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PSI-ORDINARY DICHOTOMY OF THE SOLUTIONS OF IMPULSIVE DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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Abstract: In the paper a dependence is established between the ψ-ordinary dichotomy of a homogeneous impulsive differential equation in a Banach space and the existence of ψ-bounded solution of the appropriate nonhomogeneous equation.

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1. Introduction

The mathematical theory of impulsive differential equations is much richer in problems in comparison with the corresponding theory of ordinary differential equations. That is why the impulsive differential equations are adequate apparatus for mathematical simulation of numeruous processes and phenomena which are studied in biology, phisics and technology.

The problem of ψ-boundedness and ψ-stability of the solutions of differential equations in finite dimensional Euclidean spaces has been studied by many authors, as e.q. Akinyele [1], Constantin [7]. In these papers, the function

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ψ is a scalar continuous function (and increasing, differentiable and bounded in [1], nondecreasing and such that $\psi(t) \geq 1$ on R_+ in [7]). In the papers of Diamandescu [9],[10],[11] and Boi [3],[4],[5] ψ is a nonnegative continuous diagonal matrix function.

Inspired by the famous monographs of Coppel [6], Daleckii and Krein [8] and Massera and Schaeffer [15], where the important notion of exponential and ordinary dichotomy is considered in details, Diamandescu [9]-[11] and Boi [3]-[5] introduced and studied the ψ-dichotomy for linear differential equations in finite dimensional Euclidean space. The concept of ψ-dichotomy for arbitrary Banach spaces is studied in [13] and [14]. In this case $\psi(t)$ is an arbitrary bounded invertible linear operator, instead of the restriction to be a nonnegative diagonal matrix.

The goal of the present paper is to etablish a dependence between the ψ-ordinary dichotomy of a homogeneous impulsive equation in a Banach space and the existence of a ψ-bounded on the semi-axis solution of the corresponding nonhomogeneous impulsive equation.

2. Preliminaries

Let X be an arbitrary Banach space with norm $|.|$ and let $LB(X)$ be the space of all linear bounded operators acting in X with the norm $||.||$ and identity I. Denote $R_+ = [0, \infty)$.

We consider the nonhomogenous impulsive equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + f(t) \quad (t \neq t_n) \tag{1}$$

$$x(t_n + 0) = Q_n x(t_n) + h_n \quad (n = 1, 2, 3, ...) \tag{2}$$

where the operator valued function $A(.) : R_+ \to LB(X)$ and the function $f(.) : R_+ \to X$ are strong measurable and Bochner integrable on the finite subintervals of R_+, $Q_n \in LB(X)$ $(n = 1, 2, 3, ...)$, $T = \{t_n\}_{n=1}^{\infty}$ is a sequence of points on the semi-axis R_+ satisfying the condition

$$0 < t_1 < t_2 < ..., \ \lim_{n\to\infty} t_n = \infty.$$

The corresponding homogenous impulsive equation is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x \quad (t \neq t_n) \tag{3}$$

$$x(t_n + 0) = Q_n x(t_n) \quad (t = 1, 2, 3, ...). \tag{4}$$

By a solution of the impulsive equation (1), (2) (or (3), (4)) we will understand a piecewise continuous function $x(t)$ with points of discontinuity of first kind $t_1, t_2, ...$ which satisfies (1) (or (3)) for $t \neq t_n$ and (2) (or (4)) otherwise.

Let $RL(X)$ be the subspace of all invertible operators in $LB(X)$ and let $\psi(.) : R_+ \to RL(X)$ be a continuous for any $t \in R_+$ operator-function.

Definition 1. *A function $u(.) : R_+ \to X$ is said to be ψ-bounded on R_+ if $\psi(t)u(t)$ is bounded on R_+.*

A function $f(.) : R_+ \to X$ is said to be ψ-Bochner integrable on R_+ if it is measurable and $\int\limits_0^\infty |\psi(\tau)f(\tau)|d\tau < \infty$.

Let $C_\psi(X, T)$ denote the space of all ψ-bounded on R_+ functions with values in X which are continuous for $t \neq t_n$, have discontinuities of the first kind for $t = t_n$ and are continuous from the left for $t = t_n$, which is a Banach space with the norm

$$|||f|||_{C_\psi} = \sup_{t \in R_+} |\psi(t)f(t)|.$$

Let $L_\psi(X)$ denote the Banach space of all ψ-Bochner integrable on R_+ functions with values in X with the norm

$$|||f|||_{L_\psi} = \int_0^\infty |\psi(s)f(s)|\mathrm{d}s.$$

Let H_ψ denote the set

$$H_\psi(X) = \{h = \{h_n\}_{n=1}^\infty : \sum_{j=1}^\infty |\psi(t_j + 0)h_j| < \infty\}$$

