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FUZZY TRANSLATIONS OF FUZZY $\beta-$IDEALS OF $\beta-$ALGEBRAS

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Abstract: In this paper, we introduce the notion of fuzzy translations,fuzzy extensions and fuzzy multiplications of fuzzy $\beta-$ideals of $\beta-$algebras and investigate some of their properties.

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1. Introduction

In 1966, Y. Imai and K. Iseki ([7], [8], [9]) introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK algebras is a proper subclass of the class of BCI algebras. In 2002,J.Neggers and H.S. Kim [12] introduced the notion of $\beta-$algebras. In 2012,Y.H.Kim[11] investigated some properties of $\beta-$algebras.

Lofti A. Zadeh [15] introduced the theory of fuzzy sets. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups in 1977, by Rosenfeld [13]. O.G. Xi [14] applied the concept of fuzzy

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sets to BCK algebras and got some results in 1991.W.A.Dudek and Y.B.Jun [6] fuzzified the ideals in BCC-algebras in 2001. In 2010, Y.B.Jun [10] discussed the fuzzy translations of fuzzy ideals in BCK/BCI-algebras.This motivated us to study the notion of fuzzy translations in $\beta-$algebras.

In our paper[1]) we have introduced the notion of fuzzy $\beta-$subalgebras.We discussed normal fuzzy$\beta-$subalgebras [2] and fuzzy $\beta-$ideals[3] on $\beta-$algebras. In this paper, we introduce the notion of fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy $\beta-$ideals of $\beta-$algberas and investigate elgant results.

2. Preliminaries

In this section we recall some basic definitions that are required in the sequel.

Definition 2.1. [12] A $\beta-$algebra is a non-empty set X with a constant 0 and two binary operations $+$ and $-$ satisfying the following axioms:

1. $x - 0 = x$.
2. $(0 - x) + x = 0$.
3. $(x - y) - z = x - (z + y)\ \forall\ x, y, z\ \in\ X$.

Example 2.2. Let $X = \{0, 1, 2, 3\}$ be a set with constant 0 and two binary operations + and - are defined on X with the Caylay's table

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

-	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X, +, -, 0)$ is a β-algebra.

Henceforth by a$\beta-$algebra X, we mean a $\beta-$algebra $(X, +, -, 0)$ derived from a group or a $\beta-$algebra $(X, +, -, 0)$derived from a $B-$algebra $(X, -, 0)$.

Definition 2.3. [12] A non-empty subset A of a $\beta-$algebra $(X, +, -, 0)$ is called a $\beta-$subalgebra of X, if

1. $x + y \in A, \forall\ x, y \in A$and
2. $x - y \in A, \forall\ x, y \in A$.

Example 2.4. Consider the $\beta-$algebra $(X,+,-,0)$ in example 2.2. The subsets $\{0,1\},\{0,2\}$ and $\{0,3\}$ are $\beta-$subalgebras of X. But the subset $A=\{0,1,2\}$ is not a $\beta-$sub algebra of X, since $1+2=3\notin A$.

Definition 2.5. [1] Let μ be a fuzzy set in a $\beta-$algebra X. Then μ is called a fuzzy $\beta-$subalgebra of X if

1. $\mu(x+y)\geq min\{\mu(x),\mu(y)\}\quad \forall x,y\in X$.
2. $\mu(x-y)\geq min\{\mu(x),\mu(y)\}\quad \forall x,y\in X$.

Definition 2.6. [3] A non-empty subset I of a $\beta-$algebra $(X,+,-,0)$ is called a $\beta-$ideal of X, if

1. $0\in I$,
2. $x+y\in I,\forall x,y\in I$, and
3. if $x-y$ *and* $y\in I$ then $x\in I\ \forall\ x,y\in X$.

Example 2.7. In example 2.2 of $\beta-$algebra X, the subset $I_1=\{0,1\}$ is a $\beta-$ ideal of X. But $I_2=\{0,1,3\}$ is not a $\beta-$ideal of X(since 1+3=2$\notin I_2$.)

Definition 2.8. [3] Let μ be a fuzzy set in a $\beta-$algebra X. Then μ is called a fuzzy $\beta-$ideal of X if

1. $\mu(0)\geq\mu(x)\ \forall\ x\in X$,
2. $\mu(x+y)\geq min\{\mu(x),\mu(y)\}\quad \forall x,y\in X$, and
3. $\mu(x)\geq min\{\mu(x-y),\mu(y)\}\quad \forall x,y\in X$.

