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ON Q-FUZZY IDEALS IN ORDERED SEMIGROUPS

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Abstract: In this paper we shows that in ordered groupoids the Q-fuzzy right (resp. Q-fuzzy left) ideals are Q-fuzzy quasi-ideals, in ordered semigroups the Q-fuzzy quasi-ideals are Q-fuzzy bi-ideals, and in regular ordered semigroups the Q-fuzzy quasi-ideals and the Q-fuzzy bi-ideals coincide and show that if S is an ordered semigroup, then a Q-fuzzy subset f is a Q-fuzzy quasi-ideal of S if and only if there exist a Q-fuzzy right ideal g and a Q-fuzzy left ideal h of S such that $f = g \cap h$.

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1. Introduction

A fuzzy set theory was conceptualized by Professor L. A. Zadeh at the University of California in 1965, [14] as a generalization of abstract set theory. Zadehs initiation is virtually a complete paradigm shift that initially gained popularity in the Far East and its successful applications has gained further ground almost round the globe. Rosenfeld [11] used the ideal of fuzzy set to introduce the no-

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tions of fuzzy subgroups. The ideal of fuzzy subsemigroup was also introduced by Kuroki [7], [9]. In [8], Kuroki characterized several classes of semigroups in terms of fuzzy left, fuzzy right and fuzzy bi-ideals. Xie [12] introduced the idea of extensions of fuzzy ideals in semigroups. The concept of fuzzy generalized bi-ideals of an ordered semigroup is introduced by Xie and Tang [13] and characteried regular ordered semigroups by means of fuzzy left ideals, fuzzy right ideals and fuzzy (generalized) bi-ideals. In [10], Majumder introduced the concept of Q-fuzzification of ideals of Γ-semigroups and some important properties have been investigated. In this paper we shows that in ordered groupoids the Q-fuzzy right (resp. Q-fuzzy left) ideals are Q-fuzzy quasi-ideals, in ordered semigroups the Q-fuzzy quasi-ideals are Q-fuzzy bi-ideals, and in regular ordered semigroups the Q-fuzzy quasi-ideals and the Q-fuzzy bi-ideals coincide and show that if S is an ordered semigroup, then a Q-fuzzy subset f is a Q-fuzzy quasi-ideal of S if and only if there exist a Q-fuzzy right ideal g and a Q-fuzzy left ideal h of S such that $f = g \cap h$.

2. Preliminaries

Throughout this paper, unless stated otherwise, S stands for an ordered semigroup. A function f from $S \times Q$ to the real closed interval $[0, 1]$ is called *Q-fuzzy subset* of S, where Q is a non-empty set. The ordered semigroup S itself is a Q-fuzzy subset of S, its characteristic function, also denoted by S, is defined as follows:

$$S : S \times Q \longrightarrow [0, 1] \mid (x, q) \mapsto S(x, q) := 1,$$

for all $x \in S$ and $q \in Q$.

Let f and g be two Q-fuzzy subsets of S. Then the inclusion relation $f \subseteq g$ means that

$$f(x, q) \leq g(x, q),$$

for all $x \in S$ and $q \in Q$, $f \cap g$ and $f \cup g$ are defined by

$$(f \cap g)(x, q) := \min\{f(x, q), g(x, q)\},$$

$$(f \cup g)(x, q) := \max\{f(x, q), g(x, q)\},$$

for all $x \in S$ and $q \in Q$.

Let S and Q be an ordered semigroup and a non-empty set, respectively. For $x \in S$, we define $A_x := \{(y, z) \in S \times S \mid x \leq yz\}$. The product $f \circ g$ of f

and g is defined by

$$(\forall x \in S, \forall q \in Q)(f \circ g)(x,q) := \begin{cases} \bigvee\limits_{(y,z)\in A_x} \min\{f(y,q), g(z,q)\} & \text{if } A_x \neq \emptyset, \\ 0 & \text{if } A_x = \emptyset, \end{cases}$$

We denote by $f_{A\times Q}$ the characteristic function of $A \times Q$, that is, the mapping of $S \times Q$ into $[0,1]$ defined by

$$f_{A\times Q}(x,q) := \begin{cases} 1 & \text{if } x \in A \times Q, \\ 0 & \text{if } x \notin A \times Q, \end{cases}$$

for all $(x,q) \in A \times Q$.

3. Main Results

In this section, we introduced the notion of Q-fuzzy right (resp. Q-fuzzy left) ideals, Q-fuzzy quasi-ideals, Q-fuzzy bi-ideals of ordered semigroups, and investigate related properties.

