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ON THE GENERALIZED NONLINEAR DIAMOND HEAT KERNEL

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Abstract: In this paper, we study the nonlinear heat equation

$$\frac{\partial}{\partial t}\triangle^k u(x,t) - c^2\diamondsuit^k u(x,t) = f(x,t,u(x,t)),$$

where $\triangle^k$ is the Laplacian operator iterated $k-$ times and is defined by (1.4)and $\diamondsuit^k$ is the Diamond operator iterated $k-$ times and is defined by (1.2). We obtain an interesting kernel related to the nonlinear heat equation.

Key Words: Fourier transform, spectrum, diamond operator

1. Introduction

Consider the ultra-hyperbolic operator iterated $k-$ times defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k, \quad (1.1)$$

$p+q=n$. S.E. Trione[8] has shown that the generalized function $R_{2k}(x)$ defined by (2.4) is the unique elementary solution of the operator $\Box^k$ that is $\Box^k R_{2k}(x) = \delta$ where $x \in \mathbb{R}^n$ the $n-$ dimensional Euclidean space. Also M. Aguirre Tellez[1, pp.147-149] has proved that $R_{2k}(x)$ exists only if n is odd with p odd and q even or only if n is even with p odd and q odd.

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In 1996, A. Kananthai[4] [2] first introduced the Diamond operator defined by

$$\diamondsuit^k = \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k . \tag{1.2}$$

The operator $\diamondsuit^k$ can be written as the product of the operators in the form

$$\diamondsuit^k = \triangle^k \Box^k = \Box^k \triangle^k, \tag{1.3}$$

where $\triangle^k$ is the Laplacian operator iterated $k-$ times and is defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k , \tag{1.4}$$

and $\Box^k$ is the Ultra-hyperbolic operator iterated $k-$ times and defined by(1.1).

A. Kananthai [4, p.33] has shown that the convolution $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is an elementary solution of the operator $\diamondsuit^k$, that is

$$\diamondsuit^k((-1)^k R^e_{2k}(x) * R^H_{2k}(x)) = \delta(x), \tag{1.5}$$

where $\delta(x)$ is Dirac-delta distribution and the function $R^e_{2k}(x)$ and $R^H_{2k}(v)$ are defined by (2.7) and (2.4) respectively with $\alpha = \beta = 2k, k$ is a nonnegative integer.

It is well known that for the heat equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \triangle u(x,t) \tag{1.6}$$

with the initial condition

$$u(x,0) = f(x),$$

where $\triangle$ is the Laplace operator and is defined by(1.3) with $k = 1, f(x)$ is a continuous function and $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0, \infty)$, we obtain

$$u(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4c^2 t} \right) f(y) dy \tag{1.7}$$

as the solution of (1.6). The equation (1.7) can be written in the following form

$$u(x,t) = E(x,t) * f(x),$$

where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4c^2 t} \right). \tag{1.8}$$

$E(x,t)$ is called the heat kernel, where $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and $t > 0$. Moreover, we obtain $E(x,t) \to \delta$ as $t \to 0$ where δ is the Dirac delta distribution.

Next, K. Nonlaopon and A. Kananthai [4] have studied the heat equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Box u(x,t), \tag{1.9}$$

where $\Box$ is the ultra-hyperbolic operator and is defined by (1.1) with $k = 1$. They obtain the ultrahyperbolic heat kernel

$$E(x,t) = \frac{i^q}{(4c^2 \pi t)^{n/2}} exp\left(\frac{\sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2}{4c^2 t}\right), \tag{1.10}$$

where $p + q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$ and $i = \sqrt{-1}$.

Now, the purpose of this article is to study the nonlinear equation

$$\frac{\partial}{\partial t} \triangle^k u(x,t) - c^2 \diamondsuit^k u(x,t) = f(x,t,u(x,t)), \tag{1.11}$$

where $(x,t) = (x_1, \ldots, x_n, t) \in \mathbb{R}^n \times (0, \infty), t$ is a time, c is a positive constant, $u(x,t)$ is an unknown function. We consider the equation (1.11) with the following conditions on u and f as follows

(1) $u(x,t) \in C^{(6k)}(\mathbb{R}^n)$ for any $t > 0$ where $C^{(6k)}(\mathbb{R}^n)$ is the space of continuous function with $6k$-derivatives.

