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INTERVAL-VALUED $(\widetilde{\theta}, \widetilde{\delta})$-FUZZY KU-IDEALS OF KU-ALGEBRAS

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Abstract: In this paper, we introduced the notion of interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-ideals of KU-algebras and some related properties are investigated. We proved that $U(\widetilde{\eta}; \widetilde{t})$ is a KU-ideal if and only if the interval-valued fuzzy subset $\widetilde{\eta}$ is an interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-ideal of a KU-algebra for $\widetilde{\theta} \prec \widetilde{t} \preceq \widetilde{\delta}$.

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1. Introduction

Prabpayak and Leerawat [1] introduced a new algebraic structure which is called KU-algebra. They introduced the concept of homomorphism of KU-algebras and investigated some related properties in [2].

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Zadeh [3] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches, such as groups, functional analysis, probability theory, topology and so on. Mostafa et al. [4] introduced the notion of fuzzy KU-ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU-ideals. Yaqoob et al. [5] studied the concept of cubic KU-ideals of KU-algebras. Some authors studied different aspects of fuzzy sets in algebraic structures, see [6-13].

Interval-valued fuzzy sets were proposed thirty years ago as a natural extension of fuzzy sets by Zadeh. Interval-valued fuzzy subsets have many applications in several areas. The concept of interval-valued fuzzy sets have been studied in various algebraic structures, see [14-21]. Yao introduced (λ, θ)-fuzzy normal subfields [22]. Coumaressane [23] characterized near-rings by their (λ, θ)-fuzzy quasi-ideals. Yaqoob and Ansari [24] studied bipolar (λ, δ)-fuzzy ideals in ternary semigroups. Shabir et al. [25] characterized semigroups by the properties of their fuzzy ideals with thresholds and Khan et al. [26] characterized ordered semigroups by their (λ, θ)-fuzzy bi-ideals.

In this paper, we introduced a notion of interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-ideals of KU-algebras and studied some properties of interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-ideals.

2. Review of Literature

In this section, we will recall some concepts related to KU-algebra and interval-valued fuzzy sets.

Definition 1. [1] By a KU-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ with a single binary operation $*$ that satisfies the following identities:
(ku1): $(x * y) * [(y * z) * (x * z)] = 0$,
(ku2): $x * 0 = 0$,
(ku3): $0 * x = x$,
(ku4): $x * y = 0 = y * x$ implies $x = y$,
for any $x, y, z \in X$.

In what follows, let $(X, *, 0)$ denote a KU-algebra unless otherwise specified. For brevity we also call X a KU-algebra.

Definition 2. [2] A subset S of a KU-algebra X is called a KU-subalgebra of X if $x * y \in S$, whenever $x, y \in S$.

Definition 3. [2] A non-empty subset A of a KU-algebra X is called a KU-ideal of X if it satisfies the following conditions:

(1) $0 \in A$,
(2) $x * (y * z) \in A$, $y \in A$ implies $x * z \in A$,
for all $x, y, z \in X$.

Example 4. *[5] Let $X = \{0, 1, 2, 3, 4, 5\}$ in which $*$ is defined by the following table:*

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	0	0	2	2	4	5
2	0	0	0	1	4	5
3	0	0	0	0	4	5
4	0	0	0	1	0	5
5	0	0	0	0	0	0

Clearly $(X, *, 0)$ is a KU-algebra. It is easy to show that $A = \{0, 1\}$ and $B = \{0, 1, 2, 3, 4\}$ are KU-ideals of X.

Now we will recall the concept of interval-valued fuzzy sets.

An interval number is $\widetilde{a} = [a^-, a^+]$, where $0 \le a^- \le a^+ \le 1$. Let $D[0,1]$ denote the family of all closed subintervals of $[0,1]$, i.e.,

$$D[0,1] = \{\widetilde{a} = [a^-, a^+] : a^- \le a^+, \text{ for } a^-, a^+ \in I\}.$$

We define the operations $\succeq$, $\preceq$, $=$, rmin and rmax in case of two elements in $D[0,1]$. We consider two elements $\widetilde{a} = [a^-, a^+]$ and $\widetilde{b} = [b^-, b^+]$ in $D[0,1]$. Then

(1) $\widetilde{a} \succeq \widetilde{b}$ if and only if $a^- \ge b^-$ and $a^+ \ge b^+$,
(2) $\widetilde{a} \preceq \widetilde{b}$ if and only if $a^- \le b^-$ and $a^+ \le b^+$,
(3) $\widetilde{a} = \widetilde{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$,
(4) $\text{rmin}\{\widetilde{a}, \widetilde{b}\} = [\min\{a^-, b^-\}, \min\{a^+, b^+\}]$,
(5) $\text{rmax}\{\widetilde{a}, \widetilde{b}\} = [\max\{a^-, b^-\}, \max\{a^+, b^+\}]$.