Definition 3.1. Let S and Q be an ordered groupoid and a non-empty set, respectively. A Q-fuzzy subset f of S is called a Q-fuzzy right ideal (resp. Q-fuzzy left) ideal of S if:

(1) $x \leq y$ implies $f(x,q) \geq f(y,q)$, and

(2) $f(xy,q) \geq f(x,q)$(resp. $f(xy,q) \geq f(y,q)$),

for all $x, y \in S$ and for all $q \in Q$.

Definition 3.2. Let S and Q be an ordered groupoid and a non-empty set, respectively. A Q-fuzzy subset f of S is called a Q-fuzzy quasi-ideal of S if:

(1) $x \leq y \Rightarrow f(x,q) \geq f(y,q)$,

(2) $(f \circ S) \cap (S \circ f) \subseteq f$,

for all $x, y \in S$ and for all $q \in Q$.

Definition 3.3. Let S and Q be an ordered semigroup and a non-empty set, respectively. A Q-fuzzy subsemigroup f of S is called a Q-fuzzy bi-ideal of S if:

(1) $x \leq y \Rightarrow f(x,q) \geq f(y,q)$,

(2) $f(xyz,q) \geq \min\{f(x,q), f(z,q)\}$,

for all $x, y, z \in S$ and for all $q \in Q$.

Theorem 3.4. *If S is an ordered groupoid and Q is a non-empty set, then the Q-fuzzy right (resp. left) ideals of S are Q-fuzzy quasi-ideals of S.*

Proof. Let f be a Q-fuzzy right ideal of S and $x \in S, q \in Q$. First of all,

$$((f \circ S) \cap (S \circ f))(x, q) = \min\{(f \circ S)(x, q), (S \circ f)(x, q)\}.$$

If $A_x = \emptyset$, then we have $(f \circ S)(x, q) = 0 = (S \circ f)(x, q)$ and, since f is a Q-fuzzy right ideal of S, we have $\min\{(f \circ S)(x, q), (S \circ f)(x, q)\} = 0 \leq f(x, q)$.

If $A_x \neq \emptyset$, then

$$(f \circ S)(x, q) = \bigvee_{(u,v) \in A_x} \{\min\{f(u, q), S(v, q)\}\}.$$

On the other hand, if $(u, v) \in A_x$, then $x \leq uv$ and $f(x, q) \geq f(uv, q) \geq f(u, q) = \min\{f(u, q), S(v, q)\}$. Hence, we have

$$\begin{aligned} f(x, q) \geq & \bigvee_{(u,v) \in A_x} \{\min\{f(u, q)\}\} \\ \geq & \min\{(f \circ S)(x, q), (S \circ f)(x, q)\} \\ = & ((f \circ S) \cap (S \circ f))(x, q). \end{aligned}$$

Therefore f is a Q-fuzzy quasi-ideal of S. □

Theorem 3.5. *If S is an ordered semigroup and Q is a non-empty set, then the Q-fuzzy quasi-ideals are Q-fuzzy bi-ideals of S.*

Proof. Let f be a Q-fuzzy quasi-ideal of S and $x, y, z \in S, q \in Q$. Then we have

$$f(xyz, q) \geq ((f \circ S) \cap (S \circ f))(xyz, q) = \min\{(f \circ S)(xyz, q), (S \circ f)(xyz, q)\}.$$

Since $(x, yz) \in A_{xyz}$, we have

$$\begin{aligned} (f \circ S)(xyz, q) = & \bigvee_{(u,v) \in A_{xyz}} \{\min\{f(u, q), S(v, q)\}\} \\ \geq & \min\{f(x, q), S(yz, q)\} \\ = & f(x, q). \end{aligned}$$

Since $(xy, z) \in A_{xyz}$, we have

$$(S \circ f)(xyz, q) = \bigvee_{(u,v) \in A_{xyz}} \{\min\{S(u, q), f(v, q)\}\}$$

$$\begin{aligned}&\geq \min\{S(xy,q), f(z,q)\}\\&= f(z,q).\end{aligned}$$

Thus we have

$$f(xyz,q) \geq \min\{(f\circ S)(xyz,q),(S\circ f)(xyz,q)\} \geq \min\{f(x,q), f(z,q)\}.$$

Hence f is a Q-fuzzy bi-ideal of S. □

An ordered semigroup S is called regular if for any $a \in S$ there exists $x \in S$ such that $a \leq axa$.

