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SOME PROPERTIES OF WEAK-$\bigoplus$-SUPPLEMENTED MODULE

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Abstract: In this paper we give explicit necessary and sufficient conditions for weak-$\bigoplus$-supplemented module. If M is lifting module then it is weak-$\bigoplus$-supplemented module. Moreover if we have an R-module M such that is generalized lifting module then M is weak-$\bigoplus$-supplemented module. We prove that if M is supplemented and projective then M is weak-$\bigoplus$-supplemented.

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1. Introduction and Preliminaries

Throughout, all rings are associative rings with identity, and all modules are unital left modules. The symbol $\subseteq$ denotes containment and $\subset$ proper containment for sets. If N is a submodule (respectively proper submodule) of M we write $N \leq M$ (respectively N less than M). Let N be submodule of M then N is small in M ($N \ll M$) if there is no proper submodule L of M such that $N+L=M$. Now N is called supplement of L in M, if $N+L=M$ and N minimal

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with respect to this property, therefore a module M is called supplemented if every submodule of M has a supplement which is a direct summand. A module M is called $\bigoplus$-supplemented if for any submodule N of M, there exists a direct summand K of M such that $M{=}N{+}K$ and $N{\cap}K{\ll}K$, namely every submodule N of M has a direct summand supplement in M. A module M is called lifting module (or satisfies (D_1)) if for every submodule A of M, there exists a direct summand K of M such that $K{\leq}A$ and $(A/K){\ll}(M/K)$. Clearly, every lifting module is $\bigoplus$-supplemented, but the converse is not true (see,[7]). A module M is called amply supplemented if B contains a supplement of A whenever $M{=}A{+}B$. Recall that a left R-module M is said to be semisimple if it is the direct sum of simple submodules and hence any module M is called a weak lifting module provided, for each semisimple submodule N of M, there exists a direct summand K of M such that $K{\leq}N$ and $(N/K){\ll}(M/K)$, or there exists a decomposition $M{=}M_1{\bigoplus}M_2$, such that $M_1{\leq}N$ and $M{\cap}N{\ll}M_2$.

In this article we generalize $\bigoplus$-supplemented module in order to we obtain weak-$\bigoplus$-supplemented module. We use weak lifting, $\bigoplus$-supplemented and strongly $\bigoplus$-supplemented modules to study the generalization of $\bigoplus$-supplemented module and the relation between them.

2. Main Results

Let M be an R-module, then M is called weak-$\bigoplus$-supplemented module if for each semisimple submodule N of M there exists a direct summand K of M such that $M{=}N{+}K$ and $(N{\cap}K){\ll}K$. Since there is a relation between weak lifting and weak-$\bigoplus$-supplemented module therefore we can depend on a concept lifting module in order to get some properties of weak-$\bigoplus$-supplemented module. Any lifting module is amply supplemented and every amply supplemented module is weak lifting but the converse is not true, for example the ring of integers Z is weak lifting but not amply supplemented, also lifting module is weak lifting and so weak-$\bigoplus$-supplemented. Every submodule of a module M lies above a direct summand of M. Let M be an R-module. If M is lifting then M is $\bigoplus$-supplemented and so it is weak-$\bigoplus$-supplemented.

Lemma 2.1 (5, Proposition 2.3). *.If M is weak lifting then M is weak-$\bigoplus$-supplemented module.*

Theorem 2.2. *Let M be an R-module. If M is lifting then M is weak-$\bigoplus$-supplemented module.*

Proof. Since M is a lifting module then M is amply supplemented and any supplement submodule of M is direct summand of M. Therefore if for any two submodules L and K of $M \ni L+K=M$ and K contains a supplement of L in M. Since every submodule of M lies above a direct summand of M then M is (D_1)-module therefore by [4] M is a lifting module and hence M is a weak lifting. Thus M is weak-$\bigoplus$-supplemented module. □

Theorem 2.3. *Let M be an R-module. If every amply supplemented module with every submodule of a module M lies above a direct summand, then M is weak-$\bigoplus$-supplemented module.*

Proof. Suppose M is amply supplemented module, then there exists N_1 and N_2 are submodules of M with N_1+N_2=M, $N_2 \supseteq A_i$ such that A_i are supplemented of N_1. Since every submodule of M lies above a direct summand of M, then M is (D_1)- module and this means if for every submodule A of M, there exists a direct summand K of M such that $N \leq A$ and $(A/N) \ll (M/N)$, and so M is lifting module. Hence by [Theorem 2.2] M is weak-$\bigoplus$-supplemented module. □

Lemma 2.4 (10,Theorem 3.6). *. Let M be a Noetherian R-module. If M is finitely lifting, then M is lifting.*

Theorem 2.5. *Let M be an R-module. If M satisfying the following conditions:*

1. *M is Noetherian R-module,*
2. *M is finitely lifting module.*

Then M is weak-$\bigoplus$- supplemented module.