Example 2.9. Consider the $\beta-$algebra $(X,+,-,0)$ in example 2.2. The fuzzy set $\mu:X\rightarrow[0,1]$ such that

$$\mu(x)\ =\ \begin{cases}0.8 & \text{if } x=0,2\\ 0.5 & \text{if } x=1,3\end{cases}$$

is a fuzzy $\beta-$ideal of X.

Definition 2.10. Let μ_1 and μ_2 be two fuzzy sets of a set X.If $\mu_1(x)\leq\mu_2(x)\forall\ x\in X$ then μ_2 is a fuzzy extension of μ_1.

3. Fuzzy Translations of $\beta-$Ideals of $\beta-$Algebras

In this section we introduce the notion of fuzzy translations of $\beta-$ideals of $\beta-$algebras and prove some simple theorems.

Definition 3.1. Let μ be a fuzzy set of a $\beta-$algebra X and $\alpha \in [0, \mathsf{T}]$,where $\mathsf{T} = 1 - sup\{\mu(x)/x \in X\}$. Then the fuzzy set $\mu_\alpha^T : X \to [0, 1]$ is called a fuzzy $\alpha-$translation of μ if $\mu_\alpha^T(x) = \mu(x) + \alpha \ \forall\ x \in X$.

Example 3.2. For the fuzzy set μ of X in example 2.9,T=0.2. Let α=0.1$\in [0, 0.2]$. Then the fuzzy set $\mu_\alpha^T : X \to [0, 1]$ given by

$$\mu_\alpha^T(x) = \begin{cases} 0.9 & \text{if } x = 0, 2 \\ 0.6 & \text{if } x = 1, 3 \end{cases}$$

is a fuzzy $\alpha-$translation of μ.

Theorem 3.3. If μ be a fuzzy $\beta-$ideal of X then the fuzzy $\alpha-$translation μ_α^T of μ is also a fuzzy $\beta-$ideal of $X \ \forall\ \alpha \in [0, \mathsf{T}]$.

Proof.

1. Let $\alpha \in [0, \mathsf{T}]$ and for any $x \in X$, $\mu_\alpha^T(0) = \mu(0) + \alpha \geq \mu(x) + \alpha = \mu_\alpha^T(x)$.
2. For any $x, y \in X, \mu_\alpha^T(x + y) = \mu(x + y) + \alpha \geq min\{\mu(x), \mu(y)\} + \alpha = min\{\mu(x) + \alpha, \mu(y) + \alpha\} = min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}$.
3. For any $x, y \in X, \mu_\alpha^T(x) = \mu(x) + \alpha \geq min\{\mu(x - y), \mu(y)\} + \alpha = min\{\mu(x - y) + \alpha, \mu(y) + \alpha\} = min\{\mu_\alpha^T(x - y), \mu_\alpha^T(y)\}$.

Hence μ_α^T of μ is a fuzzy $\beta-$ideal of X.

Remark 3.4. In general for any fuzzy set μ of X, the fuzzy $\alpha-$translation μ_α^T of μ need not be a fuzzy $\beta-$ideal of $X \ \forall\ \alpha \in [0, \mathsf{T}]$, as shown by the following example.

Consider the $\beta-$algebra X in example 2.2.The fuzzy set μ with $\mu(0) = 0.8, \mu(1) = 0.5, \mu(2) = \mu(3) = 0.6$ and the $\alpha(= 0.1)-$translation μ_α^T with $\mu_\alpha^T(0) = 0.9, \mu_\alpha^T(1) = 0.6, \mu_\alpha^T(2) = \mu_\alpha^T(3) = 0.7$ are not fuzzy $\beta-$ideals of X.

Theorem 3.5. Let μ be a fuzzy set of X such that the fuzzy $\alpha-$translation μ_α^T of μ is a fuzzy $\beta-$ideal of X for some $\alpha \in [0, \mathsf{T}]$. Then μ is a fuzzy $\beta-$ideal of X.

Proof. Assume that μ_α^T of μ is a fuzzy $\beta-$ideal of X for some $\alpha \in [0.\mathsf{T}]$. Let $x, y \in X$.