Theorem 3.6. *If S is a regular ordered semigroup and Q is a non-empty set, then the Q-fuzzy quasi-ideals and the Q-fuzzy bi-ideals coincide.*

Proof. Let f be a Q-fuzzy bi-ideal of S and $x \in S, q \in Q$. We will prove that

$$((f\circ S)\cap(S\circ f))(x,q) \leq f(x,q). \tag{1}$$

First of all, we have

$$((f\circ S)\cap(S\circ f))(x,q) = \min\{(f\circ S)(x,q),(S\circ f)(x,q)\}.$$

If $A_x = \emptyset$, then as we have already seen in Theorem 3.4, condition (1) is satisfied.

If $A_x \neq \emptyset$, then

$$(f\circ S)(x,q) = \bigvee_{(z,w)\in A_x}\{\min\{f(z,q), S(w,q)\}\}, \tag{2}$$

$$(S\circ f)(x,q) = \bigvee_{(u,v)\in A_x}\{\min\{S(u,q), f(v,q)\}\}. \tag{3}$$

Let $(f\circ S)(x,q) \leq f(x,q)$. Then, we have

$$\begin{aligned}f(x,q) &\geq (f\circ S)(x,q)\\&\geq \min\{(f\circ S)(x,q),(S\circ f)(x,q)\}\\&= ((f\circ S)\cap(S\circ f))(x,q),\end{aligned}$$

and condition (1) is satisfied.

Let $(f\circ S)(x,q) > f(x,q)$. Then, by (2), there exists $(z,w) \in A_x$ such that

$$\min\{f(z,q), S(w,q)\} > f(x,q) \tag{4}$$

(otherwise $f(x,q) \leq (f \circ S)(x,q)$, which is impossible). Since $(z,w) \in A_x$, we have $z, w \in S$ and $x \leq zw$. Similarly, from $\min\{f(z,q), S(w,q)\} = f(z,q)$, by (4), we obtain

$$f(z,q) > f(x,q). \tag{5}$$

We will prove that $(S \circ f)(x,q) \leq f(x,q)$, then

$$\min\{(f \circ S)(x,q), (S \circ f)(x,q)\} \leq (S \circ f)(x,q) \leq f(x,q),$$

so that $((f \circ S) \cap (S \circ f))(x,q) \leq f(x,q)$, and condition (1) is satisfied.

By (3), it is enough to prove that

$$\min\{S(u,q), f(v,q)\} \leq f(x,q), \forall (u,v) \in A_x.$$

Let $(u,v) \in A_x$. Then $x \leq uv$ for some $u, v \in S$. Since S is regular, there exists $s \in S$ such that $x \leq xsx$. It follows that $x \leq zwsuv$. Since f is a Q-fuzzy bi-ideal of S, we have

$$\min\{S(u,q), f(v,q)\} \leq f(x,q), \forall (u,v) \in A_x,$$

and, we have

$$f(x,q) \geq f(zwsuv,q) \geq \min\{f(z,q), f(v,q)\}.$$

If $\min\{f(z,q), f(v,q)\} = f(z,q)$, then $f(z,q) \leq f(x,q)$ which is impossible by (5). Thus we have $\min\{f(z,q), f(v,q)\} = f(v,q)$, then $f(x,q) \geq f(v,q) = \min\{S(u,q), f(v,q)\}$. □

In the following, using the usual definitions of ideals mentioned above, we show that the Q-fuzzy quasi-ideals of an ordered semigroup are just intersections of Q-fuzzy right and Q-fuzzy left ideals.

Lemma 3.7. *Let S and Q be an ordered semigroup and a non-empty set respectively. Let f be a Q-fuzzy subset of S. Then we have the following:*
(1) $(S \circ f)(xy,q) \geq f(y,q)$ for all $x, y \in S, q \in Q$,
(2) $(S \circ f)(xy,q) \geq (S \circ f)(y,q)$ for all $x, y \in S, q \in Q$.

