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# $\mathcal{H}(\theta)$-OPEN SETS INDUCED BY HEREDITARY CLASSES ON GENERALIZED TOPOLOGICAL SPACES

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**Abstract:** We introduce $\mathcal{H}(\theta)$)-open sets $\mathcal{H}(r)$-open sets defined by a given hereditary class, and investigate some properties and characterizations by using $\mathcal{H}(r)$-open sets. We also introduce and investigate the notions of $\mathcal{H}$-preserving mappings, strong $\mathcal{H}(\theta)$-continuous mappings and $\mathcal{H}(\theta)$-continuous mappings.

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## 1. Introduction

An *ideal* $I$ [5] on $X$ is a nonempty family $I \subseteq expX$ satisfying the following conditions:

(i) $A \subseteq B$, $B \in I$ implies $A \in I$;

(ii) $A, B \in I$ implies $A \cup B \in I$.

In [4], Jankovic and Hamlett have introduced another topology $I$-$\tau^*$ by

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using a given ideal $I$ for a given topology $\tau$.

The idea of hereditary classes was introduced by Császár [4]. A nonempty family $\mathcal{H} \subseteq expX$ is called *a hereditary class* if it satisfies the above condition (i) only (i.e. $A \subseteq B$, $B \in \mathcal{H}$ implies $A \in \mathcal{H}$).

In the paper [3], for a hereditary class $\mathcal{H}$, the operator $()^* : expX \to expX$ was introduced. An operator $c^* : expX \to expX$ was defined by using the operator $()^*$ (i.e. for $A \subseteq X$, $c^*A = A \cup A^*$), which is monotonic, enlarging and idempotent. Some properties of operators $()^*$ and $c^*$ were investigated in [3]. In this paper, we introduce $\mathcal{H}(\theta)$-open sets and $\mathcal{H}(r)$-open sets defined by a given hereditary class, and investigate some properties and characterizations by using $\mathcal{H}(r)$-open sets. We also introduce and investigate the notions of $\mathcal{H}$-preserving mappings, strong $\mathcal{H}(\theta)$-continuous mappings and $\mathcal{H}(\theta)$-continuous mappings.

## 2. Preliminaries

Let $X$ be a nonempty set and $expX$ be the power set of $X$. Then $\mu \subseteq expX$ is called a *generalized topology* (briefly GT) [1,2] on $X$ iff $\emptyset \in \mu$ and $G_i \in \mu$ for $i \in I \neq \emptyset$ implies $G = \cup_{i \in I} G_i \in \mu$. We call the pair $(X, \mu)$ a *generalized topological space* (briefly GTS) on $X$. Let $X$ be a nonempty set and $\mu$ be a GT. The elements of $\mu$ are called *$\mu$-open* [2] sets and the complements are called *$\mu$-closed* sets. The generalized-closure of a subset $S$ of $X$, denoted by $cS$, is the intersection of generalized closed sets including $S$. And the interior of $S$, denoted by $iS$, the union of generalized open sets included in $S$. We will denote $\mathcal{M}_\mu$ the union of all $\mu$-open sets in a GTS $(X, \mu)$. An operator $\gamma : expX \to expX$ is said to be *monotonic* if for $A \subseteq B \subseteq X$, $\gamma(A) \subseteq \gamma(B)$; (ii) *enlarging* if for $A \subseteq X$, $A \subseteq \gamma(A)$.

For a hereditary class $\mathcal{H}$ on $X$ and $A \subseteq X$, we define the set $A^* \subseteq X$ [3] by $x \in A^*$ iff $x \in M \in \mu$ implies $M \cap A \notin \mathcal{H}$.

If $x \notin M_\mu$, then by definition $x \in A^*$. And $x \notin A^*$ iff there exists $x \in M \in \mu$ such that $M \cap A \in \mathcal{H}$.

