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UPPER APPROXIMATION OPERATORS INDUCED BY MAPS

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Abstract: In this paper, we investigate the properties of Alexandrov fuzzy topologies and upper approximation operators induced by maps. We give their examples.

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1. Introduction

Höhle [3] introduced L-fuzzy topologies and L-fuzzy interior operators. The relationship between rough set theory and topological spaces was investigated [4-12]. Hájek [2] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Kim [5-7] investigated the properties of join (resp. meet, meet join, join meet) preserving operators and Alexandrov fuzzy topology in complete residuated lattices.

In this paper, we investigate the properties of Alexandrov fuzzy topologies and upper approximation operators induced by maps. We give their examples.

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2. Preliminaries

Definition 2.1. [1-3] A structure $(L, \vee, \wedge, \odot, \rightarrow, \bot, \top)$ is called a *complete residuated lattice* iff it satisfies the following properties:

(L1) $(L, \vee, \wedge, \bot, \top)$ is a complete lattice where $\bot$ is the bottom element and $\top$ is the top element;

(L2) $(L, \odot, \top)$ is a monoid;

(L3) It has an adjointness,i.e.

$$x \leq y \rightarrow z \text{ iff } x \odot y \leq z.$$

An operator $^* : L \rightarrow L$ defined by $a^* = a \rightarrow \bot$ is called a *strong negation* if $a^{**} = a$.

$$\top_x(y) = \begin{cases} \top, & \text{if } y = x, \\ \bot, & \text{otherwise.} \end{cases} \quad \top_x^*(y) = \begin{cases} \bot, & \text{if } y = x, \\ \top, & \text{otherwise.} \end{cases}$$

In this paper, we assume that $(L, \vee, \wedge, \odot, \rightarrow, ^*, \bot, \top)$ be a complete residuated lattice with a strong negation *.

Definition 2.2. [6,9] Let X be a set. A function $e_X : X \times X \rightarrow L$ is called:

(E1) reflexive if $e_X(x, x) = 1$ for all $x \in X$,

(E2) transitive if $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$, for all $x, y, z \in X$,

(E3) if $e_X(x, y) = e_X(y, x) = 1$, then $x = y$.

If e satisfies (E1) and (E2), (X, e_X) is a fuzzy preorder set. If e satisfies (E1), (E2) and (E3), (X, e_X) is a fuzzy partially order set (simply, fuzzy poset).

Example 2.3. (1) We define a function $e_L : L \times L \rightarrow L$ as $e_L(x, y) = x \rightarrow y$. Then (L, e_L) is a fuzzy poset.

(2) We define a function $e_{L^X} : L^X \times L^X \rightarrow L$ as $e_{L^X}(A, B) = \bigwedge_{x \in X}(A(x) \rightarrow B(x))$. Then (L^X, e_{L^X}) is a fuzzy poset from Lemma 2.4 (9).

Lemma 2.4. [1,2] Let $(L, \vee, \wedge, \odot, \rightarrow, ^*, \bot, \top)$ be a complete residuated lattice with a strong negation *. For each $x, y, z, x_i, y_i \in L$, the following properties hold.

(1) If $y \leq z$, then $x \odot y \leq x \odot z$.

(2) If $y \leq z$, then $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$.

(3) $x \rightarrow y = \top$ iff $x \leq y$.

(4) $x \rightarrow \top = \top$ and $\top \rightarrow x = x$.

(5) $x \odot y \leq x \wedge y$.

(6) $x \odot (\bigvee_{i \in \Gamma} y_i) = \bigvee_{i \in \Gamma}(x \odot y_i)$ and $(\bigvee_{i \in \Gamma} x_i) \odot y = \bigvee_{i \in \Gamma}(x_i \odot y)$.

(7) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.

(8) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$ and $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$.

(9) $(x \to y) \odot x \leq y$ and $(y \to z) \odot (x \to y) \leq (x \to z)$.

(10) $x \to y \leq (y \to z) \to (x \to z)$ and $x \to y \leq (z \to x) \to (z \to y)$.

(11) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.

(12) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$ and $(x \odot y)^* = x \to y^*$.

(13) $x^* \to y^* = y \to x$ and $(x \to y)^* = x \odot y^*$.

(14) $y \to z \leq x \odot y \to x \odot z$.

Definition 2.5. [5-7] A map $\mathcal{H} : L^X \to L^Y$ is called a *join preserving map* if it satisfies the following conditions, for all $A, A_i \in L^X$, and $\alpha \in L$,

(J1) $\mathcal{H}(\alpha \odot A) = \alpha \odot \mathcal{H}(A)$,

(J2) $\mathcal{H}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} \mathcal{H}(A_i)$.

A join preserving operator $\mathcal{H} : L^X \to L^X$ is called an *upper approximation operator* iff it satisfies the following conditions

(H1) $A \leq \mathcal{H}(A)$,

(H2) $\mathcal{H}(\mathcal{H}(A)) \leq \mathcal{H}(A)$, for all $A \in L^X$.

Definition 2.6. [6,7] An operator $\mathbf{T} : L^X \to L$ is called an *Alexandrov fuzzy topology* on X iff it satisfies the following conditions:

(T1) $\mathbf{T}(\alpha) = \top$,

(T2) $\mathbf{T}(\bigwedge_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i)$ and $\mathbf{T}(\bigvee_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i)$,

(T3) $\mathbf{T}(\alpha \odot A) \geq \mathbf{T}(A)$,

(T4) $\mathbf{T}(\alpha \to A) \geq \mathbf{T}(A)$.

Theorem 2.7. [6,7] Let $\mathbf{T} : L^X \to L$ be an Alexandrov fuzzy topology. Define $\mathbf{T}^*(A) = \mathbf{T}(A^*)$. Then $\mathbf{T}^*$ is an Alexandrov fuzzy topology.

Theorem 2.8. [6,7] Let $\mathcal{H}$ be a join preserving map. Define $\mathbf{T}_\mathcal{H} : L^X \to L$ as

$$\mathbf{T}_\mathcal{H}(A) = \bigwedge_{x\in X} (\mathcal{H}(A)(x) \to A(x)) = e_{L^X}(\mathcal{H}(A), A).$$

Then we have the following properties.

(1) $\mathbf{T}_\mathcal{H}$ is an Alexandrov fuzzy topology on X.

(2) $\mathbf{T}_\mathcal{H}(A) = \bigwedge_{x,y\in X}(\mathcal{H}(\top_x)(y) \to (A(x) \to A(y))$ such that $\mathbf{T}_\mathcal{H}(A) \geq \bigwedge_{x\neq y\in X} \mathcal{H}(\top_x)(y)$.

