International Journal of Pure and Applied Mathematics
Volume 92 No. 5 2014, 719-729
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v92i5.7

ijpam.eu

UPPER APPROXIMATION OPERATORS INDUCED BY ALEXANDROV FUZZY TOPOLOGIES

Yong Chan Kim

Department of Mathematics
Gangneung-Wonju University
Gangneung, Gangwondo, 210-702, KOREA

Abstract: In this paper, we investigate the properties of upper approximation operators induced by Alexandrov fuzzy topologies in complete residuated lattices. We give their examples.

AMS Subject Classification: 03E72, 03G10, 06A15, 06F07
Key Words: complete residuated lattices, fuzzy preorders, join preserving maps, upper approximation operators, Alexandrov fuzzy topologies

1. Introduction

Höhle [3] introduced L-fuzzy topologies and L-fuzzy interior operators. The relationship between rough set theory and topological spaces was investigated in sets [7]. Hájek [2] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool for algebraic structure of fuzzy contexts [1-6, 8,9, 12-15]. Kim [5,6] investigated the properties of join (resp. meet, meet join, join meet) preserving operators in complete residuated lattices.

In this paper, we investigate the properties of upper approximation operators induced by Alexandrov fuzzy topologies in complete residuated lattices in a sense as Höhle's L-fuzzy topologies and L-fuzzy interior operators [3]. We give their examples.

Received: March 1, 2014

2. Preliminaries

Definition 2.1. [1-3] A structure $(L, \vee, \wedge, \odot, \to, \perp, \top)$ is called a *complete residuated lattice* iff it satisfies the following properties:

(L1) $(L, \vee, \wedge, \perp, \top)$ is a complete lattice where $\perp$ is the bottom element and $\top$ is the top element;

(L2) $(L, \odot, \top)$ is a monoid;

(L3) It has an adjointness,i.e.

$$x \le y \to z \text{ iff } x \odot y \le z.$$

An operator $^* : L \to L$ defined by $a^* = a \to \perp$ is called *strong negations* if $a^{**} = a$.

$$\top_x(y) = \begin{cases} \top, & \text{if } y = x, \\ \perp, & \text{otherwise.} \end{cases} \quad \top^*_x(y) = \begin{cases} \perp, & \text{if } y = x, \\ \top, & \text{otherwise.} \end{cases}$$

In this paper, we assume that $(L, \vee, \wedge, \odot, \to, ^*, \perp, \top)$ be a complete residuated lattice with a strong negation *.

Definition 2.2. [8,9] Let X be a set. A function $e_X : X \times X \to L$ is called:

(E1) reflexive if $e_X(x, x) = 1$ for all $x \in X$,

(E2) transitive if $e_X(x, y) \odot e_X(y, z) \le e_X(x, z)$, for all $x, y, z \in X$,

(E3) if $e_X(x, y) = e_X(y, x) = 1$, then $x = y$.

If e satisfies (E1) and (E2), (X, e_X) is a fuzzy preorder set. If e satisfies (E1), (E2) and (E3), (X, e_X) is a fuzzy partially order set (simply, fuzzy poset).

Example 2.3. (1) We define a function $e_L : L \times L \to L$ as $e_L(x, y) = x \to y$. Then (L, e_L) is a fuzzy poset.

(2) We define a function $e_{L^X} : L^X \times L^X \to L$ as $e_{L^X}(A, B) = \bigwedge_{x \in X}(A(x) \to B(x))$. Then (L^X, e_{L^X}) is a fuzzy poset from Lemma 2.4 (9).

Lemma 2.4. [1,2] Let $(L, \vee, \wedge, \odot, \to, ^*, \perp, \top)$ be a complete residuated lattice with a strong negation *. For each $x, y, z, x_i, y_i \in L$, the following properties hold.

(1) If $y \le z$, then $x \odot y \le x \odot z$.

(2) If $y \le z$, then $x \to y \le x \to z$ and $z \to x \le y \to x$.

(3) $x \to y = \top$ iff $x \le y$.

(4) $x \to \top = \top$ and $\top \to x = x$.

(5) $x \odot y \le x \wedge y$.

(6) $x \odot (\bigvee_{i \in \Gamma} y_i) = \bigvee_{i \in \Gamma}(x \odot y_i)$ and $(\bigvee_{i \in \Gamma} x_i) \odot y = \bigvee_{i \in \Gamma}(x_i \odot y)$.