Here we consider that $\widetilde{0} = [0, 0]$ as least element and $\widetilde{1} = [1, 1]$ as greatest element. Let $\widetilde{a_i} \in D[0,1]$ where $i \in \Lambda$. We define

$$\underset{i\in\Lambda}{\text{rinf}}\widetilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \ \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \underset{i\in\Lambda}{\text{rsup}}\widetilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \ \sup_{i\in\Lambda} a_i^+\right].$$

An interval valued fuzzy set (briefly, IVF-set) $\widetilde{\eta}$ on X is defined as

$$\widetilde{\eta} = \left\{\left\langle x, [\eta^-(x), \eta^+(x)]\right\rangle : x \in X\right\},$$

where $\widetilde{\eta} : X \longrightarrow D[0,1]$ and $\eta^-(x) \le \eta^+(x)$, for all $x \in X$. Then the ordinary fuzzy sets $\eta^- : X \to [0,1]$ and $\eta^+ : X \to [0,1]$ are called a lower fuzzy set and an upper fuzzy set of $\widetilde{\eta}$, respectively.

3. Interval-Valued $(\widetilde{\theta}, \widetilde{\delta})$-Fuzzy KU-Ideals

In this section, we will introduce a new type of interval-valued fuzzy KU-ideals of KU-algebras.

Definition 5. [14] Let X be a KU-algebra. An interval-valued fuzzy set $\widetilde{\eta}$ in X is called an interval-valued fuzzy KU-subalgebra of X, if $\widetilde{\eta}(x * y) \succeq \mathrm{rmin}\{\widetilde{\eta}(x), \widetilde{\eta}(y)\}$, for all $x, y \in X$.

In what follows, let $\widetilde{\theta}, \widetilde{\delta} \in D[0,1]$ be such that $\widetilde{0} \prec \widetilde{\theta} \prec \widetilde{\delta} \preceq \widetilde{1}$.

Definition 6. An interval-valued fuzzy subset $\widetilde{\eta}$ of X is called an interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-subalgebra of X if

$$\mathrm{rmax}\left\{\widetilde{\eta}(x * y), \widetilde{\theta}\right\} \succeq \mathrm{rmin}\left\{\widetilde{\eta}(x), \widetilde{\eta}(y), \widetilde{\delta}\right\},$$

for all $x, y \in X$.

Example 7. *[5] Let $X = \{0, 1, 2, 3, 4\}$ be a KU-algebra in which $*$ is defined by the following table:*

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	0	1
2	0	3	0	3	4
3	0	1	2	0	1
4	0	0	0	0	0

Define an interval-valued fuzzy subset $\widetilde{\eta}$ in X as follows:

$$\widetilde{\eta}(x) = \begin{cases} [0.7, 0.9] & \text{if } x = 0 \\ [0.5, 0.6] & \text{if } x = 1, 2, 3, 4. \end{cases}$$

By routine calculations it can be seen that the interval-valued fuzzy set $\widetilde{\eta}$ is an interval-valued $([0.1, 0.2], [0.3, 0.4])$-fuzzy KU-subalgebra of X.

Proposition 8. *If an interval-valued fuzzy subset $\widetilde{\eta}$ is an interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-subalgebra of X. Then the set $\widetilde{\eta}_{\widetilde{\theta}} = \left\{x \in X : \widetilde{\eta}(x) \succ \widetilde{\theta}\right\}$ is a KU-subalgebra of X.*

Proof. Let $\widetilde{\eta}$ be an interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-subalgebra of X, and let $x, y \in X$, such that $x, y \in \widetilde{\eta}_{\widetilde{\theta}}$. Then $\widetilde{\eta}(x) \succ \widetilde{\theta}$ and $\widetilde{\eta}(y) \succ \widetilde{\theta}$. Since $\widetilde{\eta}$ is an interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-subalgebra, therefore

$$\mathrm{rmax}\left\{\widetilde{\eta}(x * y), \widetilde{\theta}\right\} \succeq \mathrm{rmin}\left\{\widetilde{\eta}(x), \widetilde{\eta}(y), \widetilde{\delta}\right\}$$

$$\succ \quad \text{rmin}\left\{\widetilde{\theta},\widetilde{\theta},\widetilde{\delta}\right\}=\widetilde{\theta}.$$

Hence $\widetilde{\eta}(x*y)\succ\widetilde{\theta}$. This shows that $x*y\in\widetilde{\eta}_{\widetilde{\theta}}$. Hence $\widetilde{\eta}_{\widetilde{\theta}}$ is a KU-subalgebra of X. □

Theorem 9. *A non-empty subset A of X is a KU-subalgebra of X if and only if the interval-valued fuzzy subset $\widetilde{\eta}$ of X defined as follows:*

$$\widetilde{\eta}(x)=\begin{cases}\succeq\widetilde{\delta} & \text{if } x\in A\\ \widetilde{\theta} & \text{if } x\notin A,\end{cases}$$

is an interval-valued $(\widetilde{\theta},\widetilde{\delta})$-fuzzy KU-subalgebra of X.

