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FILTER CHARACTERIZATIONS OF PARALINDELÖF SPACES AND META-LINDELÖF SPACES AND SOME CONSEQUENT PRODUCT THEOREMS

James E. Joseph[1], Bhamini M.P. Nayar[2 §]

[1]Emeritus, Department of Mathematics
Howard University
Washington, DC 20059, USA
[1]35 E Street NW #709
Washington, DC 20001, USA
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA

Abstract: Filter characterizations of paralindelöf and meta-Lindelöf spaces are produced. The characterizations are then used to give new proofs of some well-known properties of these spaces. For product spaces, it is also shown that if X and Y are paracompact (respectively, metacompact, paralindelöf, meta-Lindelöf) then $X \times Y$ is paracompact (respectively, metacompact, paralindelöf, meta-Lindelöf), provided that the projection $p : X \times Y \to Y$ is closed.

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1. Introduction

Characterizations for spaces which are compact, Lindelöf, countable compact

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§Correspondence author

etc. using filters or filterbases have been long available in the literature. However, in the cases of significant concepts such as paracompact spaces and metacompact spaces, such characterizations were not available in the literature until recently. This deficiency was overcome when in [5] and [9], respectively, filter and filterbase characterizations for paracompact and metacompact spaces were given. These characterizations were then effectively used to provide new proofs of some established and well-known properties of such spaces. In this article, characterizations of paralindelöf and meta-Lindelöf spaces are developed in terms of filters and filterbases, and new results are found for product spaces using these properties.

Some of new results established here for the product spaces are the following: *If X, Y are paracompact (respectively, metacompact, paralindelöf, meta-Lindelöf, Urysohn-closed, regular-closed), then $X \times Y$ is paracompact (respectively, metacompact, paralindelöf, meta-Lindelöf, Urysohn-closed, regular-closed)* provided the projection $p : X \times Y \to Y$ satisfies a closedness condition.

Recall that a family of subsets of a topological space is called *locally countable* (respectively, *locally finite*) if each point in the space has a neighborhood which intersects at most countably (respectively, finitely) many members of the family. A family of subsets Ω *refines* a family of sets Γ if for each $A \in \Omega$, $A \subseteq B$ for some $B \in \Gamma$. If a family Ω refines a family Γ, we say that Ω is a *refinement* of Γ. A space X is *paralindelöf* (respectively, *paracompact*) if each open cover Λ of X has a locally countable (respectively, locally finite) open refinement. A space X is *meta-Lindelöf* (respcetively, *metacompact*) if every open cover has a point countable (respectively, point finite) refinement. A family of subsets of a space is called *point countable* (respectively, *point finite*) if each point in the space belongs to at most countably many (respectively, finitely many) members of the family. If Ω is a filterbase, the adherence of Ω is denoted by $adh\Omega = \cap_{\Omega} clF$.

Studies of paralindelöf spaces and similar spaces are available in [1], [2] and [10]. In the second part of [10], P. J. Nyikos points out different open problems in the study of paralindelöf spaces and suggests possible avenues of investigation of these spaces, while recognizing that the study of paralindelöf spaces is a wide open area. In [2], W. G. Fleissner and G. M. Reed give a survey of spaces which are paralindelöf. In the present article, we study the class of paralindlöf spaces and the class of meta-Lindelöf spaces by providing characterizations of paralindelöf spaces and meta-Lindelöf spaces in terms of filters, in a similar fashion as in the case of paracompact spaces [9] and also as in the case of metacompact spaces [5].

It is well-known that every paracompact Hausdorff space is normal. In

[2], a series of separation axioms satisfied by paralindelöf spaces are given. In that they assume that all spaces under consideration are regular and T_1. In particular, they show that every paralindelöf space is pseudonormal, where a space is said to be *pseudonormal* [2] if every pair of disjoint closed sets, one of which is countable, can be separated by disjoint open sets. It is shown in [2] that a paralondelöf space which is σ-discrete is paracompact, where a space X is called σ-discrete if $X = \cup\{\cup Y_n : n \in \omega\}$ where Y_n is a discrete collection of singletons [2]. Whether a paralindelöf space is paracompact had been an open question for a long time. In 1982, this question was settled when C. L. Navy [8] gave an example of a paralindelöf T_3 space which was not paracompact. In [11] H. Tamano proved that a space X is paracompact if and only if $X \times \beta X$ is normal. Therefore, as noted by Nyikos [10], by the existence of a paralindelöf space which is not paracompact, there exists a paralindelöf space which is not normal since if X is paralindelöf and not paracompact, then $X \times \beta X$ is not normal, but $X \times \beta X$ is paralindelöf. Throughout this article, all spaces are assumed to be Hausdorff.