(7) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.

(8) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$ and $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$.

(9) $(x \to y) \odot x \leq y$ and $(y \to z) \odot (x \to y) \leq (x \to z)$.

(10) $x \to y \leq (y \to z) \to (x \to z)$ and $x \to y \leq (z \to x) \to (z \to y)$.

(11) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.

(12) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$ and $(x \odot y)^* = x \to y^*$.

(13) $x^* \to y^* = y \to x$ and $(x \to y)^* = x \odot y^*$.

(14) $y \to z \leq x \odot y \to x \odot z$.

Definition 2.5. [5,6] A map $\mathcal{H} : L^X \to L^Y$ is called a *join preserving map* if it satisfies the following conditions, for all $A, A_i \in L^X$, and $\alpha \in L$,

(J1) $\mathcal{H}(\alpha \odot A) = \alpha \odot \mathcal{H}(A)$,

(J2) $\mathcal{H}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} \mathcal{H}(A_i)$.

A join preserving operator $\mathcal{H} : L^X \to L^X$ is called an *upper approximation operator* iff it satisfies the following conditions

(H1) $A \leq \mathcal{H}(A)$,

(H2) $\mathcal{H}(\mathcal{H}(A)) \leq \mathcal{H}(A)$, for all $A \in L^X$.

Definition 2.6. [6] An operator $\mathbf{T} : L^X \to L$ is called an *Alexandrov fuzzy topology* on X iff it satisfies the following conditions:

(T1) $\mathbf{T}(\alpha) = \top$,

(T2) $\mathbf{T}(\bigwedge_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i)$ and $\mathbf{T}(\bigvee_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i)$,

(T3) $\mathbf{T}(\alpha \odot A) \geq \mathbf{T}(A)$,

(T4) $\mathbf{T}(\alpha \to A) \geq \mathbf{T}(A)$.

Theorem 2.7. [6] Let $\mathbf{T} : L^X \to L$ be an Alexandrov fuzzy topology. Define $\mathbf{T}^*(A) = \mathbf{T}(A^*)$. Then $\mathbf{T}^*$ is an Alexandrov fuzzy topology.

Theorem 2.8. [6] Let $\mathcal{H}$ be a join preserving map. Define $\mathbf{T}_H : L^X \to L$ as

$$\mathbf{T}_H(A) = \bigwedge_{x\in X} (\mathcal{H}(A)(x) \to A(x)) = e_{L^X}(\mathcal{H}(A), A).$$

Then we have the following properties.

(1) $\mathbf{T}_H$ is an Alexandrov fuzzy topology on X.

(2) $\mathbf{T}_H(A) = \bigwedge_{x,y\in X}(\mathcal{H}(\top_x)(y) \to (A(x) \to A(y))$ such that $\mathbf{T}_H(A) \geq \bigwedge_{x\neq y\in X} \mathcal{H}(\top_x)(y)$.

(3) If $\mathcal{H}$ is an upper approximation operator, then $\mathbf{T}_H(\mathcal{H}(\top_x)) = \top$.

(4) If $\mathcal{H}^{-1}$ is a join preserving map such that $\mathcal{H}^{-1}(\top_x)(y) = \mathcal{H}(\top_y)(x)$ for all $x, y \in X$. Define $\mathbf{T}^*_H(A) = \mathbf{T}_H(A^*)$. Then $\mathbf{T}^*_H = \mathbf{T}_{H^{-1}}$ is an Alexandrov fuzzy topology.

(5) If $\mathcal{H}$ is an upper approximation operator, then $\mathcal{H}^{-1}$ is an upper approximation operator such that

$$\mathbf{T}_H(\mathcal{H}^{-1*}(\top_x)) = \mathbf{T}_{H^{-1}}(\mathcal{H}^*(\top_x)) = \top.$$

3. Upper Approximation Operators Induced by Alexandrov Fuzzy Topologies

Theorem 3.1. Let $\mathbf{T}$ be an Alexandrov fuzzy topology on X. Define $\mathcal{H}_T : L^X \times L \to L^X$ as follows

$$\mathcal{H}_T(A, r) = \bigwedge \{B \in L^X \mid A \le B,\ \mathbf{T}(B) \ge r^*\}$$

Then we have the following properties.

(1) $\mathcal{H}_T(-, r) : L^X \to L^X$ is an upper approximation operator.