Proof. Suppose that A is a KU-subalgebra of X. Let $x,y\in X$ be such that $x,y\in A$, then $x*y\in A$. Hence $\widetilde{\eta}(x*y)\succeq\widetilde{\delta}$. Therefore

$$\text{rmax}\left\{\widetilde{\eta}(x*y),\widetilde{\theta}\right\}\succeq\widetilde{\delta}=\text{rmin}\left\{\widetilde{\eta}(x),\widetilde{\eta}(y),\widetilde{\delta}\right\}.$$

If $x\notin A$ or $y\notin A$, then $\text{rmin}\left\{\widetilde{\eta}(x),\widetilde{\eta}(y),\widetilde{\delta}\right\}=\widetilde{\theta}$. Thus

$$\text{rmax}\left\{\widetilde{\eta}(x*y),\widetilde{\theta}\right\}\succeq\widetilde{\theta}=\text{rmin}\left\{\widetilde{\eta}(x),\widetilde{\eta}(y),\widetilde{\delta}\right\}.$$

Consequently $\widetilde{\eta}$ is an interval-valued $(\widetilde{\theta},\widetilde{\delta})$-fuzzy KU-subalgebra of X.

Conversely: Let $x,y\in A$. Then $\widetilde{\eta}(x)\succeq\widetilde{\delta}$, $\widetilde{\eta}(y)\succeq\widetilde{\delta}$. As $\widetilde{\eta}$ is an interval-valued $(\widetilde{\theta},\widetilde{\delta})$-fuzzy KU-subalgebra of X, therefore

$$\begin{aligned}\text{rmax}\left\{\widetilde{\eta}(x*y),\widetilde{\theta}\right\} &\succeq \text{rmin}\left\{\widetilde{\eta}(x),\widetilde{\eta}(y),\widetilde{\delta}\right\}\\ &\succeq \text{rmin}\left\{\widetilde{\delta},\widetilde{\delta},\widetilde{\delta}\right\}=\widetilde{\delta}.\end{aligned}$$

This implies that $x*y\in A$. Hence A is a KU-subalgebra of X. □

Definition 10. For any $\widetilde{t}\in D[0,1]$. Let $\widetilde{\eta}$ be an interval-valued fuzzy set in X, the set

$$U\left(\widetilde{\eta};\widetilde{t}\right)=\left\{x\in X:\widetilde{\eta}(x)\succeq\widetilde{t}\right\},$$

is called the level set of $\widetilde{\eta}$.

Theorem 11. *A interval-valued fuzzy subset $\widetilde{\eta}$ of X is an interval-valued $(\widetilde{\theta},\widetilde{\delta})$-fuzzy KU-subalgebra of X if and only if each non-empty level subset $U(\widetilde{\eta};\widetilde{t})$ is a KU-subalgebra of X for $\widetilde{\theta}\prec\widetilde{t}\preceq\widetilde{\delta}$.*

Proof. Let $\widetilde{\eta}$ be an interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-subalgebra of X. Assume $x, y \in X$, $\widetilde{\theta} \prec \widetilde{t} \preceq \widetilde{\delta}$ and $x, y \in U(\widetilde{\eta}; \widetilde{t})$. Then $\widetilde{\eta}(x) \succeq \widetilde{t}$ and $\widetilde{\eta}(y) \succeq \widetilde{t}$. As $\widetilde{\eta}$ is an interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-subalgebra of X. Therefore

$$\begin{aligned} \text{rmax}\left\{\widetilde{\eta}(x*y), \widetilde{\theta}\right\} &\succeq \text{rmin}\left\{\widetilde{\eta}(x), \widetilde{\eta}(y), \widetilde{\delta}\right\} \\ &\succeq \text{rmin}\left\{\widetilde{t}, \widetilde{t}, \widetilde{\delta}\right\} = \widetilde{t}. \end{aligned}$$

This implies that $\widetilde{\eta}(x*y) \succeq \widetilde{t}$ and $x*y \in U(\widetilde{\eta}; \widetilde{t})$. Hence $U(\widetilde{\eta}; \widetilde{t})$ is a KU-subalgebra of X, for $\widetilde{\theta} \prec \widetilde{t} \preceq \widetilde{\delta}$.

Conversely, if there exist $x, y \in X$, such that

$$\text{rmax}\left\{\widetilde{\eta}(x*y), \widetilde{\theta}\right\} \prec \text{rmin}\left\{\widetilde{\eta}(x), \widetilde{\eta}(y), \widetilde{\delta}\right\}.$$

Then there exists $\widetilde{t}$ such that

$$\text{rmax}\left\{\widetilde{\eta}(x*y), \widetilde{\theta}\right\} \prec \widetilde{t} \preceq \text{rmin}\left\{\widetilde{\eta}(x), \widetilde{\eta}(y), \widetilde{\delta}\right\}.$$

This shows that $\widetilde{\eta}(x) \succeq \widetilde{t}$, $\widetilde{\eta}(y) \succeq \widetilde{t}$ and $\widetilde{\eta}(x*y) \prec \widetilde{t}$, so $x, y \in U(\widetilde{\eta}; \widetilde{t})$. Since $U(\widetilde{\eta}; \widetilde{t})$ is a KU-subalgebra of X. Therefore $x*y \in U(\widetilde{\eta}; \widetilde{t})$, but this is a contradiction to $\widetilde{\eta}(x*y) \prec \widetilde{t}$. Thus

$$\text{rmax}\left\{\widetilde{\eta}(x*y), \widetilde{\theta}\right\} \succeq \text{rmin}\left\{\widetilde{\eta}(x), \widetilde{\eta}(y), \widetilde{\delta}\right\}.$$

Consequently $\widetilde{\eta}$ is an interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-subalgebra of X. □

Remark 12. *An interval-valued fuzzy KU-subalgebra is an interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-subalgebra of X with $\widetilde{\theta} = [0, 0]$ and $\widetilde{\delta} = [1, 1]$.*

Corollary 13. *Every interval-valued fuzzy KU-subalgebra is an interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-subalgebra of X.*

The converse of Corollary 13 is not true in general, for this we consider the following example.