which is also a Banach space with respect to the norm

$$|||h|||_{H_\psi} = \sum_{j=1}^\infty |\psi(t_j + 0)h_j|.$$

Definition 2. *The homogenous impulsive equation (3), (4) is said to be ψ-ordinary dichotomous on R_+ if there exist a pair P_1 and $P_2 = I - P_1$ mutually complementary projections in X and a number $M > 0$ for which the inequalities*

$$||\psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)|| \leq M \qquad (0 \leq s \leq t < \infty), \tag{5}$$

$$||\psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)|| \leq M \qquad (0 \leq t \leq s < \infty) \tag{6}$$

hold , where $V(t) = V(t, 0)$ and $V(t, s)$ $(0 \leq s, t < \infty)$ is the Cauchy evolutionary operator [16, 12] of the impulsive equation (3), (4).

We shall say that condition (H) is satisfied if the following conditions hold:

H1) The operator-valued function $A(.) : R_+ \to LB(X)$ is continuous.

H2) $Q_n \in RL(X) \quad (n = 1, 2, ...)$.

3. Main Results

Theorem 1. *Let the following conditions hold:*

1. Condition (H) is satisfied.

2. Equation (3), (4) is ψ-ordinary dichotomous.

Then for any function $f \in L_\psi(X)$ and any sequence $h \in H_\psi(X)$ there exists a solution of the nonhomogeneous equation (1), (2) which is ψ-bounded on R_+.

Proof. Consider the function

$$\begin{aligned}
\tilde{x}(t) = & \int_0^t \psi(t)V(t)P_1V^{-1}(s)f(s)\mathrm{d}s - \int_t^\infty \psi(t)V(t)P_2V^{-1}(s)f(s)\mathrm{d}s + \\
& + \sum_{t_j<t} \psi(t)V(t)P_1V^{-1}(t_j+0)h_j - \sum_{t_j\geq t} \psi(t)V(t)P_2V^{-1}(t_j+0)h_j = \\
& = \int_0^t \psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)\psi(s)f(s)\mathrm{d}s - \\
& - \int_t^\infty \psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)\psi(s)f(s)\mathrm{d}s + \\
& + \sum_{t_j<t} \psi(t)V(t)P_1V^{-1}(t_j+0)\psi^{-1}(t_j+0)\psi(t_j+0)h_j - \\
& - \sum_{t_j\geq t} \psi(t)V(t)P_2V^{-1}(t_j+0)\psi^{-1}(t_j+0)\psi(t_j+0)h_j
\end{aligned}$$

We shall estimate the norm of $\tilde{x}(t)$:

$$
\begin{aligned}
|\tilde{x}(t)| \leq & \int_0^t ||\psi(t)V(t)P_1V^{-1}(s)\psi^{-1}(s)||\ |\psi(s)f(s)|\mathrm{d}s+ \\
& + \int_t^\infty ||\psi(t)V(t)P_2V^{-1}(s)\psi^{-1}(s)||\ |\psi(s)f(s)|\mathrm{d}s+ \\
& + \sum_{t_j<t} ||\psi(t)V(t)P_1V^{-1}(t_j+0)\psi^{-1}(t_j+0)||\ |\psi(t_j+0)h_j|+ \\
& + \sum_{t_j\geq t} ||\psi(t)V(t)P_2V^{-1}(t_j+0)\psi^{-1}(t_j+0)||\ |\psi(t_j+0)h_j| \leq \\
& \leq M\int_0^t |\psi(s)f(s)|\mathrm{d}s + M\int_t^\infty |\psi(s)f(s)|\mathrm{d}s+ \\
& + M\sum_{t_j<t} |\psi(t_j+0)h_j| + M\sum_{t_j\geq t} |\psi(t_j+0)h_j| = \\
& = M\int_0^\infty |\psi(s)f(s)|\mathrm{d}s + M\sum_{j=1}^\infty |\psi(t_j+0)h_j| < \infty
\end{aligned}
\tag{7}
$$

Hence $\tilde{x}(t)$ is bounded on R_+.