(2) If $r \le s$, then $\mathcal{H}_T(A, s) \le \mathcal{H}_T(A, r)$ for all $A \in L^X$.

(3) There exists a preorder $R^r_T \in L^{X \times X}$ such that

$$\mathcal{H}_T(A, r) = \bigvee_{x \in X} (A(x) \odot R^r_T(x, y)).$$

(4) If $r \le s$, then $R^r_T \ge R^s_T$ for all $A \in L^X$.

(5) If $\mathcal{H}_T(A, r_i) = B$ for all $i \in \Gamma \neq \emptyset$, then $\mathcal{H}_T(A, \bigwedge_{i \in \Gamma} r_i) = B$.

(6) Define $\mathbf{T}_{H_T} : L^X \to L$ as

$$\mathbf{T}_{H_T}(A) = \bigvee \{r_i^* \in L \mid \mathcal{H}_T(A, r_i) = A\}$$

Then $\mathbf{T}_{H_T} = \mathbf{T}$ is an Alexandrov fuzzy topology on X.

(7) There exists an Alexandrov fuzzy topology $\mathbf{T}^r$ such that

$$\mathbf{T}^r(A) = e_{L^X}(\mathcal{H}_T(A, r), A).$$

(8) If $r \le s$, then $\mathbf{T}^r \le \mathbf{T}^s$ for all $A \in L^X$.

(9) Define $\mathbf{T}_T : L^X \to L$ as

$$\mathbf{T}_T(A) = \bigvee \{r^* \in L \mid \mathbf{T}^r(A) = \top\}.$$

Then $\mathbf{T}_T = \mathbf{T} = \mathbf{T}_{H_T}$ is an Alexandrov fuzzy topology on X.

Proof. (1) First, $\mathcal{H}_T(-,r)$ is a join preserving map from the following statements. Since $\alpha \odot \bigwedge(\alpha \to B) \leq \bigwedge(\alpha \odot (\alpha \to B)) \leq \bigwedge B$, we have

$$\begin{aligned}
&\mathcal{H}_T(\alpha \odot A, r) = \bigwedge\{B \in L^X \mid \alpha \odot A \leq B,\ \mathbf{T}(B) \geq r^*\}\\
&\geq \alpha \odot \bigwedge\{\alpha \to B \in L^X \mid A \leq \alpha \to B,\ \mathbf{T}(\alpha \to B) \geq r^*\}\\
&\geq \alpha \odot \bigwedge\{C \in L^X \mid A \leq C,\ \mathbf{T}(C) \geq r^*\} = \alpha \odot \mathcal{H}_T(A, r).
\end{aligned}$$

Since $\mathbf{T}(\mathcal{H}_T(A,r)) \geq r^*$ and $\alpha \odot A \leq \alpha \odot \mathcal{H}_T(A,r)$, then $\mathbf{T}(\alpha \odot \mathcal{H}_T(A,r)) \geq r^*$ and $\mathcal{H}_T(\alpha \odot A, r) \leq \alpha \odot \mathcal{H}_T(A,r)$. Hence $\mathcal{H}_T(\alpha \odot A, r) \leq \alpha \odot \mathcal{H}_T(A,r)$.

By the definition of $\mathcal{H}_T(-,r)$, $\bigvee_{i\in\Gamma} \mathcal{H}_T(A_i, r) \leq \mathcal{H}_T(\bigvee_{i\in\Gamma} A_i, r)$.

Since $\bigvee_{i\in\Gamma} A_i \leq \bigvee_{i\in\Gamma} \mathcal{H}_T(A_i, r)$ and $\mathbf{T}(\bigvee_{i\in\Gamma} \mathcal{H}_T(A_i, r)) \geq r^*$, then

$$\mathcal{H}_T(\bigvee_{i\in\Gamma} A_i, r) \leq \bigvee_{i\in\Gamma} \mathcal{H}_T(A_i, r).$$

(H1) By the definition of $\mathcal{H}_T(-,r)$, $A \leq \mathcal{H}_T(A,r)$.

(H2) By (H1), $\mathcal{H}_T(A,r) \leq \mathcal{H}_T(\mathcal{H}_T(A,r),r)$.