**Theorem 2.1** ([3]). *Let $\mu$ be a GT in $X$ and $\mathcal{H}$ a hereditary class on $X$.*
*(1) $A \subseteq B \subseteq X$ implies $A^* \subseteq B^*$.*
*(2) $A^* \subseteq cA$.*
*(3) If $H \in \mathcal{H}$, then $H^* = X - \mathcal{M}_\mu$.*
*(4) $A^*$ is $\mu$-closed.*
*(5) If $F$ is $\mu$-closed, then $F^* \subseteq F$.*

Let $\mu$ be a GT in $X$ and $\mathcal{H}$ a hereditary class. Császár [3] introduced the operator $c^* : expX \to expX$ defined as $c^*(A) = A \cup A^*$, for $A \subseteq X$.

**Theorem 2.2** ([3]). *Let $\mu$ be a GT in $X$ and $\mathcal{H}$ a hereditary class. For $A \subseteq X$,*

$$A^* \subseteq c^*(A) \subseteq cA$$

We [6] introduced the operator $i^* : expX \to expX$ defined as $i^*(A) = X - c^*(X - A)$, for $A \subseteq X$.

**Theorem 2.3** ([6]). *Let $\mu$ be a GT in $X$ and $\mathcal{H}$ a hereditary class. For $A \subseteq X$,*
*(1) $c^*(A) = X - i^*(X - A)$.*
*(2) $iA \subseteq i^*(A) \subseteq A$.*

## 3. $\mathcal{H}(\theta)$-Open Sets

Let $\mu$ be a GT in $X$ and $\mathcal{H}$ a hereditary class. We will call the triple $(X, \mu, \mathcal{H})$ a $\mathcal{H}$-generalized topological space, denoted by $\mathcal{H}$-GTS.

**Definition 3.1.** Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$-GTS in $X$. Let us define the collection $\mathcal{H}(\theta) \subseteq P(X)$ by $A \in \mathcal{H}(\theta)$ iff for each $x \in A$, there is $M \in \mu$ such that $x \in M \subseteq c^*M \subseteq A$.

The elements of $\mathcal{H}(\theta)$ are called *$\mathcal{H}(\theta)$-open* sets and the complements are called *$\mathcal{H}(\theta)$-closed* sets.

**Theorem 3.2.** *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$-GTS in $X$. Then every $\mathcal{H}(\theta)$-open set is $\mu$-open.*

*Proof.* Obvious. □

**Theorem 3.3.** *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$-GTS in $X$. Then $\mathcal{H}(\theta)$ also is a GT included in $\mu$.*

*Proof.* Let $A = \{A_i \subseteq X : A_i \in \mathcal{H}(\theta) \text{ for } i \in J\}$. For each $x \in \cup A$, there exists a $j \in J$ such that for some $\mu$-open set $M$ containing $x$, $M \subseteq c^*M \subseteq A_j$. This implies there is $M \in \mu$ such that $x \in M \subseteq c^*M \subseteq \cup A$. Therefore, $\cup A \in \mathcal{H}(\theta)$. □

**Definition 3.4.** Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$-GTS in $X$. The $\mathcal{H}(\theta)$-closure of a subset $A$ of $X$, denoted by $c_{\mathcal{H}(\theta)}A$, is the intersection of $\mathcal{H}(\theta)$-closed sets including $A$. And the $\mathcal{H}(\theta)$-interior of $A$, denoted by $i_{\mathcal{H}(\theta)}A$, the union of $\mathcal{H}(\theta)$-open sets included in $A$.

Let us introduce the operators $\gamma_* : expX \to expX$ defined as the following:

$$\gamma_* A = \{x \in X : c^* M \cap A \neq \emptyset \text{ for every } \mu\text{-open set } M \text{ containing } x\}.$$

**Theorem 3.5.** *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$-GTS in $X$. Then $\gamma_* A$ is $\mu$-closed.*

*Proof.* Obvious. □

**Theorem 3.6.** *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$-GTS in $X$. Then*
*(1) $\gamma_* \emptyset = \emptyset$.*
*(2) $\gamma_*$ is monotonic.*
*(3) $A \subseteq cA \subseteq \gamma_* A$ for $A \subseteq X$.*

*Proof.* (1), (2) Obvious.