Example 14. *Consider Example 7, and define an interval-valued fuzzy subset $\widetilde{\eta}$ in X as follows:*

$$\widetilde{\eta}(x) = \begin{cases} [0.9, 0.98] & \text{if } x = 0, 3, 4 \\ [0.7, 0.83] & \text{if } x = 1, 2. \end{cases}$$

By routine calculations it can be seen that the interval-valued fuzzy set $\widetilde{\eta}$ is an interval-valued $([0.1, 0.3], [0.5, 0.6])$-fuzzy KU-subalgebra of X. But $\widetilde{\eta}$ is not an interval-valued fuzzy KU-subalgebra of X, because

$$\widetilde{\eta}(3*4) = \widetilde{\eta}(1) = [0.7, 0.83] \prec \min\{\widetilde{\eta}(3), \widetilde{\eta}(4)\} = [0.9, 0.98].$$

Lemma 15. *A non-empty subset A of X is KU-subalgebra of X if and only if $\widetilde{\eta}_A$ is an interval-valued $(\widetilde{\theta},\widetilde{\delta})$-fuzzy KU-subalgebra of X.*

Proof. Let A be a KU-subalgebra of X. Then $\widetilde{\eta}_A$ is an interval-valued fuzzy KU-subalgebra and by Corollary 13, $\widetilde{\eta}_A$ is an interval-valued $(\widetilde{\theta},\widetilde{\delta})$-fuzzy KU-subalgebra of X.

Conversely, let $x, y \in X$ be such that $x, y \in A$. Then $\widetilde{\eta}_A(x) = \widetilde{\eta}_A(y) = [1,1]$. Since $\widetilde{\eta}_A$ is an interval-valued $(\widetilde{\theta},\widetilde{\delta})$-fuzzy KU-subalgebra of X. Therefore

$$\begin{aligned} \mathrm{rmax}\left\{\widetilde{\eta}(x*y),\widetilde{\theta}\right\} &\succeq \mathrm{rmin}\left\{\widetilde{\eta}(x),\widetilde{\eta}(y),\widetilde{\delta}\right\} \\ &= \mathrm{rmin}\left\{[1,1],[1,1],\widetilde{\delta}\right\} = \widetilde{\delta}. \end{aligned}$$

As $\widetilde{\theta} \prec \widetilde{\delta}$, it implies that $\widetilde{\eta}_A(x*y) \succeq \widetilde{\delta}$. Thus $x*y \in A$. Therefore $\widetilde{\eta}_A$ is a KU-subalgebra of X. □

Definition 16. [14] Let X be a KU-algebra. An interval-valued fuzzy set $\widetilde{\eta}$ in X is called an interval-valued fuzzy KU-ideal of X if it satisfies the following conditions:

(IVF1) $\widetilde{\eta}(0) \succeq \widetilde{\eta}(x)$,

(IVF2) $\widetilde{\eta}(x*z) \succeq \mathrm{rmin}\{\widetilde{\eta}(x*(y*z)),\widetilde{\eta}(y)\}$,

for all $x, y, z \in X$.

Definition 17. An interval-valued fuzzy subset $\widetilde{\eta}$ of X is called an interval-valued $(\widetilde{\theta},\widetilde{\delta})$-fuzzy KU-ideal of X if it satisfies the following conditions:

(1) $\mathrm{rmax}\left\{\widetilde{\eta}(0),\widetilde{\theta}\right\} \succeq \mathrm{rmin}\left\{\widetilde{\eta}(x),\widetilde{\delta}\right\}$,

(2) $\mathrm{rmax}\left\{\widetilde{\eta}(x*z),\widetilde{\theta}\right\} \succeq \mathrm{rmin}\left\{\widetilde{\eta}(x*(y*z)),\widetilde{\eta}(y),\widetilde{\delta}\right\}$,

for all $x, y \in X$.

Example 18. *Consider Example 4, and define an interval-valued fuzzy subset $\widetilde{\eta}$ in X as follows:*

$$\widetilde{\eta}(x) = \begin{cases} [0.8, 0.9] & \text{if } x = 0, 5 \\ [0.5, 0.6] & \text{if } x = 1, 2, 3, 4. \end{cases}$$

By routine calculations it can be seen that the interval-valued fuzzy set $\widetilde{\eta}$ is an interval-valued $([0.1, 0.2], [0.3, 0.4])$-fuzzy KU-ideal of X.