1. $\mu(0)+\alpha=\mu_{\alpha}^{T}(0)\geq\mu_{\alpha}^{T}(x)=\mu(x)+\alpha$ which implies $\mu(0)\geq\mu(x)\forall\ x\in X$.

2. $\mu(x+y)+\alpha=\mu_{\alpha}^{T}(x+y)\geq min\{\mu_{\alpha}^{T}(x),\mu_{\alpha}^{T}(y)\}=min\{\mu(x)+\alpha,\mu(y)+\alpha\}=min\{\mu(x),\mu(y)\}+\alpha$ which implies $\mu(x+y)\geq min\{\mu(x),\mu(y)\}$.

3. $\mu(x)+\alpha=\mu_{\alpha}^{T}(x)\geq min\{\mu_{\alpha}^{T}(x-y),\mu_{\alpha}^{T}(y)\}=min\{\mu(x-y)+\alpha,\mu(y)+\alpha\}=min\{\mu(x-y),\mu(y)\}+\alpha$ which implies $\mu(x)\geq min\{\mu(x-y),\mu(y)\}$.

Hence μ is a fuzzy $\beta-$ideal of X.
The following lemma shows that any fuzzy $\alpha-$translation of a fuzzy $\beta-$ideal of X is a fuzzy $\beta-$subalgebra.

Theorem 3.6. Let $\alpha\in[0,\mathsf{T}]$ and μ be a fuzzy $\beta-$ideal of X. Then the fuzzy $\alpha-$translation μ_{α}^{T} of μ is a fuzzy $\beta-$subalgebra of X.

Proof. Let $\alpha\in[0,\mathsf{T}]$.

1. For any $x,y\in X,\mu_{\alpha}^{T}(x+y)=\mu(x+y)+\alpha\geq min\{\mu(x),\mu(y)\}+\alpha=min\{\mu(x)+\alpha,\mu(y)+\alpha\}=min\{\mu_{\alpha}^{T}(x),\mu_{\alpha}^{T}(y)\}$.

2. For any $x,y\in X,\mu_{\alpha}^{T}(x-y)=\mu(x-y)+\alpha\geq min\{\mu(x),\mu(y)\}+\alpha=min\{\mu(x)+\alpha,\mu(y)+\alpha\}=min\{\mu_{\alpha}^{T}(x),\mu_{\alpha}^{T}(y)\}$.

Hence μ_{α}^{T} is a fuzzy $\beta-$subalgebra of X.

Theorem 3.7. For $\alpha\in[0,\mathsf{T}]$, let the fuzzy $\alpha-$translation μ_{α}^{T} of a fuzzy set μ of X be a fuzzy $\beta-$ideal of X. Then every non empty level subset $(\mu_{\alpha}^{T})_t=\{x\in X/\mu(x)\geq t-\alpha\}\forall\ t\in Im(\mu)$ with $t>\alpha$ is a $\beta-$ideal of X.

Proof. Assume that the fuzzy $\alpha-$translationμ_{α}^{T} is a fuzzy $\beta-$ideal of X. Let $t\in Im(\mu)$ be such that $t>\alpha$.

1. Now $\mu(0)+\alpha=\mu_{\alpha}^{T}(0)\geq\mu_{\alpha}^{T}(x)=\mu(x)+\alpha\geq t\ \forall\ x\in(\mu_{\alpha}^{T})_t$. This establishes that $0\in(\mu_{\alpha}^{T})_t$.

2. Let $x,y\in(\mu_{\alpha}^{T})_t\Rightarrow\mu(x)\geq t-\alpha$ and $\mu(y)\geq t-\alpha\Rightarrow\mu(x)+\alpha\geq t$ and $\mu(y)+\alpha\geq t\Rightarrow\mu_{\alpha}^{T}(x)\geq t$ and $\mu_{\alpha}^{T}(y)\geq t$. Now $(\mu_{\alpha}^{T})(x+y)\geq min\{(\mu_{\alpha}^{T})(x),(\mu_{\alpha}^{T})(y)\}\geq min\{t,t\}=t$ which implies$\mu(x+y)+\alpha\geq t\Rightarrow\mu(x+y)\geq t-\alpha\Rightarrow x+y\in(\mu_{\alpha}^{T})_t$.