Since $\mathcal{H}_T(\mathcal{H}_T(A,r),r) = \bigwedge\{B \in L^X \mid \mathcal{H}_T(A,r) \leq B,\ \mathbf{T}(B) \geq r^*\}$ and $\mathbf{T}(\mathcal{H}_T(A,r)) \geq r^*$, then $\mathcal{H}_T(\mathcal{H}_T(A,r),r) = \mathcal{H}_T(A,r)$. Hence $\mathcal{H}_T(-,r)$ is an upper approximation operator.

(2) For $r \leq s$, since $\mathbf{T}(B) \geq r^* \geq s^*$, we have $\mathcal{H}_T(A,s) \leq \mathcal{H}_T(A,r)$ for all $A \in L^X$.

(3) Put $R^r_T(x,y) = \mathcal{H}_T(\top_x, r)(y)$. Since $A = \bigvee_{x\in X}(A(x) \odot \top_x)$ and $\mathcal{H}_T$ is a join preserving map, we have

$$\begin{aligned}
&\mathcal{H}_T(A,r)(y) = \mathcal{H}_T(\bigvee_{x\in X}(A(x) \odot \top_x), r)(y)\\
&= \bigvee_{x\in X}(A(x) \odot \mathcal{H}_T(\top_x, r)(y)) = \bigvee_{x\in X}(A(x) \odot R^r_T(x,y))
\end{aligned}$$

By (H1), $R^r_T(x,x) = \mathcal{H}_T(\top_x, r)(x) \geq \top_x(x) = \top$.

$$\begin{aligned}
&\bigvee_{y\in X}(R^r_T(x,y) \odot R^r_T(y,z))\\
&= \bigvee_{y\in X}(\mathcal{H}_T(\top_x, r)(y) \odot \mathcal{H}_T(\top_y, r)(z))\\
&= \mathcal{H}_T(\bigvee_{y\in X}(\mathcal{H}_T(\top_x, r)(y) \odot \top_y), r)(z))\\
&= \mathcal{H}_T(\mathcal{H}_T(\top_x, r), r)(z) \leq \mathcal{H}_T(\top_x, r)(z) = R^r_T(x,z).
\end{aligned}$$

Hence R^r_T is a fuzzy preorder.

(4) If $r \leq s$, by (2), $R^r_T(x,y) = \mathcal{H}_T(\top_x, r)(y) \geq R^s_T(x,y) = \mathcal{H}_T(\top_x, s)(y)$.

(5) Let $\mathcal{H}_T(A, r_i) = B$ for all $i \in \Gamma \neq \emptyset$. Since $\mathbf{T}(B) = \mathbf{T}(\mathcal{H}_T(A, r_i)) \geq r_i^*$, then $\mathbf{T}(B) \geq \bigvee_{i\in\Gamma} r_i^* = (\bigwedge_{i\in\Gamma} r_i)^*$. Hence

$$B = \mathcal{H}_T(B, \bigwedge_{i\in\Gamma} r_i) \geq \mathcal{H}_T(A, \bigwedge_{i\in\Gamma} r_i).$$

By (2), $\mathcal{H}_T(A, \bigwedge_{i\in\Gamma} r_i) \geq \mathcal{H}_T(A, r_i) = B$. Thus $\mathcal{H}_T(A, \bigwedge_{i\in\Gamma} r_i) = B$.

(6) Since $\mathbf{T}(A) = \mathbf{T}(\mathcal{H}_T(A, r_i)) \geq r_i^*$, for $\mathcal{H}_T(A, r_i) = A$,

$$\mathbf{T}_{H_T}(A) = \bigvee\{r_i^* \in L \mid \mathcal{H}_T(A, r_i) = A\} \leq \mathbf{T}(A).$$

Since $\mathcal{H}_T(A, \mathbf{T}^*(A)) = A$,

$$\mathbf{T}_{H_T}(A) = \bigvee\{r_i^* \in L \mid \mathcal{H}_T(A, r_i) = A\} \geq \mathbf{T}(A).$$

Hence $\mathbf{T}_{H_T} = \mathbf{T}$.

(7) By (1), since $\mathcal{H}_T(-, r)$ is an upper approximation operator, by Theorem 2.8, there exists an Alexandrov fuzzy topology $\mathbf{T}^r$ such that

$$\mathbf{T}^r(A) = e_{L^X}(\mathcal{H}_T(A, r), A).$$

(8) Since $\mathcal{H}_T(A, s) \leq \mathcal{H}_T(A, r)$ for $r \leq s$, $\mathbf{T}^s(A) = e_{L^X}(\mathcal{H}_T(A, s), A) \geq e_{L^X}(\mathcal{H}_T(A, r), A) = \mathbf{T}^r(A)$.