Proposition 19. *If an interval-valued fuzzy subset $\widetilde{\eta}$ is an interval-valued $(\widetilde{\theta},\widetilde{\delta})$-fuzzy KU-ideal of X. Then the set $\widetilde{\eta}_{\widetilde{\theta}}$ is a KU-ideal of X.*

Proof. Suppose that $\widetilde{\eta}$ is an interval-valued $(\widetilde{\theta},\widetilde{\delta})$-fuzzy KU-ideal of X and let $x,y,z\in X$ such that $x,y,(x*(y*z))\in\widetilde{\eta}_{\widetilde{\theta}}$. Then $\widetilde{\eta}(x)\succ\widetilde{\theta}$, $\widetilde{\eta}(y)\succ\widetilde{\theta}$ and $\widetilde{\eta}(x*(y*z))\succ\widetilde{\theta}$. Since $\widetilde{\eta}$ is an interval-valued $(\widetilde{\theta},\widetilde{\delta})$-fuzzy KU-ideal, therefore

$$\text{rmax}\left\{\widetilde{\eta}(0),\widetilde{\theta}\right\}\succeq\text{rmin}\left\{\widetilde{\eta}(x),\widetilde{\delta}\right\}\succ\text{rmin}\left\{\widetilde{\theta},\widetilde{\delta}\right\}=\widetilde{\theta},$$

and

$$\begin{aligned}\text{rmax}\left\{\widetilde{\eta}(x*z),\widetilde{\theta}\right\}&\succeq\text{rmin}\left\{\widetilde{\eta}(x*(y*z)),\widetilde{\eta}(y),\widetilde{\delta}\right\}\\&\succ\text{rmin}\left\{\widetilde{\theta},\widetilde{\theta},\widetilde{\delta}\right\}=\widetilde{\theta}.\end{aligned}$$

Hence $\widetilde{\eta}(0)\succ\widetilde{\theta}$ and $\widetilde{\eta}(x*z)\succ\widetilde{\theta}$. This shows that $0,(x*z)\in\widetilde{\eta}_{\widetilde{\theta}}$. Hence $\widetilde{\eta}_{\widetilde{\theta}}$ is a KU-ideal of X. □

Theorem 20. *A non-empty subset A of X is a KU-ideal of X if and only if the interval-valued fuzzy subset $\widetilde{\eta}$ of X defined as follows:*

$$\widetilde{\eta}(x)=\begin{cases}\succeq\widetilde{\delta} & \text{if } x\in A\\ \widetilde{\theta} & \text{if } x\notin A,\end{cases}$$

is an interval-valued $(\widetilde{\theta},\widetilde{\delta})$-fuzzy KU-subalgebra of X.

Proof. Suppose that A is a KU-ideal of X. Let $x,y,z\in X$ be such that $x,y,(x*(y*z))\in A$, then $x*z\in A$. Hence $\widetilde{\eta}(x*z)\succeq\widetilde{\delta}$. Therefore

$$\text{rmax}\left\{\widetilde{\eta}(0),\widetilde{\theta}\right\}\succeq\widetilde{\delta}=\text{rmin}\left\{\widetilde{\eta}(x),\widetilde{\delta}\right\},$$

and

$$\text{rmax}\left\{\widetilde{\eta}(x*z),\widetilde{\theta}\right\}\succeq\widetilde{\delta}=\text{rmin}\left\{\widetilde{\eta}(x*(y*z)),\widetilde{\eta}(y),\widetilde{\delta}\right\}.$$

If $x\notin A$ or $y\notin A$, then $\text{rmin}\{\widetilde{\eta}(x*(y*z)),\widetilde{\eta}(y),\widetilde{\delta}\}=\widetilde{\theta}$. Thus

$$\text{rmax}\left\{\widetilde{\eta}(0),\widetilde{\theta}\right\}\succeq\widetilde{\theta}=\text{rmin}\left\{\widetilde{\eta}(x),\widetilde{\delta}\right\},$$

and

$$\text{rmax}\left\{\widetilde{\eta}(x*z),\widetilde{\theta}\right\}\succeq\widetilde{\theta}=\text{rmin}\left\{\widetilde{\eta}(x*(y*z)),\widetilde{\eta}(y),\widetilde{\delta}\right\}.$$

Consequently $\widetilde{\eta}$ is an interval-valued $(\widetilde{\theta},\widetilde{\delta})$-fuzzy KU-ideal of X.

Conversely: Let $x, y, (x*(y*z)) \in A$. Then $\widetilde{\eta}(x) \succeq \widetilde{\delta}$, $\widetilde{\eta}(y) \succeq \widetilde{\delta}$ and $\widetilde{\eta}(x*(y*z)) \succeq \widetilde{\delta}$. As $\widetilde{\eta}$ is an interval-valued $(\widetilde{\theta},\widetilde{\delta})$-fuzzy KU-ideal of X, therefore

$$\operatorname{rmax}\left\{\widetilde{\eta}(0),\widetilde{\theta}\right\} \succeq \operatorname{rmin}\left\{\widetilde{\eta}(x),\widetilde{\delta}\right\} \succeq \operatorname{rmin}\left\{\widetilde{\delta},\widetilde{\delta}\right\} = \widetilde{\delta},$$

and

$$\begin{aligned}\operatorname{rmax}\left\{\widetilde{\eta}(x*z),\widetilde{\theta}\right\} &\succeq \operatorname{rmin}\left\{\widetilde{\eta}(x*(y*z)),\widetilde{\eta}(y),\widetilde{\delta}\right\}\\ &\succeq \operatorname{rmin}\left\{\widetilde{\delta},\widetilde{\delta},\widetilde{\delta}\right\} = \widetilde{\delta}.\end{aligned}$$