3. Let $x,y\in X$ be such that $x-y$ and $y\in(\mu_{\alpha}^{T})_t\Rightarrow\mu(x-y)\geq t-\alpha$ and $\mu(y)\geq t-\alpha\Rightarrow\mu(x-y)+\alpha\geq t$ and $\mu(y)+\alpha\geq t\Rightarrow(\mu_{\alpha}^{T})_t(x-y)\geq t$ and $(\mu_{\alpha}^{T})_t(y)\geq t$. Hence $\mu(x)+\alpha=\mu_{\alpha}^{T}(x)\geq min\{\mu_{\alpha}^{T}(x-y),\mu_{\alpha}^{T}(y)\}\geq min\{t,t\}=t$. Therefore$x\in(\mu_{\alpha}^{T})_t$. Thus $(\mu_{\alpha}^{T})_t$ is a $\beta-$ideal of X.

Theorem 3.8. Intersection and union of any two fuzzy tranlations of a fuzzy $\beta-$ideal μ of X is also a fuzzy $\beta-$ideal of X.

Proof. Let μ_α^T and $\mu_{\alpha'}^T$ be two fuzzy tranlations of a fuzzy $\beta-$ideal μ of X, where $\alpha, \alpha' \in [0, \mathsf{T}]$. Assume that $\alpha \leq \alpha'$. By theorem 3.3 μ_α^T and $\mu_{\alpha'}^T$ are fuzzy $\beta-$ideals of X. Now $(\mu_\alpha^T \cap \mu_{\alpha'}^T)(x) = min\{\mu_\alpha^T(x), \mu_{\alpha'}^T(x)\} = min\{\mu(x) + \alpha, \mu(x) + \alpha'\} = \mu(x) + \alpha = \mu_\alpha^T$. Also$(\mu_\alpha^T \cup \mu_{\alpha'}^T)(x) = max\{\mu_\alpha^T(x), \mu_{\alpha'}^T(x)\} = max\{\mu(x) + \alpha, \mu(x) + \alpha'\} = \mu(x) + \alpha' = \mu_{\alpha'}^T$. Therefore $\mu_\alpha^T \cap \mu_{\alpha'}^T$ and $\mu_\alpha^T \cup \mu_{\alpha'}^T$ are fuzzy $\beta-$ideals of X.

Theorem 3.9. Let μ and ν be two fuzzy $\beta-$ideals of X.Let $\mathsf{T} = min\{\mathsf{T}_\mu, \mathsf{T}_\nu\}$ where $\mathsf{T}_\mu = 1 - sup\{\mu(x)/x \in X\}$ and $\mathsf{T}_\nu = 1 - sup\{\nu(x)/x \in X\}$.Then the intersection of $\alpha-$translation of μ and $\alpha'-$translation of ν for some $\alpha, \alpha' \in [0, \mathsf{T}]$ is a fuzzy $\beta-$ideal of X.But the union need not be a fuzzy $\beta-$ideal of X.

Proof. Let μ and ν be two fuzzy $\beta-$ideals of X.Then by theorem 3.3 μ_α^T and $\nu_{\alpha'}^T$ are fuzzy $\beta-$ideals of X.Now by theorem 3.7 of [3] $\mu_\alpha^T \cap \nu_{\alpha'}^T$ is a fuzzy $\beta-$ideal of X.

Note: In example 4.2 the fuzzy sets $\mu = \mu_1$ and $\nu = \mu_2$ are fuzzy $\beta-$ideals of X.But$\mu_{0.1}^T \cup \nu_{0.01}^T$ is not a fuzzy $\beta-$ideals of X(since $\mu_{0.1}^T \cup \nu_{0.01}^T(3) = 0.51 \ngeq 0.6 = min\{0.6, 0.61\} = min\{(\mu_{0.1}^T \cup \nu_{0.01}^T)(3-1), (\mu_{0.1}^T \cup (\nu_{0.01}^T)(1)\})$.

Theorem 3.10. Let $f : X \to Y$ be an epimorphism between two $\beta-$algebras X and Y.And $\alpha \in [0, \mathsf{T}]$.Then inverse image of $\alpha-$translation of any fuzzy $\beta-$ideal μ of Y is same as the $\alpha-$translation of the inverse image of the fuzzy $\beta-$ideal μ.