(2) f satisfies the Lipschitz condition, that is

$$|f(x,t,u) - f(x,t,w)| \leq A|u - w|$$

where A is constant with $0 < A < 1$.

(3) $\int_0^\infty \int_{\mathbb{R}^n} |f(x,t,u(x,t))|\, dxdt < \infty$ for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $0 < t < \infty$ and $u(x,t)$ is continuous function on $\mathbb{R}^n \times (0, \infty)$.

Under such conditions of f and u and for the spectrum of

$$u(x,t) = (-1)^k R_{2k}^e(x) * (E(x,t) * f(x,t,u(x,t)))$$

as a solution of (1.11), where

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right) - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)\right) + i(\xi, x)\right] d\xi. \tag{1.12}$$

and $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ for any fixed $t > 0$ and $(-1)^k R_2^e(x)$ is defined by (2.4) with $\alpha = 2k$. The convolution $(-1)^k R_2^e(x) * E(x,t)$ is called the Diamond Heat Kernel and all properties it will be studied in detail. Before proceeding the following definitions and concepts of needed.

2. Preliminaries

Definition 2.1. Let $f(x) \in \mathrm{L}_1(\mathbb{R}^n)$-the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)\, dx \tag{2.1}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ and $dx = dx_1\, dx_2 \ldots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{f}(\xi)\, d\xi. \tag{2.2}$$

If f is a distribution with compact supports by the Equation(2.1) can be written as

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \left\langle f(x), e^{-i(\xi,x)} \right\rangle.$$

Definition 2.2. The spectrum of the kernel $E(x,t)$ of (1.9) is the bounded support of the Fourier transform $\widehat{E(\xi,t)}$ for any fixed $t > 0$.

Definition 2.3. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be a point in $\mathbb{R}^n$ and be denoted by

$$\Gamma_+ = \{\xi \in \mathbb{R}^n : \xi_1^2 + \xi_2^2 + \ldots + \xi_p^2 - \xi_{p+1}^2 - \xi_{p+2}^2 - \ldots - \xi_{p+q}^2 > 0 \ \text{ and } \ \xi_1 > 0\}.$$

The set of an interior of the forward cone, and $\overline{\Gamma}_+$ denotes the closure of Γ_+.

Let Ω be a spectrum of $E(x,t)$ defined by definition 2.2 for any fixed $t > 0$ and $\Omega \subset \overline{\Gamma}_+$. Let $\widehat{E(\xi,t)}$ be the Fourier transform of $E(x,t)$ and be defined as

$$\widehat{E(\xi,t)} = \begin{cases} \frac{1}{(2\pi)^{n/2}} \exp\left(c^2 t \left(\sum_{i=1}^{p} \xi_i^2 - \sum_{j=p+1}^{p+q} \xi_j^2 \right)^k \right) & \text{for } \xi \in \Gamma_+, \\ 0 & \text{for } \xi \notin \Gamma_+. \end{cases} \tag{2.3}$$

Definition 2.4. Let $x = (x_1, x_2, ..., x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$ and written as

$$\upsilon = x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - ... - x_{p+q}^2,$$

where $p + q = n$ is the dimension of the space $\mathbb{R}^n$.

Let $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } \upsilon > 0\}$ be the interior of the forward cone and $\overline{\Gamma}_+$ denote its closure. For any complex number α, define the function

$$R_\alpha^H(\upsilon) = \begin{cases} \frac{\upsilon^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.4}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2})\Gamma(\frac{p-\alpha}{2})}. \tag{2.5}$$

The function $R_\alpha^H(\upsilon)$ is called the ultra-hyperbolic kernel of Marcel Riesz and was introduced by Y. Nozaki [7].

It is well known that $R_\alpha^H(\upsilon)$ is an ordinary function if $Re(\alpha) \geq n$ and it is a distribution of α if $Re(\alpha) < n$. Let supp $R_\alpha^H(\upsilon)$ denote the support of $R_\alpha^H(\upsilon)$ and suppose supp $R_\alpha^H(\upsilon) \subset \bar{\Gamma}_+$, that is supp $R_\alpha^H(\upsilon)$ is compact.