This implies that $0, (x*y) \in A$. Hence A is a KU-ideal of X. □

Theorem 21. *A interval-valued fuzzy subset $\widetilde{\eta}$ of X is an interval-valued $(\widetilde{\theta},\widetilde{\delta})$-fuzzy KU-ideal of X if and only if each non-empty level subset $U(\widetilde{\eta};\widetilde{t})$ is a KU-ideal of X for $\widetilde{\theta} \prec \widetilde{t} \preceq \widetilde{\delta}$.*

Proof. Let $\widetilde{\eta}$ be an interval-valued $(\widetilde{\theta},\widetilde{\delta})$-fuzzy KU-ideal of X. Assume $x, y, z \in X$, $\widetilde{\theta} \prec \widetilde{t} \preceq \widetilde{\delta}$ and $x, y, (x*(y*z)) \in U(\widetilde{\eta};\widetilde{t})$. Then $\widetilde{\eta}(x) \succeq \widetilde{t}$, $\widetilde{\eta}(y) \succeq \widetilde{t}$ and $\widetilde{\eta}(x*(y*z)) \succeq \widetilde{t}$. As $\widetilde{\eta}$ is an interval-valued $(\widetilde{\theta},\widetilde{\delta})$-fuzzy KU-ideal of X. Therefore

$$\operatorname{rmax}\left\{\widetilde{\eta}(0),\widetilde{\theta}\right\} \succeq \operatorname{rmin}\left\{\widetilde{\eta}(x),\widetilde{\delta}\right\} \succeq \operatorname{rmin}\left\{\widetilde{t},\widetilde{\delta}\right\} = \widetilde{t},$$

and

$$\begin{aligned}\operatorname{rmax}\left\{\widetilde{\eta}(x*z),\widetilde{\theta}\right\} &\succeq \operatorname{rmin}\left\{\widetilde{\eta}(x*(y*z)),\widetilde{\eta}(y),\widetilde{\delta}\right\}\\ &\succeq \operatorname{rmin}\left\{\widetilde{t},\widetilde{t},\widetilde{\delta}\right\} = \widetilde{t}.\end{aligned}$$

This implies that $0, (x*y) \in U(\widetilde{\eta};\widetilde{t})$. Hence $U(\widetilde{\eta};\widetilde{t})$ is a KU-ideal of X.

Conversely, if there exist $x, y, z \in X$ such that

$$\operatorname{rmax}\left\{\widetilde{\eta}(0),\widetilde{\theta}\right\} \prec \operatorname{rmin}\left\{\widetilde{\eta}(x),\widetilde{\delta}\right\},$$

and

$$\operatorname{rmax}\left\{\widetilde{\eta}(x*z),\widetilde{\theta}\right\} \prec \operatorname{rmin}\left\{\widetilde{\eta}(x*(y*z)),\widetilde{\eta}(y),\widetilde{\delta}\right\}.$$

Then there exists $\widetilde{t}$ such that

$$\operatorname{rmax}\left\{\widetilde{\eta}(0),\widetilde{\theta}\right\} \prec \widetilde{t} \preceq \operatorname{rmin}\left\{\widetilde{\eta}(x),\widetilde{\delta}\right\},$$

and

$$\operatorname{rmax}\left\{\widetilde{\eta}(x*z),\widetilde{\theta}\right\} \prec \widetilde{t} \preceq \operatorname{rmin}\left\{\widetilde{\eta}(x*(y*z)),\widetilde{\eta}(y),\widetilde{\delta}\right\}.$$

This shows that $\widetilde{\eta}(x) \succeq \widetilde{t}$, $\widetilde{\eta}(y) \succeq \widetilde{t}$ and $\widetilde{\eta}(x*(y*z)) \succeq \widetilde{t}$ also $\widetilde{\eta}(0) \prec \widetilde{t}$ and $\widetilde{\eta}(x*z) \prec \widetilde{t}$, so $x, y, (x*(y*z)) \in U(\widetilde{\eta}; \widetilde{t})$. Since $U(\widetilde{\eta}; \widetilde{t})$ is a KU-ideal of X. Therefore $0, (x*z) \in U(\widetilde{\eta}; \widetilde{t})$ but this is a contradiction to $\widetilde{\eta}(0) \prec \widetilde{t}$ and $\widetilde{\eta}(x*z) \prec \widetilde{t}$. Thus

$$\text{rmax}\left\{\widetilde{\eta}(0), \widetilde{\theta}\right\} \succeq \text{rmin}\left\{\widetilde{\eta}(x), \widetilde{\delta}\right\},$$

and

$$\text{rmax}\left\{\widetilde{\eta}(x*y), \widetilde{\theta}\right\} \succeq \text{rmin}\left\{\widetilde{\eta}(x*(y*z)), \widetilde{\eta}(y), \widetilde{\delta}\right\}.$$

Consequently $\widetilde{\eta}$ is an interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-ideal of X. □

Remark 22. *An interval-valued fuzzy KU-ideal is an interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-ideal of X with $\widetilde{\theta} = [0, 0]$ and $\widetilde{\delta} = [1, 1]$.*

Corollary 23. *Every interval-valued fuzzy KU-ideal is an interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-ideal of X.*

The converse of Corollary 23 is not true in general, for this we consider the following example.