(9) Since $\mathbf{T}^r(A) = e_{L^X}(\mathcal{H}_T(A, r), A) = \top$ iff $A = \mathcal{H}_T(A, r)$, by (6), the result holds. □

Corollary 3.2. Let $\mathbf{T}$ be an Alexandrov fuzzy topology on X. Define $\mathcal{H}_{T^*} : L^X \times L \to L^X$ as follows

$$\mathcal{H}_{T^*}(A, r) = \bigwedge\{B \in L^X \mid A \leq B,\ \mathbf{T}^*(B) = \mathbf{T}(B^*) \geq r^*\}$$

Then we have the following properties.

(1) $\mathcal{H}_{T^*}(-, r) : L^X \to L^X$ is an upper approximation operator.

(2) If $r \leq s$, then $\mathcal{H}_{T^*}(A, s) \leq \mathcal{H}_{T^*}(A, r)$ for all $A \in L^X$.

(3) There exists a preorder $R^r_{T^*} \in L^{X\times X}$ such that

$$\mathcal{H}_{T^*}(A, r) = \bigvee_{x\in X} (A(x) \odot R^r_{T^*}(x, y)).$$

(4) If $r \leq s$, then $R^r_{T^*} \geq R^s_{T^*}$ for all $A \in L^X$.

(5) If $\mathcal{H}_{T^*}(A, r_i) = B$ for all $i \in \Gamma \neq \emptyset$, then $\mathcal{H}_{T^*}(A, \bigwedge_{i\in\Gamma} r_i) = B$.

(6) Define $\mathbf{T}_{H_{T^*}} : L^X \to L$ as

$$\mathbf{T}_{H_{T^*}}(A) = \bigvee\{r_i^* \in L \mid \mathcal{H}_{T^*}(A, r_i) = A\}$$

Then $\mathbf{T}_{H_{T^*}} = \mathbf{T}^*$ is an Alexandrov fuzzy topology on X.

(7) There exists an Alexandrov fuzzy topology $\mathbf{T}^{*r}$ such that

$$\mathbf{T}^{*r}(A) = e_{L^X}(\mathcal{H}_{T^*}(A, r), A).$$

(8) If $r \leq s$, then $\mathbf{T}^{*r} \leq \mathbf{T}^{*s}$ for all $A \in L^X$.
(9) Define $\mathbf{T}_{T^*} : L^X \to L$ as

$$\mathbf{T}_{T^*}(A) = \bigvee\{r^* \in L \mid \mathbf{T}^{*r}(A) = \top\}.$$

Then $\mathbf{T}_{T^*} = \mathbf{T}^* = \mathbf{T}_{H_{T^*}}$ is an Alexandrov fuzzy topology on X.

Example 3.3. Let $\mathbf{T}$ be an Alexandrov fuzzy topology on X as follows

$$\mathbf{T}(A) = \begin{cases} 1, & \text{if } A = \overline{\alpha}, \\ 0.4, & \text{otherwise.} \end{cases}$$

(1) $\mathcal{H}_T(-, r) : L^X \to L^X$ is an upper approximation operator as follows

$$\mathcal{H}_T(A, r) = \begin{cases} A, & \text{if } 0.6 \leq r \leq 1, \\ \overline{\alpha}, \ \ \alpha = \bigvee_{x \in X} A(x), & \text{if } 0 \leq r < 0.6. \end{cases}$$

(2) $\mathcal{H}_T(A, r) = \bigvee_{x \in X}(R_T^r(x, y) \odot A(x))$ with $R_T^r \in L^{X \times X}$ as follows

$$R_T^r(x, y) = \mathcal{H}_T(1_x, r)(y) = \begin{cases} 1_x(y), & \text{if } 0.6 \leq r \leq 1, \\ \overline{1}, \ \ 1 = \bigvee_{z \in X} 1_x(z), & \text{if } 0 \leq r < 0.6. \end{cases}$$