(3) For $x \in cA$, let $M$ be any $\mu$-open set containing $x$. Then $M \cap A \neq \emptyset$ and $M \subseteq c^* M$. So it implies $x \in \gamma_* A$. □

**Example 3.7.** Let $Y = \{a, b, c, d\}$, a generalized topology $\nu = \{\emptyset, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ and a hereditary class $\mathcal{H} = \{\emptyset, \{b\}\}$.
Note that $\{a, b\}^* = \{a, d\}$; $\{b, c\}^* = \{c, d\}$; $\{a, b, c\}^* = Y$;

$$c^*\{a, b\} = \{a, b, d\}; \quad c^*\{b, c\} = \{b, c, d\}; \quad c^*\{a, b, c\} = Y.$$

For a set $A = \{a, d\}$, since $A$ is $\nu$-closed, $cA = A$ but $\gamma_* A = Y$. So $cA \neq \gamma_* A$.

**Theorem 3.8.** *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$-GTS in $X$. Then $A$ is a $\mathcal{H}(\theta)$-closed set iff $\gamma_* A = A$.*

*Proof.* $A$ is a $\mathcal{H}(\theta)$-closed set iff $X - A$ is $\mathcal{H}(\theta)$-open iff $x \in X - A$ implies there exists a $\mu$-open set $M$ containing $x$ such that $M \subseteq c^* M \subseteq X - A$ iff $x \in X - A$ implies there exists a $\mu$-open set $M$ containing $x$ such that $c^* M \cap A = \emptyset$ iff $x \in X - A$ implies $x \in X - \gamma_* A$ iff $\gamma_* A \subseteq A$ iff $\gamma_* A = A$. □

**Theorem 3.9.** *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$-GTS in $X$. Then $\gamma_* A \subseteq c_{\mathcal{H}(\theta)} A$ for $A \subseteq X$.*

*Proof.* Let $x \in X - c_{\mathcal{H}(\theta)} A$. Then there exists a $\mathcal{H}(\theta)$-open set $G$ containing $x$ such that $G \cap A = \emptyset$. From $\mathcal{H}(\theta)$-openness, there exists some $\mu$-open set $M$ of $x$ such that $c^* M \subseteq G \subseteq X - A$. This implies $x \notin \gamma_* A$. □

**Example 3.10.** In Example 3.7, we have $\mathcal{H}(\theta)=\{\emptyset\}$, and $c_{\mathcal{H}(\theta)} A = Y$ for $A \subseteq X$. For $A = \{a\}$, $\gamma_* A = \{a, d\}$, and hence $\gamma_* A \neq c_{\mathcal{H}(\theta)} A$.

**Theorem 3.11.** *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$-GTS in $X$. If $V$ is a $\mathcal{H}(\theta)$-open set and $x \in V$, then there exists a $\mathcal{H}(r)$-open set $U$ such that $x \in U \subseteq c^*U \subseteq V$.*

*Proof.* Since $V$ is $\mathcal{H}(\theta)$-open in $X$, there exists a $\mu$-open set $M$ such that $x \in M \subseteq c^*M \subseteq V$. Put $U = i_\mu c^*M$; then $U$ is $\mathcal{H}(r)$-open, $M \subseteq U$ and $c^*U = c^*i_\mu c^*M \subseteq c^*M$. It implies $x \in M \subseteq U \subseteq c^*U \subseteq c^*M \subseteq V$, and hence for some $\mathcal{H}(r)$-open set $U$, $x \in U \subseteq c^*U \cap \subseteq V$. □

Since every $\mathcal{H}(r)$-open set is $\mu$-open on $X$ with a GT $\mu$, obviously the next corollary is obtained.

**Corollary 3.12.** Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$-GTS in $X$. Then $V \in \mathcal{H}(\theta)$ and $x \in V$ iff there exists a $\mathcal{H}(r)$-open set $U$ such that $x \in U \subseteq c^*U \subseteq V$.

**Definition 3.13.** Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$-GTS in $X$. $(X, \mu)$ is $\mathcal{H}$-*regular* iff for each $\mu$-open set $U$ containing $x \in X$, there exists a $\mu$-open set $V$ such that $x \in V \subseteq c^*V \subseteq U$.