Example 24. *Consider Example 4, and define an interval-valued fuzzy subset $\widetilde{\eta}$ in X as follows:*

$$\widetilde{\eta}(x) = \begin{cases} [0.9, 0.93] & \text{if } x = 0 \\ [0.6, 0.67] & \text{if } x = 1, 3 \\ [0.7, 0.8] & \text{if } x = 2, 4, 5. \end{cases}$$

By routine calculations it can be seen that the interval-valued fuzzy set $\widetilde{\eta}$ is an interval-valued $([0.1, 0.2], [0.4, 0.5])$-fuzzy KU-ideal of X. But $\widetilde{\eta}$ is not an interval-valued fuzzy KU-ideal of X, because

$$\begin{aligned} \widetilde{\eta}(2*3) &= \widetilde{\eta}(1) = [0.6, 0.67] \prec \min\{\widetilde{\eta}(2*(4*3)), \widetilde{\eta}(4)\} \\ &= \min\{\widetilde{\eta}(0), \widetilde{\eta}(4)\} = [0.7, 0.8]. \end{aligned}$$

Lemma 25. *A non-empty subset A of X is KU-ideal of X if and only if $\widetilde{\eta}_A$ is an interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-ideal of X.*

Proof. Let A be a KU-ideal of X. Then $\widetilde{\eta}_A$ is an interval-valued fuzzy KU-ideal and by Corollary 23, $\widetilde{\eta}_A$ is an interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-ideal of X.

Conversely, let $x, y, z \in X$ be such that $x, y, x*(y*z) \in A$. Then $\widetilde{\eta}_A(x) = \widetilde{\eta}_A(y) = \widetilde{\eta}_A(x*(y*z)) = [1, 1]$. Since $\widetilde{\eta}_A$ is an interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-ideal of X. Therefore

$$\text{rmax}\left\{\widetilde{\eta}(0), \widetilde{\theta}\right\} \succeq \text{rmin}\left\{\widetilde{\eta}(x), \widetilde{\delta}\right\}$$

$$= \quad \text{rmin}\left\{[1,1],\widetilde{\delta}\right\} = \widetilde{\delta},$$

and

$$\begin{aligned}\text{rmax}\left\{\widetilde{\eta}(x*z),\widetilde{\theta}\right\} &\succeq \text{rmin}\left\{\widetilde{\eta}(x*(y*z)),\widetilde{\eta}(y),\widetilde{\delta}\right\}\\ &= \text{rmin}\left\{[1,1],[1,1],\widetilde{\delta}\right\} = \widetilde{\delta}.\end{aligned}$$

As $\widetilde{\theta} \prec \widetilde{\delta}$, it implies that $\widetilde{\eta}_A(0) \succeq \widetilde{\delta}$ and $\widetilde{\eta}_A(x*z) \succeq \widetilde{\delta}$. Thus $0,(x*y) \in A$. Therefore $\widetilde{\eta}_A$ is a KU-ideal of X. □

Lemma 26. *Let $\widetilde{\eta}$ be an interval-valued $(\widetilde{\theta},\widetilde{\delta})$-fuzzy KU-ideal of X. If the inequality $x*y \le z$ hold in X. Then,*

$$\mathit{rmax}\left\{\widetilde{\eta}(y),\widetilde{\theta}\right\} \succeq \mathit{rmin}\left\{\widetilde{\eta}(x),\widetilde{\eta}(z),\widetilde{\delta}\right\}.$$

Proof. Let $\widetilde{\eta}$ be an interval-valued $(\widetilde{\theta},\widetilde{\delta})$-fuzzy KU-ideal of X. Consider that the inequality $x*y \le z$ holds in X, then $z*(x*y) = 0$. By definition of interval-valued $(\widetilde{\theta},\widetilde{\delta})$-fuzzy KU-ideal, we have

$$\text{rmax}\left\{\widetilde{\eta}(z*y),\widetilde{\theta}\right\} \succeq \text{rmin}\left\{\widetilde{\eta}(z*(x*y)),\widetilde{\eta}(x),\widetilde{\delta}\right\},$$

if we put $z = 0$, then

$$\begin{aligned}\text{rmax}\left\{\widetilde{\eta}(0*y),\widetilde{\theta}\right\} &= \text{rmax}\left\{\widetilde{\eta}(y),\widetilde{\theta}\right\}\\ &\succeq \text{rmin}\left\{\widetilde{\eta}(0*(x*y)),\widetilde{\eta}(x),\widetilde{\delta}\right\}\\ &= \text{rmin}\left\{\widetilde{\eta}(x*y),\widetilde{\eta}(x),\widetilde{\delta}\right\}, \qquad (1)\end{aligned}$$

but

$$\begin{aligned}\text{rmax}\left\{\widetilde{\eta}(x*y),\widetilde{\theta}\right\} &\succeq \text{rmin}\left\{\widetilde{\eta}(x*(z*y)),\widetilde{\eta}(z),\widetilde{\delta}\right\}\\ &= \text{rmin}\left\{\widetilde{\eta}(z*(x*y)),\widetilde{\eta}(z),\widetilde{\delta}\right\}\\ &= \text{rmin}\left\{\widetilde{\eta}(0),\widetilde{\eta}(z),\widetilde{\delta}\right\}\\ &= \text{rmin}\left\{\widetilde{\eta}(z),\widetilde{\delta}\right\}. \qquad (2)\end{aligned}$$