Proof. Let $f : X \to Y$ be an epimorphism between two $\beta-$algebras X and Y. And $\alpha \in [0, \mathsf{T}]$. Let μ be a fuzzy $\beta-$ideal of Y. Then by theorem 3.3 the $\alpha-$translation μ_α^T is a fuzzy $\beta-$ideal of Y. Now by theorem 3.6 of [4] $f^{-1}(\mu_\alpha^T)$ is a fuzzy $\beta-$ideal of X.Also $f^{-1}(\mu_\alpha^T)(x) = \mu_\alpha^T(f(x)) = \mu(f(x)) + \alpha = f^{-1}(\mu)(x) + \alpha = (f^{-1}(\mu))_\alpha^T(x)$. Hence $f^{-1}(\mu_\alpha^T) = (f^{-1}(\mu))_\alpha^T$.

Theorem 3.11. Let μ and ν be two fuzzy $\beta-$ideals of a $\beta-$algebra X.Let $\mathsf{T} = min\{\mathsf{T}_\mu, \mathsf{T}_\nu\}$ where $\mathsf{T}_\mu = 1 - sup\{\mu(x)/x \in X\}$ and $\mathsf{T}_\nu = 1 - sup\{\nu(x)/x \in X\}$and $\alpha \in [0, \mathsf{T}]$.Then the $\alpha-$translation of Cartesian product $\mu \times \nu$ of μ and ν is a fuzzy $\beta-$ideal of $X \times X$.

Proof. Let μ and ν be two fuzzy $\beta-$ideals of a $\beta-$algebra X.Let $\alpha \in [0, \mathsf{T}]$. Now by theorem 3.3 μ_α^T and ν_α^T are fuzzy $\beta-$ideal of X and by theorem 4.1 of [4] $\mu_\alpha^T \times \nu_\alpha^T$ is a fuzzy $\beta-$ideal of $X \times X$. Also $(\mu \times \nu)_\alpha^T(x, y) = (\mu \times \nu)(x, y) + \alpha = min\{\mu(x), \nu(y)\} + \alpha = min\{\mu(x) + \alpha, \nu(y) + \alpha\} = min\{\mu_\alpha^T(x), \nu_\alpha^T(y)\} = (\mu_\alpha^T \times \nu_\alpha^T)(x, y)$ $\forall (x, y) \in X \times X$. Hence $(\mu \times \nu)_\alpha^T$ is a fuzzy $\beta-$ideal of $X \times X$.

4. Fuzzy Extensions of $\beta-$Ideals of $\beta-$Algebras

In this section we introduce the notion of fuzzy extensions of $\beta-$ideals of $\beta-$algebras and prove some theorems.

Definition 4.1. Let μ_1 and μ_2 be two fuzzy sets of X such that μ_2 is a fuzzy extension of μ_1. If μ_1 is a fuzzy $\beta-$ideal of X implies that μ_2 is a fuzzy $\beta-$ideal of X, then μ_2 is called a fuzzy $\beta-$ideal extension of μ_1.

Example 4.2. Consider the $\beta-$algebra $(X, +, -, 0)$ in 2.2 The fuzzy sets μ_1 and μ_2 of X defined as below

$$\mu_1(x) = \begin{cases} 0.8 & \text{if } x = 0 \\ 0.3 & \text{if } x = 1, 3 \\ 0.5 & \text{if } x = 2 \end{cases} \qquad \mu_2(x) = \begin{cases} 0.9 & \text{if } x = 0 \\ 0.6 & \text{if } x = 1 \\ 0.5 & \text{if } x = 2, 3 \end{cases}$$

are fuzzy $\beta-$ideals of X. Clearly μ_2 is a fuzzy$\beta-$ideal extension of μ_1.But μ_2 is not a $\alpha-$translation of μ_1.

Theorem 4.3. Intersection of any two fuzzy $\beta-$ideal extensions of a fuzzy $\beta-$ideal μ of X is a fuzzy $\beta-$ideal extension of μ.

