8].

In ideal space (X, τ, I), the collection $\beta(I, \tau) = \{U - I_1 : U \in \tau, I_1 \in I\}$ is a basis for a topology $\tau^*(I)$ finer than τ[4]. When no ambiguity is present, we denote $\beta(I, \tau)$ by β and $\tau^*(I)$ by τ^*. Let (X, τ, I) be an ideal space. Then $A^*(I, \tau) = \{x \in X : A \cap U \notin I$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$. When there is no ambiguity, we will write A^* for $A^*(I, \tau)$ and call it the "local function of A". The simplest ideals are $\{\phi\}$ and $\wp(X) = \{A : A \subseteq X\}$. Observe that $A^*(\{\phi\}) = cl(A)$ and $A^*(\wp(X)\} = \phi$ for every $A \subseteq X$.

Note. $x \notin A^*$ if and only if $(U - J) \cap A = \phi$, when $U \in \tau(x)$ and $J \in I$.

Lemma 1.1. (see [10]) *Let (X, τ, I) be an ideal space and let A be a subset of X. Then:*

(i) $A^* = cl(A^*) \subseteq cl(A)$;

(ii) A is τ^*-closed if and only if $A^* \subseteq A$.

2. *I*-Compact Spaces

The concept of compactness modulo an ideal was defined by Newcomb[8] and had been studied by Rancin[9]. This concept has been further investigated by Hamlett and Jankovic[1].

Definition 2.1. A subset A of a space (X, τ, I) is said to be I- compact or compact modulo I[8] if for every cover $\{U_\lambda : \lambda \in \Lambda\}$ of A by open sets of X, there exist a finite subset $\Lambda_\circ$ of Λ such that $A - \cup\{U_\lambda : \lambda \in \Lambda_\circ\} \in I$. The space (X, τ, I) is said to be I-compact if X is I-compact. If (X, τ) is a space with an ideal $I = \{\phi\}$, then (X, τ) is compact if and only if (X, τ) is compact modulo I.

Theorem2.1. *Every g-closed subset of I-compact space is I-compact.*

Proof. Let A be g- closed subset of (X, τ, I). Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of A. Since A is g- closed, $A \subset \cup U_\lambda$ implies $cl(A) \subset \cup U_\lambda$.

Now $\{U_\lambda : \lambda \in \Lambda\} \cup \{X - cl(A)\}$ is open cover of X, which is I-compact, therefore there exists finite subset $\Lambda_\circ$ of Λ such that either $X - (\cup[U_\lambda : \lambda \in \Lambda_\circ] \cup \{X - cl(A)\}) \in I$ or $X - \cup[U_\lambda : \lambda \in \Lambda_\circ] \in I$ either $(X - (\cup[U_\lambda : \lambda \in$

$\Lambda_\circ] \cup \{X - cl(A)\}) \cap A \Rightarrow A - \cup[U_\lambda : \lambda \in \Lambda_\circ] \in I$ or $\{X - \cup[U_\lambda : \lambda \in \Lambda_\circ]\} \cap A \subset X - [U_\lambda : \lambda \in \Lambda_\circ] \in I \Rightarrow A - \cup[U_\lambda : \lambda \in \Lambda_\circ] \in I$. Hence A is I-compact.

From theorem 2.1 we have the following.

Corollary 2.1. *Every closed subset of I-compact space is I-compact.*

Corollary 2.2. *If A is I-compact in X and B an open set contained in A. Then $A - B$ is I-compact.*

Corollary 2.3. *If F is closed and K is I-compact subset of X. Then $F \cap K$ is I-compact.*