Let $x(t) = \psi^{-1}(t)\tilde{x}(t)$. Obviously $x(t)$ is ψ-bounded on R_+. It is immediately verified that the function $x(t)$ is continuous for $t \neq t_n$ and that the limit values $x(t_n+0)$ $(n = 1, 2, ...)$ exist. We shall show that the function $x(t)$ satisfies the impulsive equation (1), (2) using the well known equalities

$$
\frac{\mathrm{d}V(t,s)}{\mathrm{d}t} = A(t)V(t,s), \quad \frac{\mathrm{d}V(t,s)}{\mathrm{d}s} = V(t,s)A(s)
$$

and

$$
V(t_n+0,s) = Q_nV(t_n,s) \quad (0 \leq s \leq t_n < \infty; n = 1, 2, 3, ...).
$$

We differentiate $x(t)$ for $t \neq t_n$ and obtain

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} = & A(t)\int_0^t V(t)P_1V^{-1}(s)f(s)\mathrm{d}s + V(t)P_1V^{-1}(t)f(t)+ \\
& + V(t)P_2V^{-1}(t)f(t) - A(t)\int_t^\infty V(t)P_2V^{-1}(s)f(s)\mathrm{d}s+ \\
& + \sum_{t_j<t} A(t)V(t)P_1V^{-1}(t_j+0)h_j - \sum_{t_j\geq t} A(t)V(t)P_2V^{-1}(t_j+0)h_j = \\
& = A(t)x(t) + V(t)P_1V^{-1}(t)f(t) + V(t)P_2V^{-1}(t)f(t) = \\
& = A(t)x(t) + f(t)
\end{aligned}
$$

Analogously for $t = t_n \quad (n = 1, 2, ...)$ we obtain

$$
\begin{aligned}
x(t_n + 0) = & \int_0^{t_n} V(t_n + 0)P_1V^{-1}(s)f(s)\mathrm{d}s- \\
& - \int_{t_n}^{\infty} V(t_n + 0)P_2V^{-1}(s)f(s)\mathrm{d}s+ \\
& + \sum_{t_j < t_n} V(t_n + 0)P_1V^{-1}(t_j + 0)h_j- \\
& - \sum_{t_j \geq t_n} V(t_n + 0)P_2V^{-1}(t_j + 0)h_j =
\end{aligned}
$$

$$
\begin{aligned}
& = Q_n \int_0^{t_n} V(t_n)P_1V^{-1}(s)f(s)\mathrm{d}s- \\
& + Q_n \int_{t_n}^{\infty} V(t_n)P_2V^{-1}(s)f(s)\mathrm{d}s+ \\
& + Q_n \sum_{t_j < t_n} V(t_n)P_1V^{-1}(t_j + 0)h_j- \\
& - Q_n \sum_{t_j \geq t_n} V(t_n)P_1V^{-1}(t_j + 0)h_j+ \\
& + V(t_n)P_1V^{-1}(t_j + 0)h_j+ \\
& + V(t_n)P_2V^{-1}(t_j + 0)h_j = \\
& = Q_nx(t_n) + h_n
\end{aligned}
\tag{8}
$$

Hence the function $x(t)$ is a ψ-bounded solution of the nonhomogeneous equation (1), (2) on R_+. □

Remark 1. *For $\psi(t) = I$, i.e. for impulse equation with ordinary dichotomy, Theorem 1 is proved under weaker conditions comparing with the result obtained in [2].*

Let X_1 be the linear manifold of all $\xi \in X$ for which the function $V(t)\xi$ $(t \in R_+)$ is ψ-bounded.

Lemma 1. *Let the following conditions hold:*

1. Condition (H) is satisfied.

2. $B_\psi(X)$ is an arbitrary Banach space of functions $f(.) : R_+ \to X$ and for any function $f \in B_\psi(X)$ the nonhomogeneous equation (1), (2) has at least one

ψ-bounded on R_+ solution $x \in C_\psi(X,T)$

$$(|||x|||_{C_\psi} = \sup_{t \in R_+} |\psi(t)x(t)| < \infty).$$

3. The set X_1 is a complementary subspace of X and let X_2 is a complement of it ($X_1 + X_2 = X$).

Then to each function $f(t) \in B_\psi(X)$ there corresponds a unique ψ-bounded on R_+ solution $x(t)$ starting from X_2, i.e. $x(0) \in X_2$.

This solution satisfies the estimate

$$|||x|||_{C_\psi} \le k|||f|||_{B_\psi},$$

where $k > 0$ is a constant not depending on f.

Proof. Let P_1 and P_2 be the mutually complementary projections on the subspaces X_1 and X_2.

Denote by $C^0_\psi(X,T)$ the subspace of $C_\psi(X,T)$ consisting of the functions which satisfy the condition

$$x(t_n + 0) - Q_n x(t_n) = 0 \quad (n = 1, 2, 3, ...).$$

It is not hard to check that if $x_1(t)$ and $x_2(t)$ are two solutions of the nonhomogeneous impulsive equation (1), (2), then their difference $z(t) = x_1(t) - x_2(t)$ is a solution of the homogeneous impulsive equation (3), (4). If the solutions $x_1(t)$ and $x_2(t)$ are ψ-bounded, then $z(t)$ is ψ-bounded too, hence $z(0) \in X_1$.