Proof. Let μ_1 and μ_2 be two fuzzy $\beta-$ideal extensions of a fuzzy$\beta-$ideal μ of X. Then $\mu_1(x) \geq \mu(x)$ and $\mu_2(x) \geq \mu(x)$ $\forall\ x \in X$.Now μ is fuzzy $\beta-$ideal of X which implies μ_1 and μ_2 are fuzzy $\beta-$ideal of X.Then by theorem:3.6 of [3], $\mu_1 \cap \mu_2$ is also a fuzzy $\beta-$ideal of X. Now $(\mu_1 \cap \mu_2)(x) = min\{\mu_1(x), \mu_2(x)\} \geq min\{\mu(x), \mu(x)\} = \mu(x)$. Hence $\mu_1 \cap \mu_2$ is a fuzzy $\beta-$ideal extension of μ.

Remark 4.4. For any two fuzzy sets μ_1 and μ_2 of $X, (\mu_1 \cap \mu_2)(x) = min\{\mu_1(x), \mu_2(x)\} \leq \mu_1(x)$ and $\mu_2(x)$.Hence $\mu_1(x)$ and $\mu_2(x)$ are fuzzy extensions of $(\mu_1 \cap \mu_2)$. But $\mu_1(x)$ and $\mu_2(x)$ need not be fuzzy $\beta-$ideal extensions of $(\mu_1 \cap \mu_2)$.

Remark 4.5. Union of any two fuzzy $\beta-$ideal extensions of a fuzzy set of $\beta-$algebra need not be a fuzzy $\beta-$ideal extension. Consider the $\beta-$algebra X in the example 2.2. Let the fuzzy sets μ, μ_1 and μ_2 be defined by

$$\mu(x) = \begin{cases} 0.7 & \text{if } x = 0 \\ 0.3 & \text{if } x = 1, 3 \\ 0.5 & \text{if } x = 2 \end{cases} \qquad \mu_1(x) = \begin{cases} 0.8 & \text{if } x = 0 \\ 0.6 & \text{if } x = 1 \\ 0.5 & \text{if } x = 2, 3 \end{cases}$$

$$\mu_2(x) = \begin{cases} 0.9 & \text{if } x = 0 \\ 0.4 & \text{if } x = 1, 3 \\ 0.6 & \text{if } x = 2 \end{cases}$$

Clearly μ_1 and μ_2 are fuzzy $\beta-$ideal extensions of the fuzzy $\beta-$ideal μ. But the fuzzy set $\mu_1 \cup \mu_2$ where $(\mu_1 \cup \mu_2)(0) = 0.9, (\mu_1 \cup \mu_2)(1) = (\mu_1 \cup \mu_2)(2) = 0.6$ and $(\mu_1 \cup \mu_2)(3) = 0.5$, is not a fuzzy $\beta-$ideal extension(Since $(\mu_1 \cup \mu_2)(3) = 0.5 \ngeq 0.6 = min\{(\mu_1 \cup \mu_2)(3-2), (\mu_1 \cup \mu_2)(2)\}$).

Theorem 4.6. Let μ be a fuzzy $\beta-$ideal of X.For $\forall\ \alpha \in [0, \mathsf{T}]$ the fuzzy $\alpha-$translation μ^T_α is a fuzzy $\beta-$ideal extension of μ itself.

Proof. If μ be a fuzzy $\beta-$ideal of X then theorem 3.3 the fuzzy $\alpha-$translation μ^T_α of μ is also a fuzzy $\beta-$ideal of X $\forall\ \alpha \in [0, \mathsf{T}]$. Also $\mu^T_\alpha(x) = \mu(x) + \alpha \geq \mu(x)\ \forall x\ \in X$.Therefore μ^T_α is a fuzzy $\beta-$ideal extension of μ.

Theorem 4.7. Let μ be a fuzzy $\beta-$ideal of X. If $\alpha \geq \alpha'$ with $\alpha, \alpha' \in [0, \mathsf{T}]$,then the fuzzy $\alpha-$translation μ^T_α of μ is a fuzzy $\beta-$ideal extension of the fuzzy $\alpha'-$translation $\mu^T_{\alpha'}$ of μ.

Proof. Let μ be a fuzzy $\beta-$ideal of X.Then by theorem 3.3 the fuzzy $\alpha-$translation μ^T_α and the fuzzy $\alpha'-$translation $\mu^T_{\alpha'}$ are fuzzy $\beta-$ideal of X. Since $\alpha \geq \alpha'$ implies $\mu(x) + \alpha \geq \mu(x) + \alpha'\ \forall\ x \in X$. Therefore $\mu^T_\alpha(x) \geq \mu^T_{\alpha'}(x)\ \forall\ x \in X$. Hence μ^T_α is a fuzzy $\beta-$ideal extension of $\mu^T_{\alpha'}$.