Definition 2.5 Let $x = (x_1, x_2, \ldots, x_n)$ be a point of $\mathbb{R}^n$ and $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. The elliptic kernel of Marcel Riesz is defined as

$$R_\alpha^e(x) = \frac{|x|^{\alpha-n}}{W_n(\alpha)}, \tag{2.6}$$

where

$$W_n(\alpha) = \frac{\pi^{\frac{n}{2}}2^\alpha\Gamma(\alpha/2)}{\Gamma\left(\frac{n-\alpha}{2}\right)}, \tag{2.7}$$

with α a complex parameter and n the dimension of $\mathbb{R}^n$. It can be shown that $R_{-2k}^e(x) = (-1)^k \triangle^k \delta(x)$ where $\triangle^k$ is defined by (1.4). It follows that $R_0^e(x) = \delta(x)$. The function $R_{2k}^e(x)$ is called the elliptic kernel of Marcel Riesz.

Lemma 2.1. *Let L be the operator and defined by*

$$\mathrm{L} = \frac{\partial}{\partial t} - c^2\Box^k \tag{2.8}$$

where $\Box$ is the ultra-hyperbolic defined by (1.1) for $t \in (0, \infty)$ and c is a positive constant. Then we obtain

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t \left[\left(\sum_{i=1}^{p} \xi_i^2 - \sum_{j=p+1}^{p+q} \xi_j^2\right)\right]^k + i(\xi, x)\right] d\xi. \qquad (2.9)$$

as an elementary solution of (2.8)in the spectrum $\Omega \subset \mathbb{R}^n$ for $t > 0$.

Proof. Let

$$\mathrm{L}E(x,t) = \delta(x,t),$$

where $E(x,t)$ is an elementary solution of operator L and δ is the Dirac-delta distribution. Thus

$$\frac{\partial}{\partial t}E(x,t) - c^2\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2} - \sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^k E(x,t) = \delta(x)\delta(t).$$

If we apply the Fourier transform defined by (2.1) to both sides of the equation, we obtain

$$\frac{\partial}{\partial t}\widehat{E(\xi,t)} - c^2\left(\sum_{i=1}^{p}\xi_i^2 - \sum_{j=p+1}^{p+q}\xi_j^2\right)^k \widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\delta(t).$$

Thus

$$\widehat{E(\xi,t)} = \frac{H(t)}{(2\pi)^{n/2}} \exp\left(c^2 t\left(\sum_{i=1}^{p}\xi_i^2 - \sum_{j=p+1}^{p+q}\xi_j^2\right)^k\right),$$

where $H(t)$ is the Heaviside function. Since $H(t) = 1$ for $t > 0$. Therefore,

$$\widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}} \exp\left(c^2 t\left(\sum_{i=1}^{p}\xi_i^2 - \sum_{j=p+1}^{p+q}\xi_j^2\right)^k\right)$$

which has been already defined by (2.4). Thus

$$\begin{aligned} E(x,t) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)}\widehat{E(\xi,t)}\, d\xi \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\Omega} e^{i(\xi,x)}\widehat{E(\xi,t)}\, d\xi \end{aligned}$$

where Ω is the spectrum of $E(x,t)$. Thus from (2.4)

$$E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega \exp\left(c^2 t\left(\sum_{i=1}^{p}\xi_i^2 - \sum_{j=p+1}^{p+q}\xi_j^2\right)^k + i(\xi,x)\right)d\xi.$$

Definition 2.6 We can extend $E(x,t)$ to $\mathbb{R}^n \times \mathbb{R}$ by setting

$$E(x,t) = \begin{cases} \frac{1}{(2\pi)^n}\int_\Omega \exp\left(c^2 t\left(\sum_{i=1}^{p}\xi_i^2 - \sum_{j=p+1}^{p+q}\xi_j^2\right)^k\right), & \text{for } t>0 \\ 0, & \text{for } t\le 0. \end{cases}$$

3. Main Results

Theorem 3.1. *(The properties of the Heat Kernel $(-1)^k R_{2k}^e(x) * E(x,t)$)*

(1) $(-1)^k R_{2k}^e(x) * E(x,t)$ exists and is a tempered distribution.

(2) $(-1)^k R_{2k}^e(x) * E(x,t) \in \mathbb{C}^\infty$ is the space of continuous function and infinitely differentiable.

(3) $\lim_{t\to 0}\left((-1)^k R_{2k}^e(x) * E(x,t)\right) = (-1)^k R_{2k}^e(x)$.