(3) If $\mathcal{H}$ is an upper approximation operator, then $\mathbf{T}_\mathcal{H}(\mathcal{H}(\top_x)) = \top$.

(4) If $\mathcal{H}^{-1}$ is a join preserving map such that $\mathcal{H}^{-1}(\top_x)(y) = \mathcal{H}(\top_y)(x)$ for all $x, y \in X$. Define $\mathbf{T}^*_{\mathcal{H}}(A) = \mathbf{T}_{\mathcal{H}}(A^*)$. Then $\mathbf{T}^*_{\mathcal{H}} = \mathbf{T}_{H^{-1}}$ is an Alexandrov fuzzy topology.

(5) If $\mathcal{H}$ is an upper approximation operator, then $\mathcal{H}^{-1}$ is an upper approximation operator such that

$$\mathbf{T}_{\mathcal{H}}(\mathcal{H}^{-1*}(\top_x)) = \mathbf{T}_{H^{-1}}(\mathcal{H}^*(\top_x)) = \top.$$

3. Upper Approximation Operators Induced by Maps

Theorem 3.1 Let $\mathcal{H}_Y$ and $\mathcal{H}_Y^{-1}$ be upper approximation operators Y with $\mathcal{H}_Y^{-1}(\top_x)(y) = \mathcal{H}_Y(\top_y)(x)$ for all $x, y \in Y$. and $f : X \to Y$ be a map. Define $\mathcal{H}_X, \mathcal{H}_X^{-1} : L^X \to L^X$ as follows:

$$\mathcal{H}_X(A)(y) = \bigvee_{x \in X} (A(x) \odot \mathcal{H}_Y(\top_{f(x)})(f(y))),$$

$$\mathcal{H}_X^{-1}(A)(y) = \bigvee_{x \in X} (A(x) \odot \mathcal{H}_Y^{-1}(\top_{f(x)})(f(y))).$$

Then the following statements hold.

(1) $\mathcal{H}_X(\top_x)(y) = \mathcal{H}_Y(\top_{f(x)})(f(y)) = f^{-1}(\mathcal{H}_Y(\top_{f(x)}))(y) =$ for all $x, y \in X$.

(2) $\mathcal{H}_X^{-1}(\top_x)(y) = \mathcal{H}_X(\top_y)(x)$ for all $x, y \in X$.

(3) $\mathcal{H}_X$ and $\mathcal{H}_X^{-1}$ are upper approximation operators on X.

(4) There exist fuzzy preorders R_X, R_Y with $R_X(x, z) = \mathcal{H}_X(\top_x)(z)$ and $R_Y(y, w) = \mathcal{H}_X(\top_y)(w)$ such that

$$R_X(x, z) = R_Y(f(x), f(z)), \forall x, z \in X.$$

(5) There exist fuzzy preorders R_X^{-1}, R_Y^{-1} with $R_X^{-1}(x, z) = \mathcal{H}_X^{-1}(\top_x)(z) = R_X(z, x)$, $R_Y^{-1}(y, w) = \mathcal{H}_Y^{-1}(\top_y)(w) = R_Y(w, y)$ such that

$$R_X^{-1}(x, z) = R_Y^{-1}(f(x), f(z)), \forall x, z \in X.$$

(6) $f^{-1}(\mathcal{H}_Y(B)) \geq \mathcal{H}_X(f^{-1}(B))$ for all $B \in L^X$. If f is a surjective function, then the equality holds.

(7) $f^{-1}(\mathcal{H}_Y^{-1}(B)) \geq \mathcal{H}_X^{-1}(f^{-1}(B))$ for all $B \in L^X$. If f is a surjective function, then the equality holds.

(8) $\mathbf{T}_{\mathcal{H}_X}$ and $\mathbf{T}_{\mathcal{H}_X^{-1}}$ are Alexandrov fuzzy topologies on X such that

$$\begin{aligned}\mathbf{T}_{\mathcal{H}_X}(A) &= \bigwedge_{x\in X}(\mathcal{H}_X(A)(x)\to A(x))\\ &= \bigwedge_{x,y\in X}(\mathcal{H}_Y(\top_{f(y)})(f(x))\to(A(y)\to A(x))\\ \mathbf{T}_{\mathcal{H}_X^{-1}}(A) &= \bigwedge_{x\in X}(\mathcal{H}_X^{-1}(A)(x)\to A(x))\\ &= \bigwedge_{x,y\in X}(\mathcal{H}_Y^{-1}(\top_{f(y)})(f(x))\to(A(y)\to A(x)).\end{aligned}$$

Moreover, $\mathbf{T}_{\mathcal{H}_X^{-1}}(A)=\mathbf{T}_{\mathcal{H}_X}(A^*)$ for all $A\in L^X$.

(9) $\mathbf{T}_{\mathcal{H}_X}(f^{-1}(B))\geq\mathbf{T}_{\mathcal{H}_Y}(B)$ for all $B\in L^Y$. If f is a surjective function, then the equality holds.

(10) $\mathbf{T}_{\mathcal{H}_X^{-1}}(f^{-1}(B))\geq\mathbf{T}_{\mathcal{H}_Y^{-1}}(B)$ for all $B\in L^Y$. If f is a surjective function, then the equality holds.

(11)

$$\begin{aligned}\mathcal{H}_X(\top_x)(y) &= \bigwedge_{A\in L^X}(\mathbf{T}_{\mathcal{H}_X}(A)\to(A(x)\to A(y))\\ &= \bigwedge_{A\in L^X}(\mathbf{T}_{\mathcal{H}_X^{-1}}(A)\to(A(y)\to A(x))\\ &= \bigwedge_{z\in X}(\mathcal{H}_X(\top_z)(x)\to\mathcal{H}_X(\top_z)(y))\\ &= \bigwedge_{z\in X}(\mathcal{H}_X^{-1*}(\top_z)(x)\to\mathcal{H}_X^{-1*}(\top_z)(y))\\ &= \bigwedge_{z\in X}(\mathcal{H}_X^{-1}(\top_z)(y)\to\mathcal{H}_X^{-1}(\top_z)(x))\\ &= \bigwedge_{z\in X}(\mathcal{H}_X^{*}(\top_z)(y)\to\mathcal{H}_X^{*}(\top_z)(x))\end{aligned}$$

(12)

$$\begin{aligned}&f^{-1}(\mathcal{H}_Y(\top_{f(x)}))(z)\\ &= \bigwedge_{B\in L^Y}(\mathbf{T}_{\mathcal{H}_Y}(B)\to(B(f(z))\to B(f(x))))\\ &= \bigwedge_{B\in L^Y}(\mathbf{T}_{\mathcal{H}_Y^{-1}}(B)\to(B(f(x))\to B(f(z))))\\ &= \bigwedge_{B\in L^Y}(\mathbf{T}_{\mathcal{H}_X}(f^{-1}(B))\to(f^{-1}(B)(z)\to f^{-1}(B)(x)))\\ &= \bigwedge_{B\in L^Y}(\mathbf{T}_{\mathcal{H}_X^{-1}}(f^{-1}(B))\to(f^{-1}(B)(x)\to f^{-1}(B)(z)))\\ &= \bigwedge_{A\in L^X}(\mathbf{T}_{\mathcal{H}_X}(A)\to(A(z)\to A(x)))\\ &= \bigwedge_{A\in L^X}(\mathbf{T}_{\mathcal{H}_X^{1}}(A)\to(A(x)\to A(z)))\\ &= \mathcal{H}_X(\top_x)(z).\end{aligned}$$

Proof. (1) $\mathcal{H}_X(\top_x)(y)=\bigvee_{z\in X}(\top_x(z)\odot\mathcal{H}_Y(\top_{f(z)})(f(y)))=\mathcal{H}_Y(\top_{f(x)})(f(y))$.