Theorem 4.8. Let μ be fuzzy $\beta-$ideal of X and $\alpha' \in [0, \mathsf{T}]$. For every fuzzy $\beta-$ideal extension ν of the fuzzy $\alpha'-$translation $\mu^T_{\alpha'}$ of μ, there exist $\alpha \in [0, \mathsf{T}]$ such that $\alpha \geq \alpha'$ and ν is the fuzzy $\beta-$ideal extension of the fuzzy $\alpha-$translation μ^T_α of μ.

Proof. Let μ be fuzzy $\beta-$ideal of X and $\alpha' \in [0, \mathsf{T}]$.Then by theorem 3.3 the fuzzy $\alpha'-$translation $\mu^T_{\alpha'}$ is a fuzzy $\beta-$ideal of X. Let ν be a fuzzy $\beta-$ideal extension of $\mu^T_{\alpha'}$. Therefore $\nu(x) \geq \mu^T_{\alpha'}(x)\ \forall\ x \in X$.Then choose $\alpha = \alpha' + \min_{x \in X}\{\nu(x) - \mu^T_{\alpha'}(x)\}$. Clearly $\alpha \in [0, \mathsf{T}]$ such that $\alpha \geq \alpha'$. Then μ^T_α is a fuzzy $\alpha-$translation μ and $\nu(x) \geq \mu^T_\alpha(x)$. Hence$\nu$ is also a fuzzy $\beta-$ideal extension of the fuzzy μ^T_α.

5. Fuzzy Multiplications of $\beta-$Ideals of $\beta-$Algebras

In this section we introduce the notion of fuzzy multiplications of $\beta-$ideals of $\beta-$algebras and prove some simple theorems.

Definition 5.1. Let μ be a fuzzy set of a $\beta-$algebra X and $\gamma \in [0, 1]$. Then the fuzzy set $\mu^M_\gamma : X \to [0, 1]$ is called a fuzzy $\gamma-$ multiplication of μ if $\mu^M_\gamma(x) = \mu(x).\gamma\ \forall\ x \in X$.

Example 5.2. For the fuzzy $\beta-$ideal μ of X in example 2.9, the fuzzy set $\mu_\gamma^M : X \to [0,1]$ such that

$$\mu_\gamma^M(x) \;=\; \begin{cases} 0.08 & \text{if } x = 0, 2 \\ 0.05 & \text{if } x = 1, 3 \end{cases}$$

where γ=0.1$\in [0,1]$ is a fuzzy $\gamma-$ multiplication of μ.

Lemma 5.3. [3] If a fuzzy set $\mu : X \to [0,1]$ is a constant, then μ is a fuzzy $\beta-$ideal of X.

Proof. Straightforward.

Corollary 5.4. Let μ be a fuzzy set of X. If γ=0, then the fuzzy $\gamma-$multiplication μ_γ^M of μ is a fuzzy $\beta-$ideal of X.

Corollary 5.5. Let μ be a fuzzy set of X. If $\gamma = \frac{1}{\mu(x)}, (\mu(x) > 0) \;\forall\; x \in X$, then the fuzzy $\gamma-$multiplication μ_γ^M of μ is a fuzzy $\beta-$ideal of X.

Proof. Now $\mu_\gamma^M(x) = \mu(x).\gamma = \frac{\mu(x)}{\mu(x)} = 1 \;\forall\; x \in X$. Then by lemma 5.3 μ_γ^M is a fuzzy $\beta-$ideal of X

Theorem 5.6. If μ be a fuzzy $\beta-$ideal of X then the fuzzy $\gamma-$ multiplication μ_γ^M of μ is also a fuzzy $\beta-$ideal of $X \;\forall\; \gamma \in [0,1]$.

Proof.

1. Let $\gamma \in [0,1]$ and for any $x \in X,\; \mu_\gamma^M(0) = \mu(0).\gamma \geq \mu(x).\gamma = \mu_\gamma^M(x)$.

2. For any $x, y \in X, \mu_\gamma^M(x+y) = \mu(x+y).\gamma \geq min\{\mu(x), \mu(y)\}.\gamma$
   $= min\{\mu(x).\gamma, \mu(y).\gamma\} = min\{\mu_\gamma^M(x), \mu_\gamma^M(y)\}$.