From (1) and (2), we get $\text{rmax}\left\{\widetilde{\eta}(y),\widetilde{\theta}\right\} \succeq \text{rmin}\left\{\widetilde{\eta}(x),\widetilde{\eta}(z),\widetilde{\delta}\right\}$. □

Lemma 27. *Let $\widetilde{\eta}$ be an interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-ideal of X. If $x \leq y$, then* $\text{rmax}\left\{\widetilde{\eta}(x), \widetilde{\theta}\right\} \succeq \text{rmin}\left\{\widetilde{\eta}(y), \widetilde{\delta}\right\}$.

Proof. Let $\widetilde{\eta}$ be an interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-ideal of X. If $x \leq y$, then $y * x = 0$, this together with $0 * x = x$ and $\text{rmax}\left\{\widetilde{\eta}(0), \widetilde{\theta}\right\} \succeq \text{rmin}\{\widetilde{\eta}(y), \widetilde{\delta}\}$, we get

$$\begin{aligned}
\text{rmax}\left\{\widetilde{\eta}(x), \widetilde{\theta}\right\} &= \text{rmax}\left\{\widetilde{\eta}(0 * x), \widetilde{\theta}\right\} \\
&\succeq \text{rmin}\left\{\widetilde{\eta}(0 * (y * x)), \widetilde{\eta}(y), \widetilde{\delta}\right\} \\
&= \text{rmin}\left\{\widetilde{\eta}(0 * 0), \widetilde{\eta}(y), \widetilde{\delta}\right\} \\
&= \text{rmin}\left\{\widetilde{\eta}(0), \widetilde{\eta}(y), \widetilde{\delta}\right\} \\
&= \text{rmin}\left\{\widetilde{\eta}(y), \widetilde{\delta}\right\}.
\end{aligned}$$

This completes the proof. □

Theorem 28. *Let $\{\widetilde{\eta}_i\}_{i \in I}$ be a family of interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-ideals (resp. KU-subalgebras) of a KU-algebra X. Then $\bigcap_{i \in I} \widetilde{\eta}_i$ is an interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-ideal (resp. KU-subalgebra) of X.*

Proof. Let $\{\widetilde{\eta}_i\}_{i \in I}$ be a family of interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-ideals of a KU-algebra X. Then for any $x, y, z \in X$,

$$\begin{aligned}
\text{rmax}\left\{\left(\bigcap_{i \in I} \widetilde{\eta}_i\right)(0), \widetilde{\theta}\right\} &= \text{rmax}\left\{\left(\bigwedge_{i \in I} \widetilde{\eta}_i\right)(0), \widetilde{\theta}\right\} \\
&= \text{rmax}\left\{\bigwedge_{i \in I} \left(\widetilde{\eta}_i(0)\right), \widetilde{\theta}\right\} \\
&\succeq \text{rmin}\left\{\bigwedge_{i \in I} \left(\widetilde{\eta}_i(x)\right), \widetilde{\delta}\right\} \\
&= \text{rmin}\left\{\left(\bigwedge_{i \in I} \widetilde{\eta}_i\right)(x), \widetilde{\delta}\right\} \\
&= \text{rmin}\left\{\left(\bigcap_{i \in I} \widetilde{\eta}_i\right)(x), \widetilde{\delta}\right\},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{rmax}\left\{\left(\bigcap_{i\in I}\widetilde{\eta}_i\right)(x*z),\widetilde{\theta}\right\} &= \mathrm{rmax}\left\{\left(\bigwedge_{i\in I}\widetilde{\eta}_i\right)(x*z),\widetilde{\theta}\right\}\\
&= \mathrm{rmax}\left\{\bigwedge_{i\in I}\left(\widetilde{\eta}_i(x*z)\right),\widetilde{\theta}\right\}\\
&\succeq \mathrm{rmin}\left\{\bigwedge_{i\in I}\left\{\widetilde{\eta}_i(x*(y*z)),\widetilde{\eta}_i(y)\right\},\widetilde{\delta}\right\}\\
&= \mathrm{rmin}\left\{\begin{array}{c}\left(\bigwedge_{i\in I}\widetilde{\eta}_i\right)(x*(y*z)),\\ \left(\bigwedge_{i\in I}\widetilde{\eta}_i\right)(y),\widetilde{\delta}\end{array}\right\}\\
&= \mathrm{rmin}\left\{\begin{array}{c}\left(\bigcap_{i\in I}\widetilde{\eta}_i\right)(x*(y*z)),\\ \left(\bigcap_{i\in I}\widetilde{\eta}_i\right)(y),\widetilde{\delta}\end{array}\right\}.
\end{aligned}$$

Hence this is required. The other case can be proved in a similar way. □

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