(2) $\mathcal{H}_X^{-1}(\top_x)(y)=\mathcal{H}_Y^{-1}(\top_{f(x)})(f(y))=f^{-1}(\mathcal{H}_Y^{-1}(\top_{f(x)}))(y)$
$=\mathcal{H}_Y(\top_{f(y)})(f(x))=f^{-1}(\mathcal{H}_Y(\top_{f(y)}))(x)=\mathcal{H}_X(\top_y)(x)$ for all $x,y\in X$.

(3) $\mathcal{H}_X$ is an upper approximation operator from the following statements.

$$\begin{aligned}\mathcal{H}_X(\alpha\odot A)(y) &= \bigvee_{x\in X}((\alpha\odot A)(x)\odot\mathcal{H}_Y(\top_{f(x)})(f(y))\\ &= \alpha\odot\bigvee_{x\in X}(A(x)\odot\mathcal{H}_Y(\top_{f(x)})(f(y))\\ &= \alpha\odot\mathcal{H}_X(A)(y).\end{aligned}$$

$$\begin{aligned}\mathcal{H}_X(\bigvee_{i\in\Gamma} A_i)(y) &= \bigvee_{x\in X}((\bigvee_{i\in\Gamma} A_i)(x) \odot \mathcal{H}_Y(\top_{f(x)})(f(y))\\ &= \bigvee_{i\in\Gamma}\bigvee_{x\in X}(A_i \odot \mathcal{H}_Y(\top_{f(x)})(f(y))\\ &= \bigvee_{i\in\Gamma}\mathcal{H}_X(A_i)(y).\end{aligned}$$

$$\begin{aligned}\mathcal{H}_X(A)(y) &= \bigvee_{x\in X}(A(x) \odot \mathcal{H}_Y(\top_{f(x)})(f(y)))\\ &\geq \bigvee_{x\in X}(A(x) \odot \top_{f(x)}(f(y))\\ &= A(x) \odot \top_{f(x)}(f(x)) = A(x).\end{aligned}$$

$$\begin{aligned}&\mathcal{H}_X(\mathcal{H}_X(A))(y) = \bigvee_{x\in X}(\mathcal{H}_X(A)(x) \odot \mathcal{H}_Y(\top_{f(x)})(f(y)))\\ &= \bigvee_{x\in X}(\bigvee_{z\in X}(A(z) \odot \mathcal{H}_Y(\top_{f(z)})(f(x))) \odot \mathcal{H}_Y(\top_{f(x)})(f(y)))\\ &= \bigvee_{z\in X}(A(z) \odot \bigvee_{x\in X}(\mathcal{H}_Y(\top_{f(z)})(f(x)) \odot \mathcal{H}_Y(\top_{f(x)})(f(y)))\\ &\leq \bigvee_{z\in X}(A(z) \odot \bigvee_{y\in Y}(\mathcal{H}_Y(\top_{f(z)})(y) \odot \mathcal{H}_Y(\top_y)(f(y)))\\ &= \bigwedge_{z\in X}(A(z) \odot \mathcal{H}_Y(\mathcal{H}_Y(\top_{f(x)})(f(y)))\\ &\leq \bigwedge_{z\in X}(A(z) \odot \mathcal{H}_Y(\top_{f(x)})(f(y))) = \mathcal{H}_X(A)(y).\end{aligned}$$

Similarly, $\mathcal{H}_X^{-1}$ is an upper approximation operator.

(4) Put $R_X(x,y) = \mathcal{H}_X(\top_x)(y)$. By (1), since $\mathcal{H}_X$ is an upper approximation operator, we have

$$R_X(x,x) = \mathcal{H}_X(\top_x)(x) \geq \top_x(x) = \top$$

$$\begin{aligned}\bigvee_{y\in X}(R_X(x,y) \odot R_X(y,z)) &= \bigvee_{y\in X}(\mathcal{H}_X(\top_x)(y) \odot \mathcal{H}_X(\top_y)(z))\\ &= \mathcal{H}_X(\bigvee_{y\in X}(\mathcal{H}_X(\top_x)(y) \odot \top_y))(z)\\ &= \mathcal{H}_X(\mathcal{H}_X(\top_x))(z)\\ &\leq \mathcal{H}_X(\top_x)(z) = R_X(x,z).\end{aligned}$$

(6) For $B = \bigvee_{y\in Y}(B(y) \odot \top_y)$, we have

$$\begin{aligned}f^{-1}(\mathcal{H}_Y(B))(z) &= \mathcal{H}_Y(\bigvee_{y\in Y}(B(y) \odot \top_y))(f(z))\\ &= \bigvee_{y\in Y}(B(y) \odot \mathcal{H}_Y(\top_y)(f(z))\\ &\geq \bigvee_{x\in X}(B(f(x)) \odot \mathcal{H}_Y(\top_{f(x)})(f(z))\\ &= \bigvee_{x\in X}(f^{-1}(B)(x) \odot \mathcal{H}_X(\top_x)(z))\\ &= \mathcal{H}_X(\bigvee_{x\in X}(f^{-1}(B)(x) \to \top_x)(z))\\ &= \mathcal{H}_X(f^{-1}(B))(z).\end{aligned}$$

If f is surjective, then

$$\begin{aligned}f^{-1}(\mathcal{H}_Y(B))(z) &= \bigvee_{y\in Y}(B(y) \odot \mathcal{H}_Y(\top_y)(f(z))\\ &= \bigvee_{x\in X}(B(f(x)) \odot \mathcal{H}_Y(\top_{f(x)})(f(z))\\ &= \bigvee_{x\in X}(f^{-1}(B)(x) \odot \mathcal{H}_X(\top_x)(z))\\ &= \mathcal{H}_X(f^{-1}(B))(z).\end{aligned}$$