Proof. Since M is Noetherian module then every submodule N of M is finitely generated and M is finitely lifting then it is lifting module and by [Theorem 2.2] M is weak-$\bigoplus$-supplemented module. □

Proposition 2.6. *Let M be an R-module. If M satisfying the following conditions:*

1. *M is hollow-lifting module,*
2. *M has finite hollow dimension,*
3. *M is amply supplemented module.*

Then M is weak-$\bigoplus$-supplemented module.

Proof. Let M be an R-module. Suppose that M satisfying hollow-lifting conditions. If N coclosed submodule of M, then (M/N) has finite hollow dimension, therefore we must prove that N is a direct summand of M. Now we use induction on hollow dimension of (M/N). Let hollow dimension of (M/N) is n. If n=1, this means N is a direct summand of M because M is hollow-lifting. Suppose that hollow dimension of (M/N) is n_1 and for every coclosed submodule F of M such that (M/N) has hollow dimension less then n_1, F is a direct summand of M. Let (G/N) be coclosed in (M/N) such that $(M/N)/(G/N)$ is hollow. By [1], G is coclosed in M. Hence M=G+G_1 for some submodule G_1 of M as M is hollow-lifting. Then n_1=$G\cap(N\bigoplus G_1)$ and (M/N)=$(G/N)\bigoplus((N\bigoplus G_1)/N)$. Thus $(N\bigoplus G_1)/N$ is coclosed in (M/N). Again by [1], $(N\bigoplus G_1)$ is coclosed in M. By induction,$(N\bigoplus G_1)$ is a direct summand of M and so N is a direct summand of M. Then M is lifting. So is weak-$\bigoplus$-supplemented. □

Theorem 2.7. *Let $\bigoplus M_i$ (i=1,...,n) be a finite direct sum of M such that is weak-$\bigoplus$-supplemented modules then $\bigoplus M_i$, also weak-$\bigoplus$-supplemented.*

Proof. Let M =$M_1\bigoplus...\bigoplus M_n$. For i=1,...,n. Let p_i:$M\rightarrow M_i$ be the projection map and let L be a semisimple submodule of M. We have 0 is a supplement of $(L+M_1)+M_2+...+M^n$ in M then $(L+M_1)\cap M_i$, i=2,...,n has a direct summand supplement N in M_2, M_3,...,M_n because $(L+M_1)\cap M_i=p_2(L)$ is semisimple. Now by [2], N is a supplement of $(L+M_1)$ in M. Since

$$(L + N) \cap M_1 \cap (L + M_i) \cap M_1 = p_1(L)$$

is semisimple such that i=2,...,n implies $(L+N)\cap M_1$ has a direct summand supplement K in M_1. Again by [2], $(N+K)$ is a supplement of L in M implies $(N\bigoplus K)$ is a direct summand of M. Hence M is weak-$\bigoplus$-supplemented. □

A module M is called a generalized lifting module if the following condition satisfied:

(If M=$M_1\bigoplus M_2$ and $A\leq M$, then there exist $C_i\leq^{\oplus}M_i$ (i=1,2) such that $C_1\bigoplus C_2$ is a supplement of A in M).

Let M be an R-module, then any lifting module is a generalization lifting module and so is $\bigoplus$-supplemented. Also every generalization lifting module is $\bigoplus$-Supplemented module and the converse is true if we put some conditions on submodules of M. See the following theorem:

Theorem 2.8. *If M is a $\bigoplus$-supplemented, and satisfies in this condition that for every two direct summands N_1 and N_2 of M such that $(N_1 \cap N_2)$ is coclosed in M, implies that $(N_1 \cap N_2)$ is a direct summand of M. Then M is a generalization lifting module.*

Recall that lifting module is generalization lifting module, and so $\bigoplus$-supplemented but the GL-module is not lifting module. See the following example:

Example 2.9. Let p be any prime integer. Z-Module $(Z/pZ)\bigoplus(Z/p^3Z)$ is generalization lifting module but not lifting module [8].