Theorem2.2. *Every I-compact subset of a Hausdroff ideal space is τ^*-closed.*

Proof. Let A be I-compact subset of Hausdroff ideal space (X, τ, I). Let $x \notin A$ then $x \in X - A$. For each $y \in A$, there exist neighbourhoods U_y and V_y of x and y respectively such that $U_y \cap V_y = \phi$. Note that $x \notin cl(V_y)$. Now $\{V_y : y \in A\}$ is a τ-open cover of A which is I-compact, therefore there exists a finite subset $\Lambda_\circ$ of A such that $A - \cup[V_y : y \in \Lambda_\circ] \in I$. Now $x \notin cl(V_y)$ for each y implies $x \notin \cup_{y \in \Lambda_\circ} cl(V_y) = cl(\cup_{y \in \Lambda_\circ} V_y)$. Let $U = X - cl(\cup_{y \in \Lambda_\circ} V_y)$ and let $J = A - cl(\cup_{y \in \Lambda_\circ} V_y) \subseteq A - \cup_{y \in \Lambda_\circ} V_y = I_1$ where $I_1 \in I$ Then $U - J \in \tau^*(x)$ and $(U - J) \cap A = \phi$ implying thereby that $x \notin A^*$. Hence $A^* \subset A$, so A is τ^*-closed.

Theorem2.3. *Continuous image of I-compact space is I-compact.*

Proof. Let $f : X \to Y$ be any continuous map, where (X, τ, I) is I-compact. Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open covering of the set $f(X)$ by sets open in Y. Since f is continuous, the collection $\{f^{-1}(U_\lambda) : \lambda \in \Lambda\}$ is an open covering of X. Given that X is I-compact, there exists a finite subset $\Lambda_\circ$ of Λ such that $X - \cup[f^{-1}(U_\lambda) : \lambda \in \Lambda_\circ] \in I$. Now,$f(X - \cup[f^{-1}(U_\lambda) : \lambda \in \Lambda_\circ]) \in f(I)$. We know $f(X) - f(\cup[f^{-1}(U_\lambda) : \lambda \in \Lambda_\circ]) \subset f(X - \cup[f^{-1}(U_\lambda) : \lambda \in \Lambda_\circ])$. This implies $f(X) - f(\cup[f^{-1}(U_\lambda) : \lambda \in \Lambda_\circ]) \in f(I)$, since $f(I)$ is an ideal of Y. As $f(X) - \cup[U_\lambda : \lambda \in \Lambda_\circ] \subset f(X) - f(\cup[f^{-1}(U_\lambda) : \lambda \in \Lambda_\circ])$,so $f(X) - \cup[U_\lambda : \lambda \in \Lambda_\circ] \in f(I)$ implying thereby that continuous image of I-compact space is I-compact.

Theorem2.4. *Let (X, τ, I) be any ideal space and let A be a subset of X such that for each open set U containing A there is I-compact set B with $A \subset B \subset U$. Then A is I-compact.*

Proof. Let $\{U_\lambda : \lambda \in \Lambda\}$ be a τ_A-open cover of A where $U_\lambda = V_\lambda \cap A$ such that V_λ is open in X. By the given condition, there exists an I-compact subset B of X such that $A \subset B \subset \cup V_\lambda$. Then $\{V_\lambda \cap B : \lambda \in \Lambda\}$ is a τ_B-open cover of B. As B is I-compact, there exists a finite subset $\Lambda_\circ$ of Λ such that $B - \cup[V_\lambda \cap B : \lambda \in \Lambda_\circ] \in I_B$. Let $B - \cup[V_\lambda \cap B : \lambda \in \Lambda_\circ] = I_1 \cap B$. Here $I_1 \cap B \in I_B$, where $I_1 \in I$. Since $B = \cup[V_\lambda \cap B : \lambda \in \Lambda_\circ] \cup (I_1 \cap B)$. Then $B \cap A = (\cup[V_\lambda \cap B : \lambda \in \Lambda_\circ] \cup (I_1 \cap B)) \cap A$. $\Rightarrow A = \cup[V_\lambda \cap B \cap A : \lambda \in \Lambda_\circ] \cup (I_1 \cap B \cap A)$. $\Rightarrow A = \cup[V_\lambda \cap A : \lambda \in \Lambda_\circ] \cup (I_1 \cap A)$. $\Rightarrow A - \cup[V_\lambda \cap A : \lambda \in \Lambda_\circ] = (I_1 \cap A) \in I_A$ implying thereby that A is I-compact.