(8)

$$\begin{aligned}\mathbf{T}_{\mathcal{H}_X}(A) &= \bigwedge_{x\in X}(\mathcal{H}_X(A)(x) \to A(x))\\ &= \bigwedge_{x\in X}(\bigvee_{y\in X}(A(y) \odot \mathcal{H}_Y(\top_{f(y)})(f(x)) \to A(x))\\ &= \bigwedge_{x,y\in X}(\mathcal{H}_Y(\top_{f(y)})(f(x)) \to (A(y) \to A(x)).\end{aligned}$$

$$\begin{aligned}\mathbf{T}_{\mathcal{H}_X^{-1}}(A) &= \bigwedge_{x,y\in X}(\mathcal{H}_Y^{-1}(\top_{f(y)})(f(x)) \to (A(y) \to A(x))\\ &= \bigwedge_{x,y\in X}(\mathcal{H}_Y(\top_{f(x)})(f(y)) \to (A^*(x) \to A^*(y))\\ &= \mathbf{T}_{\mathcal{H}_X}(A^*).\end{aligned}$$

(9)

$$\begin{aligned}\mathbf{T}_{\mathcal{H}_X}(f^{-1}(B)) &= \bigwedge_{x\in X}(\mathcal{H}_X(f^{-1}(B))(x) \to f^{-1}(B)(x))\\ &\geq \bigwedge_{x\in X}(f^{-1}(\mathcal{H}_Y(B))(x) \to B(f(x)))\\ &\geq \bigwedge_{y\in Y}(\mathcal{H}_Y(B)(y) \to B(y))\\ &= \mathbf{T}_{\mathcal{H}_Y}(B).\end{aligned}$$

If f is surjective, then

$$\begin{aligned}\mathbf{T}_{\mathcal{H}_X}(f^{-1}(B)) &= \bigwedge_{x\in X}(\mathcal{H}_X(f^{-1}(B))(x) \to f^{-1}(B)(x))\\ &= \bigwedge_{x\in X}(f^{-1}(\mathcal{H}_Y(B))(x) \to B(f(x)))\\ &= \bigwedge_{y\in Y}(\mathcal{H}_Y(B)(y) \to B(y))\\ &= \mathbf{T}_{\mathcal{H}_Y}(B).\end{aligned}$$

(11) Since $a \leq (a \to b) \to b$, by Theorem 2.8(2), we have

$$\begin{aligned}\mathcal{H}_X(\top_x)(y) &\leq \bigwedge_{A\in L^X}((\mathcal{H}_X(\top_x)(y) \to (A(x) \to A(y)) \to (A(x) \to A(y)))\\ &\leq \bigwedge_{A\in L^X}(\bigwedge_{s,t}((\mathcal{H}_X(\top_s)(t) \to (A(s) \to A(t))) \to (A(x) \to A(y)))\\ &= \bigwedge_{A\in L^X}(\mathbf{T}_{\mathcal{H}_X}(A) \to (A(x) \to A(y)))\end{aligned}$$

$$\mathcal{H}_X(\top_x)(y) = \mathcal{H}_X^{-1}(\top_y)(x) \quad \leq \bigwedge_{A\in L^X}(\mathbf{T}_{\mathcal{H}_X^{-1}}(A) \to (A(y) \to A(x))).$$

Since

$$\begin{aligned}&\mathbf{T}_{\mathcal{H}_X}(\mathcal{H}(\top_x)) = \mathbf{T}_{\mathcal{H}_X}(\mathcal{H}^{-1*}(\top_x))\\ &= \mathbf{T}_{\mathcal{H}_X^{-1}}(\mathcal{H}^*(\top_x)) = \mathbf{T}_{\mathcal{H}_X^{-1}}(\mathcal{H}^{-1}(\top_x)) = \top,\end{aligned}$$

$$\begin{aligned}\mathcal{H}_X(\top_x)(y) &\leq \bigwedge_{A\in L^X}(\mathbf{T}_{\mathcal{H}_X}(A) \to (A(x) \to A(y)))\\ &\leq \bigwedge_{z\in X}(\mathbf{T}_{\mathcal{H}_X}(\mathcal{H}(\top_z)) \to (\mathcal{H}_X(\top_z)(x) \to \mathcal{H}_X(\top_z)(y)))\\ &= \bigwedge_{z\in X}(\mathcal{H}_X(\top_z)(x) \to \mathcal{H}_X(\top_z)(y))\\ &\leq \top_x(x) \to \mathcal{H}_X(\top_x)(y) = \mathcal{H}_X(\top_x)(y),\end{aligned}$$

$$\begin{aligned}\mathcal{H}_X(\top_x)(y) &\leq \bigwedge_{A\in L^X}(\mathbf{T}_{\mathcal{H}_X}(A)\to(A(x)\to A(y)))\\ &\leq \bigwedge_{z\in X}(\mathbf{T}_{\mathcal{H}_X}(\mathcal{H}^{-1*}(\top_z))\to(\mathcal{H}_X^{-1*}(\top_z)(x)\to\mathcal{H}_X^{-1*}(\top_z)(y)))\\ &= \bigwedge_{z\in X}(\mathcal{H}_X^{-1*}(\top_z)(x)\to\mathcal{H}_X^{-1*}(\top_z)(y))\\ &\leq \mathcal{H}_X^{-1*}(\top_y)(x)\to\top_y^{**}(y)\\ &= \mathcal{H}_X^{-1**}(\top_y)(x)=\mathcal{H}_X(\top_x)(y),\end{aligned}$$

$$\begin{aligned}\mathcal{H}_X(\top_x)(y) &\leq \bigwedge_{A\in L^X}(\mathbf{T}_{\mathcal{H}_X^{-1}}(A)\to(A(y)\to A(x)))\\ &\leq \bigwedge_{z\in X}(\mathbf{T}_{\mathcal{H}_X^{-1}}(\mathcal{H}_X^*(\top_z))\to(\mathcal{H}_X^*(\top_z)(y)\to\mathcal{H}_X^*(\top_z)(x)))\\ &= \bigwedge_{z\in X}(\mathcal{H}_X^*(\top_z)(y)\to\mathcal{H}_X^*(\top_z)(x))\\ &\leq \mathcal{H}_X^*(\top_x)(y)\to\top_x^*(x)=\mathcal{H}_X(\top_x)(y),\end{aligned}$$

$$\begin{aligned}\mathcal{H}_X(\top_x)(y) &\leq \bigwedge_{A\in L^X}(\mathbf{T}_{\mathcal{H}_X^{-1}}(A)\to(A(y)\to A(x)))\\ &\leq \bigwedge_{z\in X}(\mathbf{T}_{\mathcal{H}_X^{-1}}(\mathcal{H}_X^{-1}(\top_z))\to(\mathcal{H}_X^{-1}(\top_z)(y)\to\mathcal{H}_X^{-1}(\top_z)(x)))\\ &= \bigwedge_{z\in X}(\mathcal{H}_X^{-1}(\top_z)(y)\to\mathcal{H}_X^{-1}(\top_z)(x))\\ &\leq \mathcal{H}_X^{-1}(\top_y)(y)\to\mathcal{H}_X^{-1}(\top_y)(x)\\ &\leq \top_y(y)\to\mathcal{H}_X^{-1}(\top_y)(x)=\mathcal{H}_X(\top_x)(y).\end{aligned}$$