(4) $(-1)^k R_2^e(x) * E(x,t)$ is bounded and $|E(x,t)| \le \frac{2^{2-n}}{\pi^{n/2}}\frac{M(t)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}$, for $t > 0$, where $M(t)$ is a function of t in the spectrum Ω and Γ denotes the Gamma function. Thus $E(x,t)$ is bounded for any fixed $t>0$.

(5) $\frac{\partial}{\partial t}\triangle^k\left((-1)^k R_{2k}^e(x) * E(x,t)\right) - c^2\Diamond^k\left((-1)^k R_{2k}^e(x) * E(x,t)\right) = 0$

Proof. (1) Since $E(x,t)$ and $(-1)^k R_{2k}^e(x)$ are tempered distribution with compact support. Thus $(-1)^k R_{2k}^e(x) * E(x,t)$ exists and is a tempered distribution.

(2) We have

$$\frac{\partial^n}{\partial x^n}\left((-1)^k R_{2k}^e(x) * E(x,t)\right) = (-1)^k R_{2k}^e(x) * \frac{\partial^n}{\partial x^n}E(x,t)$$

Since $E(x,t)$ is infinitely differentiable and $(-1)^k R^e_{2k}(x) * E(x,t) \in \mathbb{C}^\infty$.

(3) By (2.4), we have

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left(c^2 t \left(\sum_{i=1}^{p} \xi_i^2 - \sum_{j=p+1}^{p+q} \xi_j^2 \right)^k + i(\xi, x) \right) d\xi.$$

Since $E(x,t)$ exists, then

$$\begin{aligned} \lim_{t\to 0} E(x,t) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)}\, d\xi \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\Omega} e^{i(\xi,x)}\, d\xi \\ &= \delta(x) \end{aligned}$$

By the continuity of the convolution,

$$(-1)^k R^e_{2k}(x) * E(x,t) \to (-1)^k R^e_{2k}(x) * \delta \quad as \quad t \to 0.$$

Thus

$$\lim_{t\to 0} \left((-1)^k R^e_{2k}(x) * E(x,t) \right) = (-1)^k R^e_{2k}(x).$$

(4) We have

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left(c^2 t \left(\sum_{i=1}^{p} \xi_i^2 - \sum_{j=p+1}^{p+q} \xi_j^2 \right)^k + i(\xi, x) \right) d\xi.$$

$$|E(x,t)| \leq \frac{1}{(2\pi)^n} \int_\Omega \exp\left(c^2 t \left(\sum_{i=1}^{p} \xi_i^2 - \sum_{j=p+1}^{p+q} \xi_j^2 \right)^k \right) d\xi.$$

By changing to bipolar coordinates

$$\xi_1 = r\omega_1,\ \xi_2 = r\omega_2, \ldots,\ \xi_p = r\omega_p \quad \text{and}$$

$$\xi_{p+1} = s\omega_{p+1},\ \xi_{p+2} = s\omega_{p+2}, \ldots,\ \xi_{p+q} = s\omega_{p+q}$$

where $\sum_{i=1}^{p} \omega_i^2 = 1$ and $\sum_{j=p+1}^{p+q} \omega_j^2 = 1$. Thus

$$|E(x,t)| \leq \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t\left(r^2 - s^2\right)\right] r^{p-1} s^{q-1}\, dr\, ds\, d\Omega_p\, d\Omega_q$$

where $d\xi = r^{p-1}s^{q-1}\, dr\, ds\, d\Omega_p\, d\Omega_q$, $d\Omega_p$ and Ω_q are the elements of the surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. Since $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ and we suppose $0 \leq r \leq R$ and $0 \leq s \leq T$ where R and T are constants, we obtain

$$\begin{aligned} |E(x,t)| &\leq \frac{\Omega_p\,\Omega_q}{(2\pi)^n} \int_0^R \int_0^T \exp\left[c^2 t\left(r^2 - s^2\right)\right] r^{p-1} s^{q-1}\, ds\, dr \\ &= \frac{\Omega_p\,\Omega_q}{(2\pi)^n} M(t) \quad \text{for any fixed } t > 0 \text{ in the spectrum } \Omega \\ &= \frac{2^{2-n}}{\pi^{n/2}} \frac{M(t)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} \end{aligned}$$