**Theorem 3.14.** *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$-GTS in $X$. If $(X, \mu)$ is a $\mathcal{H}$-regular GTS, then the following hold:*
*(1) $c_\mu B = \gamma_* B$ for $B \subseteq X$.*
*(2) Every $\mu$-open set is $\mathcal{H}(\theta)$-open.*

*Proof.* (1) Let $x \in \gamma_* B$ and $U$ any $\mu$-open set containing $x$. Then from $\mathcal{H}$-regularity, there exists a $\mu$-open set $V$ such that $x \in V \subseteq c^*V \subseteq U$. Since $x \in \gamma_* B$, $c^*V \cap B \neq \emptyset$. Thus $U \cap B \neq \emptyset$ so that $x \in c_\mu B$. Hence by Theorem 3.6, $c_\mu B = \gamma_* B$.

(2) Let $V$ be a $\mu$-open set. From (1), $Y - V = c_\mu(Y - V) = \gamma_*(Y - V)$, and so by Theorem 3.8 $Y - V$ is $\mathcal{H}(\theta)$-closed. Finally, $V$ is $\mathcal{H}(\theta)$-open. □

**Theorem 3.15.** *Let $(X, \mu, \mathcal{H})$ be a $\mathcal{H}$-GTS in $X$. If $(X, \mu)$ is a $\mathcal{H}$-regular GTS, then $\mu = \mathcal{H}(\theta)$.*

*Proof.* It follows from Theorem 3.3 and (2) of Theorem 3.14. □

## 4. $\mathcal{H}$-Preserving Mappings

Let $X$ and $Y$ be two non-empty sets and let $\mu$ and $\nu$ be two GT's on $X$ and $Y$, respectively. In the sequel, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hereditary classes of $X$ and $Y$, respectively.

**Definition 4.1.** Let $(X, \mu, \mathcal{H}_1)$ and $(Y, \nu, \mathcal{H}_2)$ be $\mathcal{H}$-GTS's. Then a mapping $f : (X, \mu, \mathcal{H}_1) \to (Y, \nu, \mathcal{H}_2)$ is said to be *$\mathcal{H}$-preserving* if for $H \in \mathcal{H}_2$, $f^{-1}(H) \in \mathcal{H}_1$.

**Theorem 4.2.** *Let $f : X \to Y$ be a mapping on $\mathcal{H}$-GTS's $(X, \mu, \mathcal{H}_1)$ and $(Y, \nu, \mathcal{H}_2)$. If $f$ is $(\mu, \nu)$-continuous and $\mathcal{H}$-preserving, then*
*(1) $(f^{-1}(B))^* \subseteq f^{-1}(B^*)$ for $B \subseteq Y$;*
*(2) $f(A^*) \subseteq (f(A))^*$ for $A \subseteq X$.*

*Proof.* (1) For each $x \notin f^{-1}(B^*)$, $f(x) \notin B^*$ and so there exists $M \in \mu$ such that $M \cap B \in \mathcal{H}_2$. Since $f^{-1}(M) \in \mu$ and $f$ is $\mathcal{H}$-preserving, $f^{-1}(M \cap B) = f^{-1}(M) \cap f^{-1}(B) \in \mathcal{H}_1$, and hence $x \notin f^{-1}(B)^*$.

(2) For $A \subseteq X$, by (1), $A^* \subseteq (f^{-1}(f(A)))^* \subseteq f^{-1}((f(A))^*)$. Hence $f(A^*) \subseteq (f(A))^*$. □

**Theorem 4.3.** *Let $f : X \to Y$ be a mapping on $\mathcal{H}$-GTS's $(X, \mu, \mathcal{H}_1)$ and $(Y, \nu, \mathcal{H}_2)$. If $f$ is $(\mu, \nu)$-continuous and $\mathcal{H}$-preserving, then*
*(1) $c^*(f^{-1}(B)) \subseteq f^{-1}(c^*B)$ for $B \subseteq Y$;*
*(2) $f(c^*A) \subseteq c^*(f(A))$ for $A \subseteq X$.*

*Proof.* (1) For $B \subseteq Y$, $c^*(f^{-1}(B)) = f^{-1}(B) \cup (f^{-1}(B))^* \subseteq f^{-1}(B) \cup f^{-1}(B^*) = f^{-1}(B \cup B^*) = f^{-1}(c^*B)$.