(12)

$$\begin{aligned}&f^{-1}(\mathcal{H}_Y(\top_{f(x)}))(z)=\mathcal{H}_Y(\top_{f(x)})(f(z))\\ &=\bigwedge_{B\in L^Y}(\mathbf{T}_{\mathcal{H}_Y}(B)\to(B(f(x))\to B(f(y))))\\ &\geq\bigwedge_{B\in L^Y}(\mathbf{T}_{\mathcal{H}_X}(f^{-1}(B))\to(f^{-1}(B)(x)\to f^{-1}(B)(y)))\\ &\geq\bigwedge_{A\in L^X}(\mathbf{T}_{\mathcal{H}_X}(A)\to(A(x)\to A(y)))=\mathcal{H}_X(\top_x)(z).\end{aligned}$$

Since $f^{-1}(\mathcal{H}_Y(\top_{f(x)}))=\mathcal{H}_X(\top_x)$ for all $x\in X$, the equality holds. Other case is similarly proved.

(5), (7) and (10) are similarly proved. □

Example 3.2. Let $(L=[0,1],\odot,\to,^*)$ be a complete residuated lattice with a strong negation * defined by

$$x\odot y=(x+y-1)\vee 0,\ \ x\to y=(1-x+y)\wedge 1,\ x^*=1-x.$$

Let $X=\{x,y,z\}$ and $A,B\in L^X$ as follows:

$$A(x)=0.9,A(y)=0.8,A(z)=0.3,\ \ B(x)=0.3,B(y)=0.7,B(z)=0.8$$

(1) Let $X=\{a,b,c,d\},Y=\{x,y,z\}$ be a set and $f:X\to Y$ be a map as follows:

$$f(a)=f(b)=x,f(c)=y,f(d)=z.$$

Let $\mathcal{H}_Y : L^Y \to L^Y$ be a join generating map such that

$$\begin{pmatrix} \mathcal{H}_Y(1_x)(x) = 1 & \mathcal{H}_Y(1_x)(y) = 0.8 & \mathcal{H}_Y(1_x)(z) = 0.6 \\ \mathcal{H}_Y(1_y)(x) = 0.7 & \mathcal{H}_Y(1_y)(y) = 1 & \mathcal{H}_Y(1_y)(z) = 0.3 \\ \mathcal{H}_Y(1_z)(x) = 0.5 & \mathcal{H}_Y(1_z)(y) = 0.6 & \mathcal{H}_Y(1_z)(z) = 1. \end{pmatrix}$$

Since $1_x \leq \mathcal{H}_Y(1_x)$ for all $x, y \in X$ and $A = \bigvee_{z\in X}(A(z) \odot 1_z)$, then

$$\begin{aligned} \mathcal{H}_Y(A)(x) &= \mathcal{H}_Y(\bigvee_{z\in X}(A(z) \odot 1_z))(x) = \bigvee_{z\in X}(A(z) \odot \mathcal{H}_Y(1_z)(x) \\ &\geq A(x) \odot 1_x(x) = A(x). \end{aligned}$$

Since $\bigvee_{y\in X}(\mathcal{H}_Y(1_x)(y) \odot \mathcal{H}_Y(1_y)(z) = \mathcal{H}_Y(1_x)(z)$ for all $x, y \in X$,

$$\begin{aligned} &\mathcal{H}_Y(\mathcal{H}_Y(A))(z) = \bigvee_{y\in X}(\mathcal{H}_Y(A)(y) \odot 1_y))(z) \\ &= \bigvee_{y\in X}(\bigvee_{x\in X}(A(x) \odot \mathcal{H}_Y(1_x)(y)) \odot \mathcal{H}_Y(1_y)(z))) \\ &= \bigvee_{x\in X}(A(x) \odot \bigvee_{y\in X}(\mathcal{H}_Y(1_x)(y)) \odot \mathcal{H}_Y(1_y)(z)) \\ &= \bigvee_{x\in X}(A(x) \odot \mathcal{H}_Y(1_x)(z)) = \mathcal{H}_Y(A)(z). \end{aligned}$$

Hence $\mathcal{H}_Y$ is an upper approximation operator.

We obtain a join generating map $\mathcal{H}_Y^{-1} : L^Y \to L^Y$ with $\mathcal{H}_Y^{-1}(1_x)(y) = \mathcal{H}_Y(1_y)(x)$ as follows

$$\begin{pmatrix} \mathcal{H}_Y^{-1}(1_x)(x) = 1 & \mathcal{H}_Y^{-1}(1_x)(y) = 0.7 & \mathcal{H}_Y^{-1}(1_x)(z) = 0.5 \\ \mathcal{H}_Y^{-1}(1_y)(x) = 0.8 & \mathcal{H}_Y^{-1}(1_y)(y) = 1 & \mathcal{H}_Y^{-1}(1_y)(z) = 0.6 \\ \mathcal{H}_Y^{-1}(1_z)(x) = 0.6 & \mathcal{H}_Y^{-1}(1_z)(y) = 0.3 & \mathcal{H}_Y^{-1}(1_z)(z) = 1. \end{pmatrix}$$

Since $\bigvee_{y\in X}(\mathcal{H}^{-1}(1_x)(y) \odot \mathcal{H}^{-1}(1_y)(z)) = \mathcal{H}^{-1}(1_x)(z)$ and $1_x \leq \mathcal{H}^{-1}(1_x)$ for all $x, y \in X$, by a similar method, $\mathcal{H}_Y^{-1}$ is an upper approximation operator.