where

$$M(t) = \int_0^R \int_0^T \exp\left[c^2 t\left(r^2 - s^2\right)\right] r^{p-1} s^{q-1}\, ds\, dr$$

is a function of t, $\Omega_p = \dfrac{2\pi^{p/2}}{\Gamma\left(\frac{p}{2}\right)}$ and $\Omega_q = \dfrac{2\pi^{p/2}}{\Gamma\left(\frac{q}{2}\right)}$. Thus, for any fixed $t > 0$, $E(x,t)$ is bounded. Since $R_2^H(x)$ is a tempered distribution with compact support, $R_2^H(x)$ is bounded. Thus $R_2^H(x) * E(x,t)$ is bounded.

(5) Since

$$\begin{aligned} \triangle^k\left((-1)^k R_{2k}^e(x) * E(x,t)\right) &= \triangle(-1)^k R_{2k}^e(x) * E(x,t) \\ &= \delta * E(x,t) \\ &= E(x,t) \end{aligned}$$

and

$$\begin{aligned} \diamondsuit^k\left((-1)^k R_{2k}^e(x) * E(x,t)\right) &= \square^k \triangle^k\left((-1)^k R_{2k}^e(x) * E(x,t)\right) \\ &= \square^k\left(\triangle^k(-1)^k R_{2k}^e(x) * E(x,t)\right) \\ &= \square^k\left(\delta * E(x,t)\right) \\ &= \square^k E(x,t) \end{aligned}$$

Thus

$$
\begin{aligned}
\frac{\partial}{\partial t}\triangle^k\left((-1)^k R^e_{2k}(x)*E(x,t)\right)-c^2\diamondsuit^k\left((-1)^k R^e_{2k}(x)*E(x,t)\right)
&=\frac{\partial}{\partial t}E(x,t)-c^2\Box^k E(x,t)\\
&=\left(\frac{\partial}{\partial t}-c^2\Box^k\right)E(x,t)\\
&=0
\end{aligned}
$$

where $E(x,t)$ is defined by (2.9).

Theorem 3.2. *Given the equation*

$$\frac{\partial}{\partial t}\triangle^k u(x,t)-c^2\diamondsuit^k u(x,t)=f(x,t,u(x,t)) \tag{3.1}$$

for $(x,t)\in\mathbb{R}^n\times(0,\infty)$, *$k$ is a positive number and with the following conditions on u and f as follows*

(1) $u(x,t)$ is the space of continuous function on $\mathbb{R}^n\times(0,\infty)$.

(2) f satisfies the Lipschitz condition,

$$|f(x,t,u)-f(x,t,w)|\le A|u-w|$$

where A is constant with $0<A<1$.

(3) $\int_0^\infty\int_{\mathbb{R}^n}|f(x,t,u(x,t))|\,dxdt<\infty$ *for* $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$, $0<t<\infty$ *and $u(x,t)$ is continuous function on $\mathbb{R}^n\times(0,\infty)$.*

Then we obtain the convolution

$$u(x,t)=((-1)R^e_{2k}(x)*E(x,t))*f(x,t,u(x,t)) \tag{3.2}$$

*as a solution of (3.3) for $x\in\Omega_0$ where Ω_0 is a compact subset of $\mathbb{R}^n$ for $0\le t\le T$ with T is constant and $(-1)R^e_{2k}(x)$ is given by (2.4) and $E(x,t)$ is given by (**??**).*

Proof. By (3.1), we have

$$\frac{\partial}{\partial t}\triangle^k u(x,t)-c^2\diamondsuit^k u(x,t)=f(x,t,u(x,t)) \tag{3.3}$$

The above can be written in the form

$$\frac{\partial}{\partial t}\triangle^k u(x,t) - c^2\Box^k\triangle^k u(x,t) = f(x,t,u(x,t)). \tag{3.4}$$

Setting $v(x,t) = \triangle^k u(x,t)$, the equation (3.4) reduce to

$$\frac{\partial}{\partial t} v(x,t) - c^2\Box^k v(x,t) = f(x,t,u(x,t)). \tag{3.5}$$