3. For any $x, y \in X, \mu_\gamma^M(x) = \mu(x).\gamma \geq min\{\mu(x-y), \mu(y)\}.\gamma$
   $= min\{\mu(x-y).\gamma, \mu(y).\gamma\} = min\{\mu_\gamma^M(x-y), \mu_\gamma^M(y)\}$.

Hence μ_γ^M of μ is a fuzzy $\beta-$ideal of X.

Theorem 5.7. Let μ be a fuzzy set of X such that the fuzzy $\gamma-$ multiplication μ_γ^M of μ is a fuzzy $\beta-$ideal of X for some $\gamma \in (0,1]$ then μ is a fuzzy $\beta-$ideal of X.

Proof. Assume that μ_γ^M of μ is a fuzzy $\beta-$ideal of X for some $\gamma \in (0,1]$. Let $x, y \in X$.

1. $\mu(0).\gamma = \mu_\gamma^M(0) \geq \mu_\gamma^M(x) = \mu(x).\gamma$ which implies $\mu(0) \geq \mu(x) \forall\; x \in X$.

2. $\mu(x+y).\gamma = \mu_\gamma^M(x+y) \geq min\{\mu_\gamma^M(x), \mu_\gamma^M(y)\} = min\{\mu(x).\gamma, \mu(y).\gamma\} = min\{\mu(x), \mu(y)\}.\gamma$ which implies $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$.

3. $\mu(x).\gamma = \mu_\gamma^M(x) \geq min\{\mu_\gamma^M(x-y), \mu_\gamma^M(y)\} = min\{\mu(x-y).\gamma, \mu(y).\gamma\} = min\{\mu(x-y), \mu(y)\}.\gamma$ which implies $\mu(x) \geq min\{\mu(x-y), \mu(y)\}$.

Hence μ is a fuzzy $\beta-$ideal of X.

Theorem 5.8. Let $\gamma \in [0,1]$ and μ be a fuzzy $\beta-$ideal of X. Then the fuzzy $\gamma-$ multiplication μ_γ^M of μ is a fuzzy $\beta-$subalgebra of X.

Proof. Let $\gamma \in [0,1]$.

1. For any $x, y \in X$, $\mu_\gamma^M(x+y) = \mu(x+y).\gamma \geq min\{\mu(x), \mu(y)\}.\gamma$ $= min\{\mu(x).\gamma, \mu(y).\gamma\} = min\{\mu_\gamma^M(x), \mu_\gamma^M(y)\}$.

2. For any $x, y \in X$, $\mu_\gamma^M(x-y) = \mu(x-y).\gamma \geq min\{\mu(x), \mu(y)\}.\gamma$ $= min\{\mu(x).\gamma, \mu(y).\gamma\} = min\{\mu_\gamma^M(x), \mu_\gamma^M(y)\}$.

Hence μ_γ^M is a fuzzy $\beta-$subalgebra of X.

Theorem 5.9. Let μ be a fuzzy set of $X, \alpha \in [0, \top]$ and $\gamma \in (0,1]$. If μ_γ^M is a fuzzy $\beta-$ideal of X, then the fuzzy $\alpha-$translationμ_α^T is a fuzzy $\beta-$ideal extension of μ_γ^M.

Proof. Let$\alpha \in [0, \top], \gamma \in (0,1]$ and μ_γ^M is a fuzzy $\beta-$ideal of X. Then by theorem 5.7 the fuzzy setμ is a fuzzy $\beta-$ideal of X.And by theorem3.3 the $\alpha-$translationμ_α^T is a fuzzy $\beta-$ideal of X. Now $\mu_\alpha^T(x) = \mu(x) + \alpha \geq \mu(x) \geq \mu(x).\gamma = \mu_\gamma^M$.Hence μ_α^T is a fuzzy $\beta-$ideal extension of μ_γ^M.

Remark 5.10. If $\gamma = 0$ then by theorem 5.4 the fuzzy $\gamma-$multiplication μ_γ^M of μ is a fuzzy $\beta-$ideal of X.Also μ_α^T is a fuzzy extension of μ_γ^M but not a fuzzy $\beta-$ideal extension(since by the Remark:3.4,μ_α^T need not be a fuzzy $\beta-$ideal of X).

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