2. Initial Results

We begin this section by defining the following concepts.

Definition 2.1. A family Ω of subsets of a space X is said to be *locally countably ultimately dominating (l.c.u.d.)* if for each $x \in X$ there exists an open set U containing x such that U is contained in all but countably many members of Ω. A family Ω of subsets of a space X is said to be *countably point dominating (c.p.d.)* if each $x \in X$ is a member of all but countably many members of Ω.

It is to be noted that if a family $\mathcal{F}$ of subsets of a space X is locally countable (respectively point countable), then the family $\mathcal{B} = \{B \subseteq X : B = X - U, U \in \mathcal{F}\}$ is a l.c.u.d. (respectively, c.p.d.) family.

Definition 2.2. A filter Ω on a space is said to be of *type PL* (respectively, *type ML*) if each l.c.u.d. (respectively, c.p.d.) subcollection of Ω has non-empty adherence. A base for a filter of *type PL* (respectively, *of type ML*) will be referred to as a *filterbase of type PL* (respectively, a *filterbase of type ML*).

Before we state our initial theorem, the following Lemma is offered without proof as the proof is straight forward.

Lemma 2.1. A filter Ω is of *typePL* (respectively, of *typeML*) if and only if every l.c.u.d. (respectively, c.p.d) closed subcollection of the filter has non-empty adherence.

Theorem 2.1. A space is paralindelöf (meta-Lindelöf) if and only if every filter of $typePL(typeML)$ on the space has non-empty adherence.

Proof. Suppose that X is paralindelöf and that Ω is a filter on X such that $adh\Omega = \emptyset$. Then $\mathcal{F} = \{X - clF : F \in \Omega\}$ is an open cover of the paralindelöf space X. Hence there is a locally countable open refinement $\mathcal{K}$ for $\mathcal{F}$. Let $\mathcal{K}^* = \{X - U : U \in \mathcal{K}\}$. Then $\mathcal{K}^*$ is l.c.u.d. and is a subcollection of Ω. $\mathcal{K}^*$ is a subcollection of Ω since if $B \in \mathcal{K}^*$, then $B = X - U$ for some $U \in \mathcal{K}$ and for some $F \in \Omega, F \subseteq clF \subseteq X - (X - clF) \subseteq X - U = B$. Therefore, $B \in \Omega$. Also $\mathcal{K}^*$ is l.c.u.d., since $\mathcal{K}$ is locally countable. Furthermore, $adh\mathcal{K}^* = \emptyset$, since $\cap_{\mathcal{K}^*} clB = \cap_{\mathcal{K}} cl(X - U) = \emptyset$ since $\mathcal{K}$ is a refinement of $\mathcal{F}$ and is a cover of X. Therefore, Ω is a filter of not $typePL$. Hence every filter of $typePL$ has non-empty adherence.