Convolving both sides of (3.5) with $E(x,t)$, we obtain

$$E(x,t) * \left[\frac{\partial}{\partial t} v(x,t) - c^2\Box^k v(x,t)\right] = E(x,t) * f(x,t,u(x,t))$$

or

$$E(x,t) * \left[\frac{\partial}{\partial t} - c^2\Box^k\right] v(x,t) = E(x,t) * f(x,t,u(x,t)).$$

By properties of convolution

$$v(x,t) * \left(\frac{\partial}{\partial t} - c^2\Box^k\right) E(x,t) = E(x,t) * f(x,t,u(x,t)).$$

so

$$v(x,t) * \delta(x) = E(x,t) * f(x,t,u(x,t)).$$

or

$$v(x,t) = E(x,t) * f(x,t,u(x,t)).$$

$$\triangle^k u(x,t) = E(x,t) * f(x,t,u(x,t)).$$

Convolving both sides of the above equation with $(-1)^k R^e_{2k}(x)$, we obtain

$$(-1)^k R^e_{2k}(x) * \triangle^k u(x,t) = (-1)^k R^e_{2k}(x) * \left(E(x,t) * f(x,t,u(x,t))\right).$$

$$\triangle^k (-1)^k R^e_{2k}(x) * u(x,t) = (-1)^k R^e_{2k}(x) * \left(E(x,t) * f(x,t,u(x,t))\right).$$

By definition 2.5, we obtain the solution of (3.1).

$$u(x,t) = (-1)^k R^e_{2k}(x) * \left(E(x,t) * f(x,t,u(x,t))\right). \tag{3.6}$$

or

$$u(x,t) = (-1)^k R^e_{2k}(x) * \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} E(r,s) f(x-r, t-s, u(x-r,t-s))\, dr ds,$$

where $E(r,s)$ is given by (2.9). We next show that $u(x,t)$ is bounded on $\mathbb{R}^n \times (0,\infty)$. We have

$$|u(x,t)| \leq (-1)^k R^e_{2k}(x) \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |E(r,s)||f(x-r,t-s,u(x-r,t-s))|drds$$
$$\leq (-1)^k R^e_{2k}(x)|E(r,s)|N$$

where $N = \displaystyle\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |f(x-r,t-s,u(x-r,t-s))|drds$. By condition (3) in Theorem 3.1 and $(-1)^k R^e_{2k}(x)$ is a tempered distribution with compact support, we conclude $u(x,t)$ is bounded on $\mathbb{R}^n \times (0,\infty)$.

Lastly, we will to show that $u(x,t)$ is unique. Suppose there is another solution $w(x,t)$ of (3.1). We next to show that $u(x,t)$ is unique. Let $w(x,t)$ be another solution of (2.1). Let the operator

$$L = \frac{\partial}{\partial t}\triangle^k - c^2 \Diamond^k$$

then (3.1) can be written in the form

$$Lu(x,t) = f(x,t,u(x,t))$$

Thus

$$Lu(x,t) - Lw(x,t) = f(x,t,u(x,t)) - f(x,t,w(x,t)).$$

By the condition (2) of the theorem 3.1,

$$|Lu(x,t) - Lw(x,t)| \leq A|u(x,t) - w(x,t)|. \tag{3.7}$$

Let $\Omega_0 \times (0,T]$ the compact subset of $\mathbb{R}^n \times [0,\infty)$ and $L : C^{(4k)}(\Omega_0) \to C^{(4k)}(\Omega_0)$ for$0 \leq t \leq T$.

Now $\left(C^{(2k)}(\Omega_0), ||..||\right)$ is a Banach space where $u(x,t) \in C^{(4k)}(\Omega_0)$ for $0 \leq t \leq T$and $||..||$ is given by

$$\|u(x,t)\| = \sup_{\substack{x\in\Omega_0 \\ 0<t\leq T}} |u(x,t)|.$$

Then, from (3.7)with $0 < A < 1$, the operator L is a contraction mapping on $C^{(4k)}(\Omega_0)$. Since $\left(C^{(2k)}(\Omega_0), ||..||\right)$ i a Banach space and $L : C^{(2k)}(\Omega_0) \to C^{(2k)}(\Omega_0)$ is a contraction mapping on $C^{(2k)}(\Omega_0)$, by Contraction Theorem [5, p.300], we obtain the operator L has a fixe point and has uniqueness property. Thus $u(x,t) = w(x,t)$. It follows that $u(x,t)$ is unique.

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