Corollary 2.4. *If every open subset of X is I-compact, then every subset of X contained in open subset is I-compact.*

Theorem2.5. *If A and B are I- compact in ideal space (X, τ, I), then $A \cup B$ is I- compact in X.*

Proof. Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of $A \cup B$ in X. Then $\{U_\lambda : \lambda \in \Lambda\}$ is an open cover of A and B. Given A and B are I-compact, there exists $I_1, I_2 \in I$ and finite subset $\Lambda_\circ$ and Λ_1 of Λ such that $A - \cup[U_{\lambda_i} : \lambda_i \in \Lambda_\circ] = I_1$ and $B - \cup[U_{\lambda_K} : \lambda_K \in \Lambda_1] = I_2$. $A = \cup[U_{\lambda_i} : \lambda_i \in \Lambda_\circ] \cup I_1$ and $B = \cup[U_{\lambda_K} : \lambda_K \in \Lambda_1] \cup I_2$. Now,$A \cup B = (\cup[U_{\lambda_i} : \lambda_i \in \Lambda_\circ]) \cup (\cup[U_{\lambda_K} : \lambda_K \in \Lambda_1]) \cup (I_1 \cup I_2)$. $A \cup B = \cup[U_{\lambda_i} \cup U_{\lambda_K} : \lambda_i \in \Lambda_\circ, \lambda_K \in \Lambda_1] \cup (I_1 \cup I_2)$

This implies $A \cup B = \cup[U_{\lambda_i} \cup U_{\lambda_K} : \lambda_i \in \Lambda_\circ, \lambda_K \in \Lambda_1] \cup I$ where $I_1 \cup I_2 = I$ $(A \cup B) - \cup[U_{\lambda_i} \cup U_{\lambda_K} : \lambda_i \in \Lambda_\circ, \lambda_K \in \Lambda_1] \in I$ implying thereby that is I-compact in X.

Corollary 2.5. *Finite union of I-compact space X is I-compact.*

Theorem2.6. *The following are equivalent for a space (X, τ, I)*

(a) (X, τ, I) is I-compact.

(b) (X, τ^*, I) is I-compact.

(c) For any family $\{F_\lambda : \lambda \in \Lambda\}$ of closed sets of X such that $\cap\{F_\lambda : \lambda \in \Lambda\} = \phi$, there exists a finite subset $\Lambda_\circ$ of Λ such that $\cap\{F_\lambda : \lambda \in \Lambda_\circ\} \in I$.

Proof. $(a) \Rightarrow (b)$ Let $\{U_\lambda : \lambda \in \Lambda\}$ be a τ^*-open cover of X such that $U_\lambda = V_\lambda - E_\lambda$, where V_λ open in X and $E_\lambda \in I$. Now $\{V_\lambda : \lambda \in \Lambda\}$ is an open cover of X and hence there exists a finite subset $\Lambda_\circ$ of Λ such that $X - \cup\{V_\lambda : \lambda \in \Lambda_\circ\} \in I$. This implies that $X - \cup\{U_\lambda : \lambda \in \Lambda_\circ\} \subset (X - \cup\{V_\lambda : \lambda \in \Lambda_\circ\}) \cup [\cup\{E_\lambda : \lambda \in \Lambda_\circ\}] \in I$. Therefore, (X, τ^*, I) is I-compact.

$(b) \Rightarrow (a)$ It follows from $\tau \subset \tau^*$.

$(a) \Rightarrow (c)$ Let $\{F_\lambda : \lambda \in \Lambda\}$ be a family of closed sets of X such that $\cap\{F_\lambda : \lambda \in \Lambda\} = \phi$.

Then $\{X - F_\lambda : \lambda \in \Lambda\}$ is an open cover of X. By (a) since (X, τ, I) is I-compact, there exists a finite subset $\Lambda_\circ$ of Λ such that $X - \cup\{X - F_\lambda : \lambda \in \Lambda_\circ\} \in I$. This implies that $\cap\{F_\lambda : \lambda \in \Lambda_\circ\} \in I$.

$(c) \Rightarrow (a)$ Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of X, then $\{X - U_\lambda : \lambda \in \Lambda\}$ is a collection of closed sets and $\cap\{X - U_\lambda : \lambda \in \Lambda\} = \phi$. Hence there exists a finite subset $\Lambda_\circ$ of Λ such that $\cap\{X - U_\lambda : \lambda \in \Lambda_\circ\} \in I$.This implies that $X - \cup\{U_\lambda : \lambda \in \Lambda_\circ\} \in I$.This shows (X, τ, I) is I-compact.

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