By Theorem 3.1 (3), we obtain upper approximation operators such that $\mathcal{H}_X(1_a)(b) = \mathcal{H}_Y(1_{f(a)})(f(b))$ and $\mathcal{H}_X^{-1}(1_a)(b) = \mathcal{H}_Y^{-1}(1_{f(a)})(f(b))$ as follows

$$\begin{matrix} \mathcal{H}_X(1_a)(a) = 1 & \mathcal{H}_X(1_a)(b) = 1 & \mathcal{H}_X(1_a)(c) = 0.8 & \mathcal{H}_X(1_a)(d) = 0.6 \\ \mathcal{H}_X(1_b)(a) = 1 & \mathcal{H}_X(1_b)(b) = 1 & \mathcal{H}_X(1_b)(c) = 0.8 & \mathcal{H}_X(1_b)(d) = 0.6 \\ \mathcal{H}_X(1_c)(a) = 0.7 & \mathcal{H}_X(1_c)(b) = 0.7 & \mathcal{H}_X(1_c)(c) = 1 & \mathcal{H}_X(1_c)(d) = 0.3 \\ \mathcal{H}_X(1_d)(a) = 0.5 & \mathcal{H}_X(1_d)(b) = 0.5 & \mathcal{H}_X(1_d)(c) = 0.6 & \mathcal{H}_X(1_d)(d) = 1 \end{matrix}$$

$$\begin{matrix} \mathcal{H}_X^{-1}(1_a)(a) = 1 & \mathcal{H}_X^{-1}(1_a)(b) = 1 & \mathcal{H}_X^{-1}(1_a)(c) = 0.7 & \mathcal{H}_X^{-1}(1_a)(d) = 0.5 \\ \mathcal{H}_X^{-1}(1_b)(a) = 1 & \mathcal{H}_X^{-1}(1_b)(b) = 1 & \mathcal{H}_X^{-1}(1_b)(c) = 0.7 & \mathcal{H}_X^{-1}(1_b)(d) = 0.5 \\ \mathcal{H}_X^{-1}(1_c)(a) = 0.8 & \mathcal{H}_X^{-1}(1_c)(b) = 0.8 & \mathcal{H}_X^{-1}(1_c)(c) = 1 & \mathcal{H}_X^{-1}(1_c)(d) = 0.6 \\ \mathcal{H}_X^{-1}(1_d)(a) = 0.6 & \mathcal{H}_X^{-1}(1_d)(b) = 0.6 & \mathcal{H}_X^{-1}(1_d)(c) = 0.3 & \mathcal{H}_X^{-1}(1_d)(d) = 1. \end{matrix}$$

Let $A, B \in L^X$ as follows:

$$A(x) = 0.9, A(y) = 0.8, A(z) = 0.3, \quad B(x) = 0.3, B(y) = 0.7, B(z) = 0.8.$$

Since $\mathcal{H}_Y(A)(y) = \bigvee_{x\in X}(A(x) \odot \mathcal{H}(1_x)(y))$,

$$\mathcal{H}_Y(A) = (0.9, 0.8, 0.5),\ \mathcal{H}_Y(B) = (0.4, 0.7, 0.8),$$

$$\mathcal{H}_Y^{-1}(A) = (0.9, 0.8, 0.4),\ \mathcal{H}_Y^{-1}(B) = (0.5, 0.7, 0.8).$$

Moreover, by Theorem 3.1,

$$\mathbf{T}_{\mathcal{H}_Y}(A) = e_{L^X}(\mathcal{H}_Y(A), A) = 0.8,\ \mathbf{T}_{\mathcal{H}_Y}(B) = e_{L^X}(\mathcal{H}_Y(B), B) = 0.9.$$

$$\mathbf{T}_{\mathcal{H}_Y^{-1}}(A) = e_{L^X}(\mathcal{H}_Y^{-1}(A), A) = 0.9,\ \mathbf{T}_{\mathcal{H}_Y^{-1}}(B) = e_{L^X}(\mathcal{H}_Y^{-1}(B), B) = 0.8.$$

$$\begin{array}{c} \mathcal{H}_Y(\mathcal{H}_Y(1_x)) = \mathcal{H}_Y(1_x) = (1, 0.8, 0.6) \\ \mathcal{H}_Y(\mathcal{H}_Y(1_y)) = \mathcal{H}_Y(1_y) = (0.7, 1, 0.3) \\ \mathcal{H}_Y(\mathcal{H}_Y(1_z)) = \mathcal{H}_Y(1_z) = (0.5, 0.6, 1). \end{array}$$

$$\begin{array}{l} 0.6 = \mathcal{H}_Y(\top_z)(y) = \bigwedge_{x\in X}(\mathbf{T}_{\mathcal{H}_X}(\mathcal{H}_Y(1_x)) \to (\mathcal{H}_Y(1_x)(z) \to \mathcal{H}_Y(1_x)(y)) \\ = (0.6 \to 0.8) \wedge (0.3 \to 0.1) \wedge (1 \to 0.6) = 0.6. \end{array}$$

$$\begin{array}{l} 0.3 = \mathcal{H}_X(\top_c)(d) = \bigwedge_{x\in X}(\mathbf{T}_{\mathcal{H}_X}(\mathcal{H}_X^{-1*}(1_x)) \to (\mathcal{H}_X^{-1*}(1_x)(c) \to \mathcal{H}_X^{-1*}(1_x)(d)) \\ = (0.5 \to 0.7) \wedge (0.6 \to 1) \wedge (1 \to 0.3) = 0.3. \end{array}$$

$$\mathcal{H}_Y(B) = \begin{pmatrix} B(x) \vee (B(y) - 0.3) \vee (B(z) - 0.5) \\ (B(x) - 0.2) \vee B(y) \vee (B(z) - 0.4) \\ (B(x) - 0.4) \vee (B(y) - 0.7) \vee B(z) \end{pmatrix}$$

$$\mathcal{H}_X(A) = \begin{pmatrix} A(a) \vee A(b) \vee (A(c) - 0.3) \vee (A(d) - 0.5) \\ A(a) \vee A(b) \vee (A(c) - 0.3) \vee (A(d) - 0.5) \\ (A(a) - 0.2) \vee (A(b) - 0.2) \vee A(c) \vee (A(d) - 0.4) \\ (A(a) - 0.4) \vee (A(b) - 0.4) \vee (A(c) - 0.7) \vee A(d) \end{pmatrix}$$

$$\begin{array}{l} \mathcal{H}_X(f^{-1}(B)) \\ = \begin{pmatrix} B(f(a)) \vee B(f(b)) \vee (B(f(c) - 0.3) \vee (B(f(d) - 0.5) \\ B(f(a)) \vee B(f(b)) \vee (B(f(c) - 0.3) \vee (B(f(d) - 0.5) \\ (B(f(a)) - 0.2) \vee (B(f(b)) - 0.2) \vee B(f(c)) \vee (B(f(d)) - 0.4) \\ (B(f(a)) - 0.4) \vee (B(f(b)) - 0.4) \vee (B(f(c)) - 0.7) \vee B(f(d)) \end{pmatrix} \end{array}$$

Hence $f^{-1}(\mathcal{H}_Y(B)) = \mathcal{H}_X(f^{-1}(B))$.