(3) $\mathbf{T}^r$ is an Alexandrov fuzzy topology on X as follows

$$\mathbf{T}^r(A) = \begin{cases} \mathbf{T}_1(A), & \text{if } 0.6 \leq r \leq 1, \\ \mathbf{T}_2(A), & \text{if } 0 \leq r < 0.6, \end{cases}$$

where

$$\mathbf{T}_1(A) = 1, \ \forall A \in L^X, \ \ \mathbf{T}_2(A) = \begin{cases} 1, & \text{if } A = \overline{\alpha}, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, $\mathbf{T}_T(A) = \bigvee\{r^* \in L \mid \mathbf{T}^r(A) = 1\} = \mathbf{T}_{M_T}(A) = \mathbf{T}(A) = \mathbf{T}^*(A)$ for all $A \in L^X$. □

Example 3.4. Let $(L = [0, 1], \odot, \to, ^*)$ be a complete residuated lattice with a strong negation defined by

$$a \odot b = (a + b - 1) \vee 0, \ \ a \to b = (1 - a + b) \wedge 1, \ a^* = 1 - a.$$

(1) Let $X = \{x, y, z\}$ be a set. Define a map $\mathbf{T} : [0, 1]^X \to [0, 1]$ as

$$\mathbf{T}(A) = (1 - A(x) + A(z)) \wedge 1 = A(x) \to A(z).$$

Trivially, $\mathbf{T}(\overline{\alpha}) = \alpha$

Since $\alpha \odot A(x) \to \alpha \odot A(z) \geq A(x) \to A(z)$, $\mathbf{T}(\alpha \odot A) \geq \mathbf{T}(A)$. Since $(\alpha \to A(x)) \to (\alpha \to A(z)) \geq A(x) \to A(z)$, $\mathbf{T}(\alpha \to A) \geq \mathbf{T}(A)$. By Lemma 2.4 (8), $\mathbf{T}(\bigvee_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i)$ and $\mathbf{T}(\bigwedge_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i)$. Hence $\mathbf{T}$ is an Alexandrov fuzzy topology.

By Theorem 2.8(1), we obtain an upper approximation operator $\mathcal{H}_T(-,r) : L^X \to L^X$ as follows:

$$\mathcal{H}_T(1_x, r)(z) = \bigwedge\{B(z) \mid B \geq 1_x,\ \mathbf{T}(B) \geq r^*\}$$

Since $B(x) = 1$ and $\mathbf{T}(B) = 1 - 1 + B(z) \geq 1 - r$, then $B(z) \geq 1 - r$. We have $\mathcal{H}_T(1_x, r)(z) = 1 - r$.

$$\mathcal{H}_T(1_x, r)(x) = \bigwedge\{B(x) \mid B \geq 1_x,\ \mathbf{T}(B) \geq r^*\} = 1$$

$$\mathcal{H}_T(1_x, r)(y) = \bigwedge\{B(y) \mid B \geq 1_x,\ \mathbf{T}(B) \geq r^*\} = 0$$

$$\mathcal{H}_T(1_z, r)(x) = \bigwedge\{B(x) \mid B \geq 1_z,\ \mathbf{T}(B) \geq r^*\}$$

Since $B(z) = 1$ and $\mathbf{T}(B) = (1 - B(x) + 1) \wedge 1 = 1$, then $\mathcal{H}_T(1_z, r)(x) = 0$.

$$\begin{pmatrix} \mathcal{H}_T(1_x, r)(x) = 1 & \mathcal{H}_T(1_x, r)(y) = 0 & \mathcal{H}_T(1_x, r)(z) = 1 - r \\ \mathcal{H}_T(1_y, r)(x) = 0 & \mathcal{H}_T(1_y, r)(y) = 1 & \mathcal{H}_T(1_y, r)(z) = 0 \\ \mathcal{H}_T(1_z, r)(x) = 0 & \mathcal{H}_T(1_z, r)(y) = 0 & \mathcal{H}_T(1_z, r)(z) = 1 \end{pmatrix}$$

For $A = \bigvee_{x\in X}(A(x) \odot \top_x)$, we have

$$\mathcal{H}_T(A, r)(y) = \bigvee_{x\in X} (A(x) \odot \mathcal{H}_T(\top_x, r)(y)).$$

$$\mathcal{H}_T(A, r) = (A(x), A(y), A(z) \vee (A(x) - r))$$

If $A(x) - r \leq A(z)$, then $\mathcal{H}_T(A, r) = A$. Thus

$$\begin{aligned} \mathbf{T}_{H_T}(A) &= \bigvee\{r^* \in L \mid \mathcal{H}_T(A, r) = A\} \\ &= (1 - A(x) + A(z)) \wedge 1 = \mathbf{T}(A). \end{aligned}$$