Conversely, let every filter of $typePL$ on X has non-empty adherence. Suppose that Ω is an open cover of X which has no locally countable refinement. Therefore, Ω has no countable subcover. Consider $\mathcal{U} = \{X - \cup_\Gamma F : F \in \Omega,$ and Γ is countable$\}$. Then $\mathcal{U}$ is a base for a filter on X with empty adherence because, $\cap_{\mathcal{U}} clU = \cap_\Omega cl(X - \cup_\Gamma F) = \cap_\Omega cl(\cap_\Gamma (X - F)) \subseteq \cap_\Omega cl(X - F) = \cap_\Omega (X - F) = \emptyset$, where Γ is countable. Hence $\mathcal{U}$ is not a filterbase of $typePL$. Consequently, in view of Lemma 2.1, there is a closed, l.c.u.d. subcollection Λ, of the filter generated by $\mathcal{U}$, with empty adherence. Then $\mathcal{F} = \{X - F : F \in \Lambda\}$ is locally countable, since Λ is a l.c.u.d. family. Also, $\cup_\Lambda (X - F) = X$. Since Λ is a subcollection of the filter generated by $\mathcal{U}$, for each $F \in \Lambda$, choose a countable collection $\Omega(F) \subseteq \Omega$ such that $X - \cup_{\Omega(F)} A \subseteq F$ so that $X - F \subseteq \cup_{\Omega(F)} A$. For each $F \in \Lambda$, let $\mathcal{H}(F) = \{A \cap (X - F) : A \in \Omega(F)\}$. Let $\mathcal{R} = \cup_{F \in \Lambda} \mathcal{H}(F)$. Now, $\mathcal{R}$ is an open refinement of Ω, since F is closed and $A \in \Omega(F) \subseteq \Omega$. Moreover, $\mathcal{R}$ is a cover of X since $\cup_{\mathcal{R}} V = \cup_{F \in \Lambda}([(X - F) \cap (\cup_{\Omega(F)} A)] = \cup_{F \in \Lambda}(X - F) = X$. To show that $\mathcal{R}$ is locally countable, for $x \in X$, let G be an open subset of X containing x such that $G \cap (X - F) = \emptyset$ for all but countable many $F \in \Lambda$, since Λ is a l.c.u.d. family. Let Σ be the countable subfamily of Λ such that $G \cap (X - F) \neq \emptyset$. Therefore, if $V \in \mathcal{R}$, and $V \cap G \neq \emptyset$, $V \in \mathcal{H}(F), F \in \Sigma$, then $G \cap (A \cap (X - F)) \neq \emptyset$ since $V = A \cap (X - F)$, where $F \in \Sigma \subset \Lambda$. Hence, G intersects at most countably many members of $\mathcal{R}$. Therefore, $\mathcal{R}$ is a locally countable open refinement of Ω. □

In view of the Theorem 2.1 and Lemma 2.1, the following Theorem is immediate.

Theorem 2.2. A space is paralindelöf if and only if every closed filter of $typePL$ on the space has non-empty adherence.

The characterizations for meta-Lindelöf spaces can be obtained by replacing the filter of $typePL$ with filter of $typeML$. The proof will follow along the same line of argument and construction.

Theorem 2.3. A space X is meta-Lindelöf if and only if every filter (closed filter) of $typeML$ has non-empty adherence.

We shall use the above characterization and the following Lemma 2.2 to give new proofs of some properties of paralindelöf and meta-lindelöf spaces and continuous images of them, for example, as can be seen in the following theorems.

Lemma 2.2. If $f : X \to Y$ is continuous and Ω is a filter of $typePL(typeML)$ on X, then the filter generated by $\{f(F) : F \in \Omega\}$ is a filter of $typePL(typeML)$ on Y.

Proof. Let Γ be a closed l.c.u.d. subcollection of the filter generated by $\{f(F) : F \in \Omega\}$. Then $\{f^{-1}(H)|H \in \Gamma\}$ is a closed l.c.u.d. subcollection of Ω in X. Hence $\cap_{H\in\Gamma} f^{-1}(H) \neq \emptyset$. Since $f(\cap_{H\in\Gamma} f^{-1}(H)) \subseteq \cap_{H\in\Gamma} H$, $adh\Gamma \neq \emptyset$. □

Following is an easy consequence of Lemma 2.2.

Lemma 2.3. If X is a space and $A \subseteq B \subseteq X$, then any filter of $typePL$ ($typeML$) on A is a filter of $typePL(typeML)$ on B.

Proof. The proof is clear, in view of Lemma 2.2, since the identity function from A to B is continuous. □

Note that on a space X, a filter Ω of *type PL(type ML)* has non-empty adherence if and only if a base $\mathcal{B}$ of Ω has non-empty adherence. In view of this fact, Theorem 2.1 and Theorem 2.2 above, the following characterization follows:

Theorem 2.4. A space X is paralindelöf (respectively, meta-Lindelöf) if and only if every filterbase (closed filterbase) of *type PL(respectively, type ML)* has non-empty adherence.

Theorem 2.5. A closed subspace of a paralindelöf (respectively, meta-Lindelöf) space is paralindelöf (respectively, meta-Lindelöf).