$$\begin{aligned}&\mathbf{T}_{\mathcal{H}_Y}(B) = \bigwedge_{x\in Y}(\mathcal{H}_Y(B)(x) \to B(x))\\&= (1.3 + B(x) - B(y)) \wedge (1.5 + B(x) - B(z))\\&\wedge(1.2 + B(y) - B(x)) \wedge (1.4 + B(y) - B(z))\\&\wedge(1.4 + B(z) - B(x)) \wedge (1.7 + B(z) - B(y)) \wedge 1\end{aligned}$$

$$\begin{aligned}&\mathbf{T}_{\mathcal{H}_X}(A) = \bigwedge_{a\in X}(\mathcal{H}_X(A)(a) \to A(a))\\&= (1 - A(b) + A(a)) \wedge (1.3 + A(a) - A(c)) \wedge (1.5 + A(a) - A(d))\\&\wedge(1 - A(a) + A(b)) \wedge (1.3 + A(b) - A(c)) \wedge (1.5 + A(b) - A(d))\\&\wedge(1.2 + A(c) - A(a)) \wedge (1.2 + A(c) - A(b)) \wedge (1.4 + A(c) - A(d))\\&\wedge(1.4 + A(d) - A(a)) \wedge (1.4 + A(d) - A(a)) \wedge (1.7 + A(d) - A(c)) \wedge 1\end{aligned}$$

$$\mathcal{H}_Y^{-1}(B) = \begin{pmatrix} B(x) \vee (B(y) - 0.2) \vee (B(z) - 0.4)\\ (B(x) - 0.3) \vee B(y) \vee (B(z) - 0.7)\\ (B(x) - 0.5) \vee (B(y) - 0.4) \vee B(z)\end{pmatrix}$$

$$\mathcal{H}_X^{-1}(A) = \begin{pmatrix} A(a) \vee A(b) \vee (A(c) - 0.2) \vee (A(d) - 0.4)\\ A(a) \vee A(b) \vee (A(c) - 0.2) \vee (A(d) - 0.4)\\ (A(a) - 0.3) \vee (A(b) - 0.3) \vee A(c) \vee (A(d) - 0.7)\\ (A(a) - 0.5) \vee (A(b) - 0.5) \vee (A(c) - 0.6) \vee A(d)\end{pmatrix}$$

$$\begin{aligned}&\mathcal{H}_X^{-1}(f^{-1}(B))\\&= \begin{pmatrix} B(f(a)) \vee B(f(b)) \vee (B(f(c) - 0.2) \vee (B(f(d) - 0.4)\\ B(f(a)) \vee B(f(b)) \vee (B(f(c) - 0.2) \vee (B(f(d) - 0.4)\\ (B(f(a)) - 0.3) \vee (B(f(b)) - 0.3) \vee B(f(c)) \vee (B(f(d)) - 0.7)\\ (B(f(a)) - 0.5) \vee (B(f(b)) - 0.5) \vee (B(f(c)) - 0.6) \vee B(f(d))\end{pmatrix}\end{aligned}$$

Hence $f^{-1}(\mathcal{H}_Y^{-1}(B)) = \mathcal{H}_X^{-1}(f^{-1}(B))$.

$$\begin{aligned}&\mathbf{T}_{\mathcal{H}_Y^{-1}}(B) = \bigwedge_{x\in Y}(\mathcal{H}_Y^{-1}(B)(x) \to B(x))\\&= (1.2 + B(x) - B(y)) \wedge (1.4 + B(x) - B(z))\\&\wedge(1.3 + B(y) - B(x)) \wedge (1.7 + B(y) - B(z))\\&\wedge(1.5 + B(z) - B(x)) \wedge (1.4 + B(z) - B(y)) \wedge 1\end{aligned}$$

$$\begin{aligned}&\mathbf{T}_{\mathcal{H}_X^{-1}}(A) = \bigwedge_{a\in X}(\mathcal{H}_X^{-1}(A)(a) \to A(a))\\&= (1 - A(b) + A(a)) \wedge (1.2 + A(a) - A(c)) \wedge (1.4 + A(a) - A(d))\\&\wedge(1 - A(a) + A(b)) \wedge (1.2 + A(b) - A(c)) \wedge (1.4 + A(b) - A(d))\\&\wedge(1.3 + A(c) - A(a)) \wedge (1.3 + A(c) - A(b)) \wedge (1.7 + A(c) - A(d))\\&\wedge(1.5 + A(d) - A(a)) \wedge (1.4 + A(d) - A(a)) \wedge (1.7 + A(d) - A(c)) \wedge 1.\end{aligned}$$

(2) Put X, $Y = \{x, y, z\}$ and $\mathcal{H}_Y$ in (1). Let $g : X \to Y$ be a map as follows:

$$g(a) = g(b) = x, g(c) = g(d) = y.$$

By Theorem 3.1 (3), we obtain upper approximation operator such that $\mathcal{H}_X(1_a)(b) = \mathcal{H}_Y(1_{g(a)})(g(b))$ and $\mathcal{H}_X^{-1}(1_a)(b) = \mathcal{H}_Y^{-1}(1_{g(a)})(g(b))$ as follows

$$\begin{array}{cccc}
\mathcal{H}_X(1_a)(a) = 1 & \mathcal{H}_X(1_a)(b) = 1 & \mathcal{H}_X(1_a)(c) = 0.8 & \mathcal{H}_X(1_a)(d) = 0.8 \\
\mathcal{H}_X(1_b)(a) = 1 & \mathcal{H}_X(1_b)(b) = 1 & \mathcal{H}_X(1_b)(c) = 0.8 & \mathcal{H}_X(1_b)(d) = 0.8 \\
\mathcal{H}_X(1_c)(a) = 0.7 & \mathcal{H}_X(1_c)(b) = 0.7 & \mathcal{H}_X(1_c)(c) = 1 & \mathcal{H}_X(1_c)(d) = 1 \\
\mathcal{H}_X(1_d)(a) = 0.7 & \mathcal{H}_X(1_d)(b) = 0.7 & \mathcal{H}_X(1_d)(c) = 1 & \mathcal{H}_X(1_d)(d) = 1
\end{array}$$