Moreover, we obtain

$$\begin{aligned} \mathbf{T}^r(A) &= \bigwedge\nolimits_{x\in X}(\mathcal{H}_T(A, r)(x) \to A(x)) \\ &= A(z) \vee (A(x) - r) \to A(z) \\ &= (A(z) \to A(z)) \wedge ((A(x) - r) \to A(z)) \\ &= (1 + r - A(x) + A(z)) \vee 0. \end{aligned}$$

$$\begin{aligned}\mathbf{T}_T(A) &= \bigvee\{1-r \in L \mid \mathbf{T}^r(A)=1\}\\ &= (1-A(x)+A(z))\wedge 1.\end{aligned}$$

Hence $\mathbf{T}_T = \mathbf{T}_{H_T} = \mathbf{T}$.

Since $R^r_T(x,y) = \mathcal{H}_T(1_x, r)(y)$, then $\mathcal{H}_T(A,r)(y) = \bigvee_{x\in X}(A(x)\odot R^r_T(x,y))$ with

$$R^r_T = \begin{pmatrix} 1 & 0 & 1-r\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix}$$

(2) By (1), we obtain a map $\mathbf{T}^*: [0,1]^Y \to [0,1]$ as

$$\mathbf{T}^*(A) = (1-A^*(x)+A^*(z))\wedge 1 = (1-A(z)+A(x))\wedge 1.$$

We obtain a join preserving map $\mathcal{H}_{T^*}(-,r): L^X \to L^X$ as follows:

$$\mathcal{H}_{T^*}(1_x, r)(z) = \bigwedge\{B(z)\in L^X \mid B \geq 1_x,\ \mathbf{T}^*(B)\geq r^*\}$$

Since $B(x)=1$ and $\mathbf{T}^*(B) = (1-B(z)+1)\wedge 1 = 1$, then $\mathcal{H}_{T^*}(1_x,r)(z)=0$.

$$\mathcal{H}_{T^*}(1_z, r)(y) = \bigwedge\{B(y)\in L^X \mid B \geq 1_z,\ \mathbf{T}^*(B)\geq r^*\} = 0$$

$$\mathcal{H}_{T^*}(1_y, r)(y) = \bigwedge\{B(y)\in L^X \mid B \geq 1_y,\ \mathbf{T}^*(B)\geq r^*\} = 1$$

$$\mathcal{H}_{T^*}(1_z, r)(x) = \bigwedge\{B(x)\in L^X \mid B \geq 1_z,\ \mathbf{T}^*(B)\geq r^*\}$$

Since $B(z)=1$ and $\mathbf{T}^*(B) = 1-1+B(x) \geq 1-r$, then $B(x)\geq 1-r$. We have $\mathcal{H}_{T^*}(1_z,r)(x) = 1-r$.

$$\begin{pmatrix} \mathcal{H}_{T^*}(1_x,r)(x)=1 & \mathcal{H}_{T^*}(1_x,r)(y)=0 & \mathcal{H}_{T^*}(1_x,r)(z)=0\\ \mathcal{H}_{T^*}(1_y,r)(x)=0 & \mathcal{H}_{T^*}(1_y,r)(y)=1 & \mathcal{H}_{T^*}(1_y,r)(z)=0\\ \mathcal{H}_{T^*}(1_z,r)(x)=1-r & \mathcal{H}_{T^*}(1_z,r)(y)=0 & \mathcal{H}_{T^*}(1_z,r)(z)=1 \end{pmatrix}$$

For $A = \bigvee_{x\in X}(A(x)\odot \top_x)$, we have

$$\mathcal{H}_{T^*}(A,r)(y) = \bigvee_{x\in X}(A(x)\odot \mathcal{H}_{T^*}(\top_x, r)(y)).$$

$$\mathcal{H}_{T^*}(A,r) = (A(x)\vee(A(z)-r), A(y), A(z))$$

If $A(z)-r\leq A(x)$, then $\mathcal{H}_{T^*}(A,r)=A$. Thus

$$\begin{aligned}\mathbf{T}_{H_{T^*}}(A) &= \bigvee\{r^*\in L \mid \mathcal{H}_{T^*}(A,r)=A\}\\ &= (1-A(z)+A(x))\wedge 1 = \mathbf{T}^*(A).\end{aligned}$$