(2) It follows from (1). □

**Theorem 4.4.** *Let $f : X \to Y$ be a mapping on $\mathcal{H}$-GTS's $(X, \mu, \mathcal{H}_1)$ and $(Y, \nu, \mathcal{H}_2)$. If $f$ is $(\mu, \nu)$-continuous and $\mathcal{H}$-preserving, then $\gamma_* f^{-1}(B) \subseteq f^{-1}(\gamma_* B)$ for $B \subseteq Y$.*

*Proof.* If $x \in f^{-1}(\gamma_* B)$, then $f(x) \notin \gamma_* B$ and it implies there exists a $\nu$-open set $N$ containing $f(x) \in N$ such that $c^*N \cap A = \emptyset$. From Theorem 4.3, it follows $\emptyset = f^{-1}(c^*N \cap A) = f^{-1}(c^*N) \cap f^{-1}(A) \supset c^* f^{-1}(N) \cap f^{-1}(A)$. Since $x \in f^{-1}(N) \in \mu$, $x \notin \gamma_* f^{-1}(B)$. □

**Theorem 4.5.** *Let $f : X \to Y$ be a mapping on $\mathcal{H}$-GTS's $(X, \mu, \mathcal{H}_1)$ and $(Y, \nu, \mathcal{H}_2)$. If $f$ is $(\mu, \nu)$-continuous and $\mathcal{H}$-preserving, then $B^* \subseteq B$ implies $(f^{-1}(B))^* \subseteq f^{-1}(B)$ for $B \subseteq Y$.*

*Proof.* From Theorem 4.2, it follows $(f^{-1}(B))^* \subseteq f^{-1}(B^*) \subseteq f^{-1}(B)$ for $B \subseteq Y$. □

**Theorem 4.6.** *Let $f : X \to Y$ be a mapping on $\mathcal{H}$-GTS's $(X, \mu, \mathcal{H}_1)$ and $(Y, \nu, \mathcal{H}_2)$. If $f$ is $(\mu, \nu)$-continuous and $\mathcal{H}$-preserving, then $f^{-1}(\mathcal{M}_\nu) \subseteq \mathcal{M}_\mu$.*

*Proof.* For $H \in \mathcal{H}_2$, $(f^{-1}(H))^* = X - \mathcal{M}_\mu$ and $f^{-1}(H^*) = f^{-1}(Y - \mathcal{M}_\nu) = X - f^{-1}(\mathcal{M}_\nu)$ implies $f^{-1}(\mathcal{M}_\nu) \subseteq \mathcal{M}_\mu$. □

## 5. Strong $\mathcal{H}(\theta)$-Continuity, $\mathcal{H}(\theta)$-Continuity

**Definition 5.1.** A mapping $f : (X, \mu, \mathcal{H}_1) \to (Y, \nu, \mathcal{H}_2)$ is said to be *strong $\mathcal{H}(\theta)$-continuous* if for $x \in X$ and each $\nu$-open set $V$ containing $f(x)$, there exists a $\mu$-open set $U$ containing $x$ such that $f(c^*U) \subseteq ic^*V$.

**Theorem 5.2.** *Let $f : X \to Y$ be a mapping on $\mathcal{H}$-GTS's $(X, \mu, \mathcal{H}_1)$ and $(Y, \nu, \mathcal{H}_2)$. Then the following are equivalent:*

*(1) $f$ is strong $\mathcal{H}(\theta)$-continuous.*

*(2) For every $x \in X$ and each $\mathcal{H}(r)$-open set $V$ containing $f(x)$, there exists a $\mu$-open set $U$ containing $x$ such that $f(c^*U) \subseteq V$.*

*(3) For every $\mathcal{H}(r)$-open set $V$ of $Y$, $f^{-1}(V)$ is $\mathcal{H}(\theta)$-open.*

*(4) For every $\mathcal{H}(r)$-closed set $F$ of $Y$, $f^{-1}(F)$ is $\mathcal{H}(\theta)$-closed.*

*Proof.* (1) $\Rightarrow$ (2) Let $x \in X$ and $V$ a $\mathcal{H}(r)$-open set containing $f(x)$. Since $V$ is a $\nu$-open set, there exists a $\mu$-open set $U$ containing $x$ such that $f(c^*U) \subseteq ic^*V = V$. So we have (2).