Proof. The proof is straight forward, in view of Lemma 2.3 □

Theorem 2.6. Each subspace of a paralindelöf (respectively, meta-Lindelöf) space is paralindelöf (respectively, meta-Lindelöf) if and only if each open subspace is paralindelöf (respectively, meta-Lindelöf).

Proof. If each subspace is paralindelöf (respectively, meta-Lindelöf), then each open subspace is paralindelöf (respectively, meta-Lindelöf). Conversely, let each open subspace of a space X is paralindelöf (respectively, meta-Lindelöf). Suppose that A is a subspace of X and that Ω is a filterbase of $typePL(typeML)$ on A such that $A \cap adh\Omega = \emptyset$. Then $A \subseteq X - adh\Omega$ and Ω is a filterbase of $typePL(typeML)$ on $X - adh\Omega$. Also, $(X - adh\Omega) \cap adh\Omega = \emptyset$, which is a contradiction, since $X - adh\Omega$ is open. □

Theorem 2.7. If Y is paralindelöf (respectively, meta-Lindelöf) and $f : X \to Y$ is a continuous closed function such that $f^{-1}(w)$ is compact for each $w \in Y$, then X is paralindelöf (respectively, meta-Lindelöf).

Proof. To show that X is paralindelöf, in view of Theorem 2.4, we shall show that each closed filterbase of $typePL$ on X has non-empty adherence. Suppose that Ω is a closed filterbase of $typePL$ on X. Then $\{f(F) : F \in \Omega\}$ is a closed filterbase of $typePL$ on the paralindelöf space Y. Therefore, there exists a $w \in \cap_\Omega f(F)$ and $\Gamma = \{F \cap f^{-1}(w) : F \in \Omega\}$ is a closed filterbase on the compact set $f^{-1}(w)$. Hence, $adh\Gamma = adh\Omega \cap f^{-1}(w) \neq \emptyset$. Hence $adh\Omega \neq \emptyset$. Hence X is paralindelöf. □

Corollary 2.1. If X is compact and Y is paralindelöf, then $X \times Y$ is paralindelöf.

Proof. The projection $p : X \times Y \to Y$ is continuous and closed. Also for each $w \in Y, p^{-1}(w)$ is homeomorphic to the compact space X. Hence the result follows from Theorem 2.7. □

Remark. Each of the above results is true, if 'paralindelöf' is replaced by 'meta-Lindelöf' and the filter (filterbase) of $typeL$ be replaced by filter (filterbase) of $typeML$.

Corollary 2.2. If X is compact and Y is Lindelöf, then $X \times Y$ is paralindelöf.

Proof. Follows from Theorem 2.7 since a Lindlöf space is paralindlöf. □

Corollary 2.3. If X is compact, Y is regular and Lindelöf, then $X \times Y$ is paracompact.

Proof. Follows from Theorem 2.7 above and Theorem 4 of [9], using a similar argument as in the proof of Corollary 1, since a regular Lindelöf space is paracompact. (Corollary 1 of [9]) □

Corollary 2.4. Let X be regular and $f : X \to Y$ be a closed, continuous and onto function with $f^{-1}(y)$ compact for each $y \in Y$. Then X is paralindelöf if and only if Y is paralindelöf.

Proof. Follows from Theorem 2.7 above and the Corollary 3.2 of [1] which states that if X is paralindelöf and $f : X \to Y$ is a continuous closed onto function with $f^{-1}(y)$ Lindelöf for each $y \in Y$, then Y is paralindelöf. □

Lemma 2.4. Let Ω be a filterbase of $typePL$ with the property that countable intersection of members of Ω belongs to Ω and let $\mathcal{U} = \{U | U = \cap_\Gamma F, F \in \Omega, \Gamma \text{countable } \}$. Then $\mathcal{U}$ is a filterbase of $typePL$.