If $x(t)$ is a solution of (1), (2) lying in $C^0_\psi(X,T)$, then $\tilde{x}(t) = x(t) - V(t)P_1x(0)$ is also a solution of (1), (2) lying in $C^0_\psi(X,T)$ with initial value $\tilde{x}(0) = P_2x(0) \in X_2$. From the conditions of the lemma it follows that for $f(t) \in B_\psi(X)$ the equation (1), (2) has a solution $\tilde{x}(t) \in C^0_\psi(X,T)$ satisfying the equality $P_1\tilde{x}(0) = 0$. But this solution is unique, since the difference of two such solutions would be a ψ-bounded solution of the homogeneous equation which is initially in X_2, which is possible only for the zero solution. Thus an operator $\tilde{K} : B_\psi(X) \to C^0_\psi(X,T)$ is defined which associates with each element $f \in B_\psi(X)$ a solution of equation (1), (2). From the Banach's closed graph theorem it follows that this operator is continuous, i.e. there exists a number k for which

$$|||\tilde{K}f|||_{C^0_\psi} \le k|||f|||_{B_\psi}.$$

Lemma 1 is proved. □

Theorem 2. *Let the following conditions hold:*

1. Condition (H) is satisfied.

2. The set X_1 is a complementary subspace of X

3. The nonhomogeneous equation (1), (2) has a ψ-bounded on R_+ solution for any function $f \in L_\psi(X)$ and $h = \{h_n\}_{n=1}^{\infty} = 0$.

Then the homogeneous equation (3), (4) is ψ-ordinary dichotomous.

Proof. We shall show that the estimate $||\psi(t)G(t,s)\psi^{-1}(s)|| \le K$ is valid for $0 \le s, t < \infty$, where K is a constant and

$$G(t,s) = \begin{cases} V(t)P_1V^{-1}(s) & (0 \le s \le t < \infty), \\ -V(t)P_2V^{-1}(s) & (0 \le t \le s < \infty). \end{cases} \tag{9}$$

For this purpose let us consider the function f defined as follows

$$f(t) = \begin{cases} \psi^{-1}(t)z & s-a \le t \le s \\ 0 & \text{otherwise} \end{cases}$$

where $s \ge a > 0$ are fixed points and $z \in X$ and $|z| = 1$. Obviously $f \in L_\psi(X)$ and $|||f|||_{L_\psi} = a$.

Introduce the function $x(t)$ by the formula

$$x(t) = \int_{s-a}^{s} G(t,\tau)\psi^{-1}(\tau)z\mathrm{d}\tau.$$

For $t \ge a$ the equality $x(t) = V(t)\xi$ holds, where

$$\xi = \int_{s-a}^{s} P_1V^{-1}(\tau)\psi^{-1}(\tau)z\mathrm{d}\tau \in X_1,$$

i.e. the function $x(t)$ is ψ-bounded. For $0 \le t \le a$ the function $x(t)$ is also ψ-bounded, hence it is ψ-bounded on the whole semi-axis R_+. Moreover

$$x(0) = \int_{s-a}^{s} G(0,\tau)\psi^{-1}(\tau)z\mathrm{d}\tau = -P_2\int_{s-a}^{s} V^{-1}(tau)\psi^{-1}(\tau)z\mathrm{d}\tau \in X_2.$$

Since $x(t)$ is obviously a ψ-bounded solution, from Lemma 1 it follows that $x(t)$ is the unique ψ-bounded on R_+ solution of (1), (2):

$$x(t) = \int_{0}^{\infty} G(t,\tau)f(\tau)\mathrm{d}\tau = \int_{s-a}^{s} G(t,\tau)\psi^{-1}(\tau)z\mathrm{d}\tau$$

and there exists a constant $k > 0$ for which the followings estimate holds

$$|||x|||_{C_\psi} \leq k|||f|||_{B_\psi},$$

i.e.

$$|\psi(t)x(t)| = |\int_{s-a}^{s} \psi(t)G(t,\tau)\psi^{-1}(\tau)z\mathrm{d}\tau| \leq ka.$$

It follows that

$$|\frac{1}{a}\int_{s-a}^{s} \psi(t)G(t,\tau)\psi^{-1}(\tau)z\mathrm{d}\tau| \leq k.$$

Let $a \to 0$. Then the following inequality holds

$$|\psi(t)G(t,\tau)\psi^{-1}(\tau)z| \leq k \quad (|z| = 1),$$

hence

$$||\psi(t)G(t,\tau)\psi^{-1}(t)|| \leq k \quad (0 \leq s, t < \infty).$$

Theorem 2 is proved. □

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