Theorem 2.10. *Let M be an R-module. If M is generalization lifting module then it is weak-$\bigoplus$-supplemented module*

A module M is called a strongly $\bigoplus$-supplemented module if every supplement submodule of M is a direct summand of M. All strongly $\bigoplus$-supplemented modules are $\bigoplus$-supplemented. And so every $\bigoplus$-supplemented is weak-$\bigoplus$-supplemented module, but the converse is not true in general. See the following example:

Example 2.11. Let R be a local Artinian ring with radical W such that W^2=0, Q=R=W is commutative, dim(QW)=2 and dim(WQ)=1. Then the indecomposable injective right R-module U=[($R\bigoplus R$)/D]R with D=(ur,-vr)—$r \in R$ in [2] is a weak-$\bigoplus$-supplemented module, but is not $\bigoplus$-supplemented.

Given a right R-module M, the socle of M is defined as the sum of all the simple submodules of M_R. Now we introduce another example to show M is weak-$\bigoplus$-supplemented module. See the following example:

Example 2.12. Let N be a nonzero semisimple submodule of M. Then N=Soc(M). Since Soc(M) is simple and $N \ll M$ then Soc(M)$\ll M$. Hence M is weak-$\bigoplus$-supplemented module.

Theorem 2.13. *Let M be an R-module. If M supplemented module and every supplement submodule of M lies above a direct summand then M is weak-$\bigoplus$-supplemented module.*

Proof. Suppose V be supplement submodule of M and supplement of U in M. But we have every supplement submodule of M lies above a direct summand then there exist M_1 submodule of M and M_2 submodule of M such that M=$M_1 \bigoplus M_2$, M_1 submodule of V and $(V \cap M_2)$ small in M_2. Therefore V=$V \cap M$=$M_1 \bigoplus$V$\cap M_2$ and since $V \cap M_2$ small in M, then

$$M = U + V = U + V \cap M_2 + M_1 = U + M_1.$$

Also, since V is a supplement of U and V=M_1. Thus M=$V\bigoplus M_2$ and V is a direct summand of M. Hence M is strongly $\bigoplus$-supplemented. But every strongly $\bigoplus$-supplemented module is $\bigoplus$-supplemented and then M is weak-$\bigoplus$-supplemented module. □

Proposition 2.14. *Let M be an R-module. If M is supplemented and projective then M is weak-$\bigoplus$-supplemented.*

Proof. Suppose M is projective R-module such that M=U+V. We must prove that the β:$U\bigoplus V\rightarrow M$ is splits where β is epimorphism. Since M projective module then there exists a mapping δ:$M\bigoplus U\rightarrow V$ and then βoδ=1_M is the identity mapping implies β:$U\bigoplus V\rightarrow M$ is splits. Hence β splits and so π-projective. Now let M=N+K and A be a supplement of N in M. Also, let $g\in$End(M)$\ni$Img(g)$\subseteq K$ and Img(1-g)$\subseteq N$, since we have $g(N)\subseteq N$, M=N+$g(B)$ and $g(N\cap B)$=$N\cap g(B)$ implies b-n =(1-g)(b)$\in N$). Since

$$N \cap B \ll A, \quad N \cap g(A)) \ll g(A),$$

and then $g(A)$ is a supplement of N $\ni$ $g(A)\subseteq K$. Hence M is amply supplemented module. Let N be a semisimple submodule of M, there exists a submodule K of M $\ni$ M=N+K and $N\cap B\ll K$. Now there exists a submodule T of $M\ni M$=T+K, $T\cap K\ll N$. Since T is semisimple ($T\cap K$)=0 and hence M=$T\bigoplus K$. Then M is weak lifting [5], but a weak lifting is weak-$\bigoplus$-supplemented module. □

Proposition 2.15. *For a prufer ring, any finitely generated torsion free supplemented R-module is weak-$\bigoplus$-supplemented module.*

Proof. Suppose R is a Prufer ring, then every finitely generated torsion free R-module is projective (See [6]). Since every projective module is π-projective, then every finitely generated torsion free supplemented R-module is strongly $\bigoplus$-supplemented and this implies M is $\bigoplus$-supplemented. Hence M is weak-$\bigoplus$-supplemented module. □

Corollary 2.16. *Every (D_1) module is weak-$\bigoplus$-supplemented module.*

Corollary 2.17. *Every amply supplemented module is weak-$\bigoplus$- supplemented.*

Corollary 2.18. *Every strongly-$\bigoplus$-supplemented module is a weak-$\bigoplus$-supplemented.*

The following implications are now clear for a module M:

Lifting module $\Rightarrow$ Generalization lifting module $\Rightarrow$ $\bigoplus$-Supplemented module

$\Downarrow$

Strongly $\bigoplus$-supplemented $\Rightarrow$ Weak-$\bigoplus$-Supplemented module.

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