Proof. Consider that Ω is a filterbase of $typePL$ with the indicated property and let $\mathcal{U} = \{U | U = \cap_\Gamma F, F \in \Omega, \Gamma \text{ countable}\}$. Suppose that $\mathcal{B}$ is a l.c.u.d. subcollection of the filter generated by $\mathcal{U}$. Then $\mathcal{B}$ is a subcollection of the filter generated by Ω, since for each $B \in \mathcal{B}$, there is a countable subfamily Γ such that $U = \cap_\Gamma F \subseteq B$, $F \in \Omega$. Since $\cap_\Gamma F \in \Omega$, Γ countable, this implies that $\mathcal{B}$ is a subcollection of the filter generated by Ω and hence has an adherent point. Hence $\mathcal{U}$ is a filterbase of $typePL$. □

Lemma 2.5. If $f : X \to Y$ is continuous and if Ω is a filterbase of $typePL$ on X with the property that countable intersections of members of Ω belong to Ω, then $\{f(F) : F \in \Omega\}$ is a filterbase of $typePL$ on Y with property that countable intersections of members of $\{f(F)\}$ belong to $\{f(F)\}$.

Proof. In view of Lemmas 2.2 and 2.4, if Ω is a filterbase of $typePL$ on X, then $\mathcal{F} = \{f(\cap_\Gamma F) | F \in \Omega, \Gamma \text{countable}\}$ is a filterbase of $typePL$ on Y. Moreover, $f(\cap_\Gamma F) \subseteq \cap_\Gamma f(F)$ and $f(\cap_\Gamma F) \in \mathcal{F}$. Therefore, $\cap_\Gamma f(F) \in \mathcal{F}$, for $F \in \Omega$ and Γ countable. □

Theorem 2.8. If Y is paralindelöf, $f : X \to Y$ is continuous and closed with $f^{-1}(v)$ Lindelöf for each $v \in Y$, then X is paralindelöf.

Proof. Suppose that Ω is a closed filterbase of $typePL$ on X. Then $\mathcal{F} = \{f(F) : F \in \Omega\}$ is a closed filterbase of $typePL$ on Y. Since Y is paralindlöf, $adh\mathcal{F} \neq \emptyset$. Hence, there exists a $w \in Y$ such that $w \in \cap_\Omega f(F) \subseteq \cap_\Gamma f(F)$, where Γ is a countable sub-collection of Ω. Consider $\mathcal{W} = \{f^{-1}(w) \cap f^{-1}(\cap_\Gamma f(F))\}$. Then $\mathcal{W}$ is a family of closed subsets of $f^{-1}(w)$ with the countable intersection property and $f^{-1}(w)$ is Lindelöf. Therefore, there is a $p \in f^{-1}(w)$ such that $p \in \cap_\mathcal{W} W \subset \cap_\Omega F$. Hence $adh\Omega \neq \emptyset$. Hence X is paralindelöf. □

Corollary 2.5. Let X be regular and $f : X \to Y$ be closed continuous onto function with $f^{-1}(y)$ Lindelöf for each $y \in Y$. Then X is paralindelöf if and only if Y is paralindelöf.

Proof. Follows from Theorem 2.8 above and the Corollary 3.2 of [1]. □

Note that the Corollary 2.5 improves on the Corollary 2.4, since every compact space is Lindelöf.

3. Product Theorems

In this section we shall use the characterizations of paracompact spaces, metacompact spaces, paralindelöf spaces and meta-Lindelöf spaces in terms of filters introduced here and in [5] and [9] to prove that the product of these spaces preserves the property, provided that the projection satisfies a closedness condition.

In [5] and [9] respectively, characterizations for metacompact spaces and paracompact spaces are given using filters and filterbases. It is shown in [9] that *a space X is paracompact if and only if every filterbase (or closed filterbase) of typeP has non-empty adherence.* A filterbase Ω is defined to be of *typeP* [9] if each locally ultimately dominating (l.u.d.) filter subbase coarser than Ω has non-empty adherence. A family is *locally ultimately dominating (l.u.d)* if for each $x \in X$ there is an open set about x contained in all but finitely many elements of Ω. *A space X is metacompact [5] if and only if every filter (or closed filter) of typeM on X has non-empty adherence.* A filter on a space is of *typeM* [5] if every point dominating (p.d.) subcollection of the filter has nonempty adherence. A collection Ω of subsets of a space is *point dominating (p.d.)* if each $x \in X$ is a member of all but finitely many members of Ω.

Note that the characterization of paracompact spaces given in [9] in terms of filter bases can be easily stated in terms of filters of *type P*, where a filter Ω is defined to be of *type P* if every l.u.d. subcollection of Ω has a non-empty adherence. The proof of Theorem 3.1 can essentially be done as the proof of Theorem 1 of [9].