Proof. (1) Let $x, y \in S$ and $q \in Q$. Since $(x,y) \in A_{xy}$, we have

$$(S \circ f)(xy,q) = \bigvee_{(w,z) \in A_{xy}} \{\min\{S(w,q), f(z,q)\}\} \geq \min\{S(x,q), f(y,q)\} = f(y,q).$$

(2) Let $x, y \in S$ and $q \in Q$. If $A_y = \emptyset$, then $(S \circ f)(y, q) = 0$. Since $(S \circ f)$ is a Q-fuzzy subset of S, we have $(S \circ f)(xy, q) \geq 0 = (S \circ f)(y, q)$. If $A_x \neq \emptyset$, then

$$(S \circ f)(y, q) = \bigvee_{(w,z) \in A_y} \{\min\{S(w, q), f(z, q)\}\}.$$

On the other hand,

$$(S \circ f)(xy, q) \geq \min\{S(w, q), f(z, q)\}, \forall (w, z) \in A_y. \tag{6}$$

Indeed, let $(w, z) \in A_y$. Since $(x, y) \in A_{xy}$, we have

$$(S \circ f)(xy, q) = \bigvee_{(s,t) \in A_{xy}} \{\min\{S(s, q), f(t, q)\}\}.$$

Since $(w, z) \in A_y$, we have $y \leq wz$, then $xy \leq xwz$, and $(xw, z) \in A_{xy}$. Hence we have

$$(S \circ f)(xy, q) \geq \min\{S(xw, q), f(z, q)\} = f(z, q) = \min\{S(w, q), f(z, q)\}.$$

By (6), we have

$$(S \circ f)(xy, q) \geq \bigvee_{(w,z) \in A_y} \{\min\{S(w, q), f(z, q)\}\} = (S \circ f)(y, q). \quad \square$$

In a similar way we can prove the following lemmas:

Lemma 3.8. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Let f be a Q-fuzzy subset of S. Then we have the following:*
(1) $(S \circ f)(xy, q) \geq f(x, q)$ for all $x, y \in S, q \in Q$,
(2) $(S \circ f)(xy, q) \geq (S \circ f)(x, q)$ for all $x, y \in S, q \in Q$.

Lemma 3.9. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Let f be a Q-fuzzy subset of S and $x \leq y$. Then we have $(S \circ f)(x, q) \geq (S \circ f)(y, q)$, for all $q \in Q$.*

Proof. Let $x, y \in S$ and $q \in Q$. Then, if $A_y = \emptyset$, then $(S \circ f)(y, q) = 0$. Since $S \circ f$ is a Q-fuzzy subset of S, we have $(S \circ f)(x, q) \geq 0$, then $(S \circ f)(x, q) \geq (S \circ f)(y, q)$. If $A_y \neq \emptyset$, then

$$(S \circ f)(y, q) = \bigvee_{(w,z) \in A_y} \{\min\{S(w, q), f(z, q)\}\} = \bigvee_{(w,z) \in A_y} \{f(z, q)\}.$$

On the other hand,

$$(S \circ f)(x,q) \geq f(z,q), \forall (w,z) \in A_y. \tag{7}$$

Indeed, let $(w,z) \in A_y$. Since $x \leq y \leq wz$, we have $(w,z) \in A_x$. Then

$$(S \circ f)(xy,q) = \bigvee_{(s,t)\in A_{xy}} \{\min\{S(s,q), f(t,q)\}\} \geq \min\{S(w,q), f(z,q)\} = f(z,q).$$

Thus, by (7), we have

$$(S \circ f)(x,q) \geq \bigvee_{(w,z)\in A_y} \{f(z,q)\} = (S \circ f)(y,q).$$

The proof is completed. □

Lemma 3.10. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Let f be a Q-fuzzy subset of S and $x \leq y$. Then we have $(f \circ S)(x,q) \geq (f \circ S)(y,q)$, for all $q \in Q$.*

Lemma 3.11. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Let f be a Q-fuzzy subset of S such that $x \leq y$, we have $f(x,q) \geq f(y,q)$ for all $x, y \in S, q \in Q$. Then the Q-fuzzy subset $f \cup (S \circ f)$ is a Q-fuzzy left ideal of S.*

Proof. Let $x, y \in S$ and $q \in Q$. By, Theorem 3.9, we have $(f \cup (S \circ f))(xy,q) \geq (f \cup (S \circ f))(y,q)$. Let now $x \leq y$. Then $(f \cup (S \circ f))(x,q) \geq (f \cup (S \circ f))(y,q)$. Indeed: Since f is a Q-fuzzy subset of S and $x \leq y$, by Lemma 3.7, we get $(S \circ f)(x,q) \geq (S \circ f)(y,q)$ and, by hypothesis, $f(x,q) \geq f(y,q)$. Then

$$\begin{aligned}(f \cup (S \circ f))(x,q) &= \max\{f(x,q), (S \circ f)(x,q)\} \\ &\geq \max\{f(y,q), (S \circ f)(y,q)\} \\ &= (f \cup (S \circ f))(y,q).\end{aligned}$$
□

In a similar way we can prove the following:

Lemma 3.12. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Let f be a Q-fuzzy subset of S such that $x \leq y$, we have $f(x,q) \geq f(y,q)$ for all $x, y \in S, q \in Q$. Then the Q-fuzzy subset $f \cup (f \circ S)$ is a Q-fuzzy right ideal of S.*

Lemma 3.13. *Let S and f, g, h be an ordered semigroup and Q-fuzzy subsets of S, respectively. Then*

$$f \cap (g \cup h) = (f \cap g) \cup (f \cap h).$$

Proof. Let $x \in S$ and $q \in Q$. Then we have

$$\begin{aligned}(f \cap (g \cup h))(x,q) =& \min\{f(x,q), (g \cup h)(x,q)\} \\ =& \min\{f(x,q), \max\{g(x,q), h(x,q)\}\} \\ =& \max\{\min\{f(x,q), g(x,q)\}, \min\{f(x,q), h(x,q)\} \\ =& \max\{(f \cap g)(x,q), (f \cap h)(x,q)\} \\ =& ((f \cap g) \cup (f \cap h))(x,q).\end{aligned}$$ □

Corollary 3.14. *. Let S and Q be an ordered semigroup. Then the set of all Q-fuzzy subsets of S is a distributive lattice.*

Theorem 3.15. *Let S and Q be an ordered semigroup and a non-empty set, respectively. Then a Q-fuzzy subset f of S is a Q-fuzzy quasi-ideal of S if and only if there exist a Q-fuzzy right ideal g and a Q-fuzzy left ideal h of S such that $f = g \cap h$.*

Proof. ($\Rightarrow$). By Lemma 3.11 and Lemma 3.12, $f \cup (S \circ f)$ is a Q-fuzzy left ideal and $f \cup (f \circ S)$ is a Q-fuzzy right ideal of S. Moreover, we have

$$f = (f \cup (S \circ f)) \cap (f \cup (f \circ S)).$$

In fact, by Corolary 3.14, we have

$$\begin{aligned}(f \cup (S \circ f)) \cap (f \cup (f \circ S)) =& ((f \cup (S \circ f)) \cap f) \cup ((f \cup (S \circ f)) \cap (f \circ S)) \\ =& (f \cap f) \cup ((S \circ f) \cap f) \cup (f \cap (f \circ S)) \cup ((S \circ f) \\ & \cap (f \circ S)) \\ =& f \cup ((S \circ f) \cap f) \cup (f \cap (f \circ S)) \cup ((S \circ f) \\ & \cap (f \circ S)).\end{aligned}$$

Since f is a Q-fuzzy quasi-ideal of S, we have $(f \circ S) \cap (S \circ f) \subseteq f$. Besides, $(S \circ f) \cap f \subseteq f$ and $f \cap (f \circ S) \subseteq f$. Hence

$$(f \cup (S \circ f)) \cap (f \cup (f \circ S)) = f.$$

($\Leftarrow$). Let $x \in S$ and $q \in Q$. Then

$$((f \circ S) \cap (S \circ f))(x,q) \leq f(x,q) \tag{8}$$

In fact, $((f\circ S)\cap(S\circ f))(x,q)=\min\{(f\circ S)(x,q),(S\circ f)(x,q)\}$. If $A_x=\emptyset$, then $(f\circ S)(x,q)=0=(S\circ f)(x,q)$. Thus, in this case condition (8) is satisfied. If $A_x\neq\emptyset$, then

$$(f\circ S)(x,q)=\bigvee_{(y,z)\in A_x}\{\min\{f(y,q),S(z,q)\}\}=\bigvee_{(y,z)\in A_x}\{f(y,q)\}. \quad (9)$$

We have

$$f(y,q)\leq h(x,q),\forall(y,z)\in A_x. \quad (10)$$

Indeed, for $(y,z)\in A_x$, we have $x\leq yz$ and $h(x,q)\geq h(yz,q)\geq h(y,q)$ because h is a Q-fuzzy left ideal of S. Thus, applying (10) to (9), we obtain

$$(f\circ S)(x,q)=\bigvee_{(y,z)\in A_x}\{f(y,q)\}\leq h(x,q).$$

In a similar way, we get $(S\circ f)(x,q)\leq g(x,q)$. Hence

$$\begin{aligned}((f\circ S)\cap(S\circ f))(x,q) &= \min\{(f\circ S)(x,q),(S\circ f)(x,q)\}\\ &\leq \min\{h(x,q),g(xq)\}\\ &= (h\cap g)(x,q)\\ &= f(x,q),\end{aligned}$$

which completes the proof of (8). □

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