Moreover, we obtain

$$\begin{array}{ll}\mathbf{T}^{*r}(A) & = \bigvee_{x\in X}(\mathcal{H}_{T^*}(A,r)(x) \to A(x)) \\ & = (A(x) \vee (A(z)-r)) \to A(x) \\ & = (1+r-A(z)+A(x)) \wedge 1.\end{array}$$

For $B(x)=0.9, B(y)=0.3, B(z)=0.2$, $\mathbf{T}^{*0.5}(B)=(1+0.5-B(z)+B(x))\vee 0 = 1$.

$$\begin{array}{ll}\mathbf{T}_{T^*}(A) & = \bigvee\{1-r\in L \mid \mathbf{T}^{*r}(A)=1\} \\ & = (1-A(z)+A(x)) \wedge 1.\end{array}$$

Hence $\mathbf{T}_{T^*} = \mathbf{T}_{H_{T^*}} = \mathbf{T}^*$.

Since $R^r_{T^*}(x,y) = \mathcal{H}_{T^*}(1_x,r)(y)$, then $\mathcal{H}_{T^*}(A,r)(y) = \bigvee_{x\in X}(A(x)\odot R^r_{T^*}(x,y))$ with

$$R^r_{T^*} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1-r & 0 & 1 \end{pmatrix}$$

□

References

[1] R. Bělohlávek,*Fuzzy Relational Systems*, Kluwer Academic Publishers, New York , (2002), **doi:** 10.1007/978-1-4615-0633-1.

[2] P. Hájek, *Metamathematices of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht (1998), **doi:** 10.1007/978-94-011-5300-3.

[3] U. Höhle, S.E. Rodabaugh,(1999), *Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, The Handbooks of Fuzzy Sets Series 3*, Kluwer Academic Publishers, Boston.

[4] Fang Jinming, I-fuzzy Alexandrov topologies and specialization orders, *Fuzzy Sets and Systems,* **158**(2007), 2359-2374, **doi:** 10.1016/j.fss.2007.05.001.

[5] Y.C. Kim, Alexandrov L-topologies and L-join meet approximation operators *International Journal of Pure and Applied Mathematics,* **91**(1)(2014), 113-129 **doi:** 10.12732/ijpam.v9lili.12.

[6] Y.C. Kim, Join preserving maps, fuzzy preorders and Alexandrov fuzzy topologies, submit to International Journal of Pure and Applied Mathematics

[7] J. Kortelainen, On relationships between modified sets, topological spaces and rough sets, *Fuzzy Sets and Systems,* **61**(1994), 91-95, **doi:** 10.1016/0165-0114(94)90288-7.

[8] H. Lai, D. Zhang, Fuzzy preorder and fuzzy topology, *Fuzzy Sets and Systems,* **157** (2006), 1865-1885, **doi:** 10.1016/j.fss.2006.02.013.

[9] H. Lai, D. Zhang, Concept lattices of fuzzy contexts: Formal concept analysis vs. rough set theory, *Int. J. Approx. Reasoning,* **50** (2009), 695-707, **doi:** 10.1016/j.ijar.2008.12.002.

[10] Z. Pawlak, Rough sets, *Int. J. Comput. Inf. Sci.*, **11** (1982), 341-356, , **doi:** 10.1007/BF01001956.

[11] Z. Pawlak, Rough probability, *Bull. Pol. Acad. Sci. Math.*, **32**(1984), 607-615.

[12] A. M. Radzikowska, E.E. Kerre, A comparative study of fuzy rough sets, *Fuzzy Sets and Systems*, **126**(2002), 137-155, **doi:** 10.1016/so165-0114(01)00032-x.

[13] S. P. Tiwari, A.K. Srivastava, Fuzzy rough sets, fuzzy preorders and fuzzy topologies, *Fuzzy Sets and Systems*, **210**(2013), 63-68, **doi:** 10.1016/j.fss.2012.06.001.

[14] Y.H. She, G.J. Wang, An axiomatic approach of fuzzy rough sets based on residuated lattices, *Computers and Mathematics with Applications*, **58** (2009), 189-201, **doi:** 10.1016/j.camwa.2009.03.100.

[15] Zhen Ming Ma, Bao Qing Hu, Topological and lattice structures of L-fuzzy rough set determined by lower and upper sets, *Information Sciences*, **218**(2013), 194-204, **doi:** 10.1016/j.ins.2012.06.029.