Theorem 3.1. A space X is paracompact if and only if every filter of *type P* on the space has non-empty adherence.

Below we prove that product of two paracompact (respectively, metacompact, paralindelöf, meta-Lindelöf) spaces is paracompact (respectively, metacompact, paralindelöf, meta-Lindelöf), provided that the projection $p : X \times Y \to Y$ is a closed function.

Theorem 3.2. Let X and Y be paracompact spaces and the projection $p : X \times Y \to Y$ be closed. Then $X \times Y$ is paracompact.

Proof. Suppose that Ω is a closed filterbase of $typeP$ on $X \times Y$. Then $\{p(F) : F \in \Omega\}$ is a closed filterbase of $typeP$ on the paracompact space Y (Lemma 2 [9]), since p is continuous and closed. Hence there is an adherent point $w \in \cap_\Omega p(F)$, $w \in Y$. Consider the function $g : X \times Y \to X \times \{w\}$ defined by $g((x, y)) = (x, w)$. Then g is continuous and hence $g(\Omega)$ is a filterbase of $typeP$ on $X \times \{w\}$, which is homeomorphic to the paracompact space X. Therefore, $adh_\Omega g(F) \neq \emptyset$. However, $adh_\Omega g(F) = \cap_\Omega (F \cap (X \times \{w\})) \subseteq adh\Omega$. Hence $adh\Omega \neq \emptyset$. Thus $X \times Y$ is paracompact.

Replacing the phrase "filterbase of $typeP$" with "filter of $typeM$", filterbase of "$typePL$" and "filterbase of $typeML$", respectively, we can prove the following three theorems, since continuous image of a filter of $typeM$ is a filter of $typeM$ [5], continuous image of a filter of $typePL$ is a filter of $typePL$ and continuous image of filter of $typeML$ is a filter of $typeML$, in view of Lemma 2.2.

Theorem 3.3. Let X and Y be metacompact spaces and the projection $p : X \times Y \to Y$ is closed. Then $X \times Y$ is metacompact.

Theorem 3.4. Let X and Y be paralindelöf spaces and the projection $p : X \times Y \to Y$ is closed. Then $X \times Y$ is paralindelöf.

Theorem 3.5. Let X and Y be meta-Lindelöf spaces and the projection $p : X \times Y \to Y$ is closed. Then $X \times Y$ is meta-Lindelöf. □

In the characterizations of the paracompact spaces as well as metacompact spaces using filters provided in [9] and [5] respectively, and in Theorem 3.1 above, if we replace the phrase "a filter of *type P*" (respectively, "a filter of *type M*") with the phrase "a countable filter of *type P*" (respectively, "a countable filter of *type M*") we have the following characterizations for countably paracompact spaces and for countably metacompact spaces. The proofs will follow in the same lines of argument as in Theorem 1 of [9] and as in Theorem 1 of [5].

Theorem 3.5. A space X is countably paracompact if and only if every countable filter of *type P* has nonempty adherence.

Theorem 3.6. A space X is countably metacompact if and only if every countable filter of *type M* on X has non-empty adherence.

Theorem 3.2 can be extended to countably paracompact spaces and countably metacompact spaces, following the same lines of proof.

Theorem 3.7. Let X and Y be countably paracompact (respectively, metacompact) spaces and the projection $p : X \times Y \to Y$ is closed. Then $X \times Y$

is countably paracompact (respectively, metacompact).

A Urysohn space is *Urysohn-closed* if and only if the $adh_u\Omega \neq \emptyset$ for each filter Ω where $adh_u\Omega = adh \bigcup_\Omega \Lambda(F), \Lambda(F)$ consists of the open sets containing the closed neighborhood of an element of Ω [3].

Lemma 3.1. The function $g : X \to Y$ Is u-continuous if and only if $g(adh_u\Omega) \subset adh_u g(\Omega)$ for every filter Ω.

Proof. The proof may be found in [7]. Every continuous function is u-continuous [7]. □

A function $g : X \to Y$ is *u-closed* [3] if $cl_u g(A) = g(A)$ for each $A \subset X$, where the *u-closure* of $A \subseteq X$, denoted as $cl_u(A), = \{x : clV \cap A \neq \emptyset, V \in \Lambda(x)\}$. For a set $A, cl_u(A)$ is closed.