$$\begin{array}{cccc}
\mathcal{H}_X^{-1}(1_a)(a) = 1 & \mathcal{H}_X^{-1}(1_a)(b) = 1 & \mathcal{H}_X^{-1}(1_a)(c) = 0.7 & \mathcal{H}_X^{-1}(1_a)(d) = 0.7 \\
\mathcal{H}_X^{-1}(1_b)(a) = 1 & \mathcal{H}_X^{-1}(1_b)(b) = 1 & \mathcal{H}_X^{-1}(1_b)(c) = 0.7 & \mathcal{H}_X^{-1}(1_b)(d) = 0.7 \\
\mathcal{H}_X^{-1}(1_c)(a) = 0.8 & \mathcal{H}_X^{-1}(1_c)(b) = 0.8 & \mathcal{H}_X^{-1}(1_c)(c) = 1 & \mathcal{H}_X^{-1}(1_c)(d) = 1 \\
\mathcal{H}_X^{-1}(1_d)(a) = 0.8 & \mathcal{H}_X^{-1}(1_d)(b) = 0.8 & \mathcal{H}_X^{-1}(1_d)(c) = 1 & \mathcal{H}_X^{-1}(1_d)(d) = 1
\end{array}$$

$$\mathcal{H}_X(A) = \begin{pmatrix} A(a) \vee A(b) \vee (A(c) - 0.3) \vee (A(d) - 0.3) \\ A(a) \vee A(b) \vee (A(c) - 0.3) \vee (A(d) - 0.3) \\ (A(a) - 0.2) \vee (A(b) - 0.2) \vee A(c) \vee A(d) \\ (A(a) - 0.2) \vee (A(b) - 0.2) \vee A(c) \vee A(d) \end{pmatrix}$$

$$\begin{aligned} &\mathcal{H}_X(g^{-1}(B)) \\ &= \begin{pmatrix} B(g(a)) \vee B(g(b)) \vee (B(g(c) - 0.3) \vee (B(g(d) - 0.3) \\ B(g(a)) \vee B(g(b)) \vee (B(g(c) - 0.3) \vee (B(g(d) - 0.3) \\ (B(g(a)) - 0.2) \vee (B(g(b)) - 0.2) \vee B(g(c)) \vee B(g(d)) \\ (B(g(a)) - 0.2) \vee (B(g(b)) - 0.2) \vee B(g(c)) \vee B(g(d)) \end{pmatrix} \end{aligned}$$

$$\mathcal{H}_X^{-1}(A) = \begin{pmatrix} A(a) \vee A(b) \vee (A(c) - 0.2) \vee (A(d) - 0.2) \\ A(a) \vee A(b) \vee (A(c) - 0.2) \vee (A(d) - 0.2) \\ (A(a) - 0.3) \vee (A(b) - 0.3) \vee A(c) \vee A(d) \\ (A(a) - 0.3) \vee (A(b) - 0.3) \vee A(c) \vee A(d) \end{pmatrix}$$

$$\begin{aligned} &\mathcal{H}_X^{-1}(g^{-1}(B)) \\ &= \begin{pmatrix} B(g(a)) \vee B(g(b)) \vee (B(g(c) - 0.2) \vee (B(g(d) - 0.2) \\ B(g(a)) \vee B(g(b)) \vee (B(g(c) - 0.2) \vee (B(g(d) - 0.2) \\ (B(g(a)) - 0.3) \vee (B(g(b)) - 0.3) \vee B(g(c)) \vee B(g(d)) \\ (B(g(a)) - 0.3) \vee (B(g(b)) - 0.3) \vee B(g(c)) \vee B(g(d)) \end{pmatrix} \end{aligned}$$

Hence $g^{-1}(\mathcal{H}_Y(B)) \geq \mathcal{H}_X(g^{-1}(B))$. For $B(x) = 0.9, B(y) = 0.7, B(z) = 0.1$, then

$$\begin{array}{l}\mathcal{H}_Y(B) = (0.9, 0.7, 0.5),\ \mathcal{H}_Y^{-1}(B) = (0.9, 0.7, 0.4)\\ g^{-1}(\mathcal{H}_Y(B)) = (0.9, 0.9, 0.7, 0.7) = g^{-1}(\mathcal{H}_Y^{-1}(B))\\ \mathcal{H}_X(g^{-1}(B)) = (0.9, 0.9, 0.7, 0.7) = \mathcal{H}_X^{-1}(g^{-1}(B))\\ \mathbf{T}_{\mathcal{H}_Y}(B) = \bigwedge_{x\in Y}(\mathcal{H}_Y(B)(x) \to B(x)) = 0.6\\ \mathbf{T}_{\mathcal{H}_X}(g^{-1}(B)) = \bigwedge_{a\in X}(\mathcal{H}_X(g^{-1}(B))(a) \to g^{-1}(B)(a)) = 1.\\ \mathbf{T}_{\mathcal{H}_Y^{-1}}(B) = \bigwedge_{x\in Y}(\mathcal{H}_Y^{-1}(B)(x) \to B(x)) = 0.7\\ \mathbf{T}_{\mathcal{H}_X^{-1}}(g^{-1}(B)) = \bigwedge_{a\in X}(\mathcal{H}_X^{-1}(g^{-1}(B))(a) \to g^{-1}(B)(a)) = 1.\end{array}$$

Since g is not a surjective function,

$$\mathbf{T}_{\mathcal{H}_X}(g^{-1}(B)) \neq \mathbf{T}_{\mathcal{H}_Y}(B),\ \mathbf{T}_{\mathcal{H}_X^{-1}}(g^{-1}(B)) \neq \mathbf{T}_{\mathcal{H}_Y^{-1}}(B).$$

$$\begin{array}{l}\mathbf{T}_{\mathcal{H}_X}(A) = \bigwedge_{a\in X}(\mathcal{H}_X(A)(a) \to A(a))\\ = (1 - A(b) + A(a)) \wedge (1.3 + A(a) - A(c)) \wedge (1.3 + A(a) - A(d))\\ \wedge(1 - A(a) + A(b)) \wedge (1.3 + A(b) - A(c)) \wedge (1.3 + A(b) - A(d))\\ \wedge(1.2 + A(c) - A(a)) \wedge (1.2 + A(c) - A(b)) \wedge (1 + A(c) - A(d))\\ \wedge(1.2 + A(d) - A(a)) \wedge (1.2 + A(d) - A(a)) \wedge (1 + A(d) - A(c)) \wedge 1.\end{array}$$

$$\begin{array}{l}\mathbf{T}_{\mathcal{H}_X^{-1}}(A) = \bigwedge_{a\in X}(\mathcal{H}_X(A)(a) \to A(a))\\ = (1 - A(b) + A(a)) \wedge (1.2 + A(a) - A(c)) \wedge (1.2 + A(a) - A(d))\\ \wedge(1 - A(a) + A(b)) \wedge (1.3 + A(b) - A(c)) \wedge (1.3 + A(b) - A(d))\\ \wedge(1.3 + A(c) - A(a)) \wedge (1.3 + A(c) - A(b)) \wedge (1 + A(c) - A(d))\\ \wedge(1.3 + A(d) - A(a)) \wedge (1.3 + A(d) - A(a)) \wedge (1 + A(d) - A(c)) \wedge 1.\end{array}$$

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