Theorem 3.8. If X, Y are Urysohn-closed spaces and the projection $p : X \times Y \to Y$ is u-closed, then $X \times Y$ is Urysohn-closed.

Proof. Clearly $X \times Y$ is Urysohn. If Ω is a filter on $X \times Y$ then $p(\Omega)$ is a filterbase on Y. Therefore $adh_u p(\Omega) \neq \emptyset$. Since p is u-closed $\bigcap_\Omega p(F) \neq \emptyset$. If $y_0 \in \bigcap_\Omega p(F)$, then $X \times \{y_0\}$ is homeomorphic to X and $\emptyset \neq adh_u(\Omega \cap X \times \{y_0\}) \subset adh_u(\Omega)$. □

We define the following before stating the next result.

Definition 3.1. A Urysohn space is called *lower Urysohn-closed* (<u>UC</u>) if $adh_u\Omega \neq \emptyset$ for each closed filterbase Ω.

Definition 3.2. A function $g : X \to Y$ is called *lower u-closed* (*<u>u</u>-closed*) if $cl_u g(A) = g(A)$ for each closed $A \subset X$.

Theorem 3.9. If X, Y are <u>UC</u> spaces and the projection $p : X \times Y \to Y$ is <u>u</u>-closed, then $X \times Y$ is <u>UC</u>.

Proof. Clearly $X \times Y$ is Urysohn. If Ω is a closed filterbase on $X \times Y$ then $p(\Omega)$ is a closed filterbase on Y, since $p : X \times Y \to Y$ is <u>u</u>-closed and therefore, $cl_u p(F) = p(F)$ for each $F \in \Omega$ and $cl_u P(F)$ is closed. Therefore $adh_u p(\Omega) \neq \emptyset$. Since p is <u>u</u>-closed $\bigcap_\Omega p(F) \neq \emptyset$. If $y_0 \in \bigcap_\Omega p(F)$, then $X \times \{y_0\}$ is homeomorphic to X and $\emptyset \neq adh_u(\Omega \cap X \times \{y_0\}) \subset adh_u(\Omega)$. □

A function $g : X \to Y$ is *s-closed* if $cl_s g(A) = g(A)$ for each $A \subset X$. See [4], [6] for definitions and discussions of s-closure operator. It is known that $cl_s(A)$ is a closed set.

Theorem 3.10. If X, Y are regular-closed spaces and the projection $p : X \times Y \to Y$ is s-closed, then $X \times Y$ is regular-closed.

Proof. Clearly $X \times Y$ is regular. If Ω is a filter on $X \times Y$ then $p(\Omega)$ is a filterbase on Y. Therefore $adh_s p(\Omega) \neq \emptyset$. Since p is s-closed $\bigcap_\Omega p(F) \neq \emptyset$. If $y_0 \in \bigcap_\Omega p(F)$, then $X \times \{y_0\}$ is homeomorphic to X and $\emptyset \neq adh_s(\Omega \cap X \times \{y_0\}) \subset adh_s(\Omega)$. □

Definition 3.3. A regular space is called *lower regular-closed (RC)* if $adh_s \Omega \neq \emptyset$ for each closed filterbase Ω.

Definition 3.4. A function $g : X \to Y$ is *lower s-closed (s-closed* if $cl_s g(A) = g(A)$ for each closed $A \subset X$.

Theorem 3.11. If X, Y are RC spaces and the projection $p : X \times Y \to Y$ is $\underline{s}$-closed, then $X \times Y$ is RC.

Proof. Clearly $X \times Y$ is regular. If Ω is a closed filterbase on $X \times Y$ then $p(\Omega)$ is a closed filterbase on Y, since $cl_s p(F) = p(F)$ for each $F \in \Omega$ and $cl_s p(F)$ is closed. Therefore $adh_s p(\Omega) \neq \emptyset$. Since p is $\underline{s}$-closed $\bigcap_\Omega p(F) \neq \emptyset$. If $y_0 \in \bigcap_\Omega p(F)$, then $X \times \{y_0\}$ is homeomorphic to X and $\emptyset \neq adh_s(\Omega \cap X \times \{y_0\}) \subset adh_s(\Omega)$. □

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