(2) $\Rightarrow$ (1) Let $x \in X$ and $V$ a $\nu$-open set containing $f(x)$. Then $V \subseteq ic^*V \subseteq c^*V$ implies $ic^*V \subseteq ic^*ic^*V \subseteq ic^*c^*V = ic^*V$, and $ic^*V$ is $\mathcal{H}(r)$-open. By hypothesis, there exists a $\mu$-open set $U$ containing $x$ such that $f(c^*U) \subseteq ic^*V$. Hence, $f$ is strong $\mathcal{H}(\theta)$-continuous.

(2) $\Rightarrow$ (3) Let $V$ be any $\mathcal{H}(r)$-open set in $Y$. For each $x \in f^{-1}(V)$, by (2), there exists a $\mu$-open set $U$ containing $x$ such that $x \in U \subseteq c^*U \subseteq f^{-1}(V)$. So $f^{-1}(V)$ is $\mathcal{H}(\theta)$-open.

(3) $\Rightarrow$ (2), (3) $\Leftrightarrow$ (4) Obvious. □

**Definition 5.3.** A mapping $f : (X, \mu, \mathcal{H}_1) \to (Y, \nu, \mathcal{H}_2)$ is said to be *$\mathcal{H}(\theta)$-continuous* if for $x \in X$ and each $\nu$-open set $V$ containing $f(x)$, there exists a $\mu$-open set $U$ containing $x$ such that $f(c^*U) \subseteq c^*V$.

Now we get the following characterization for the $\mathcal{H}(\theta)$-continuity:

**Theorem 5.4.** *Let $f : X \to Y$ be a mapping on $\mathcal{H}$-GTS's $(X, \mu, \mathcal{H}_1)$ and $(Y, \nu, \mathcal{H}_2)$. Then the following are equivalent:*

*(1) $f$ is $\mathcal{H}(\theta)$-continuous.*

*(2) $f(\gamma_* A) \subseteq \gamma_* f(A)$ for $A \subseteq X$.*

*(3) $\gamma_* f^{-1}(B) \subseteq f^{-1}(\gamma_* B)$ for $B \subseteq Y$.*

*Proof.* (1) $\Rightarrow$ (2) Let $A \subseteq X$ and $y \in f(\gamma_* A)$. Let $y = f(x)$ for some $x \in \gamma_* A$. Then for each $\nu$-open set $V$ containing $f(x)$, from the definition of $\mathcal{H}(\theta)$-continuous mapping, there exists a $\nu$-open set $M$ containing $x$ such that $f(c^*M) \subseteq c^*V$, i.e., $x \in c^*M \subseteq f^{-1}(c^*V)$. Since $x \in \gamma_* A$, $c^*M \cap A \neq \emptyset$ and so $f^{-1}(c^*V) \cap A \neq \emptyset$. It implies $c^*V \cap f(A) \neq \emptyset$ and hence, $y \in \gamma_* f(A)$.

(2) $\Rightarrow$ (3) For $B \subseteq Y$, by (2), $f(\gamma_* f^{-1}(B)) \subseteq \gamma_* f(f^{-1}(B)) \subseteq \gamma_* B$ and so $\gamma_* f^{-1}(B) \subseteq f^{-1}(\gamma_* B)$.

(3) $\Rightarrow$ (1) Let $x \in X$ and $V$ any $\nu$-open set containing $f(x)$. Then $c^*V \cap (Y - c^*V) = \emptyset$, from the definition of operator $\gamma_*$, $f(x) \notin \gamma_*(Y - c^*V)$ and so $x \notin f^{-1}(\gamma_*(Y - c^*V))$. By (3), we have $x \notin \gamma_* f^{-1}(Y - c^*V)$, and there exists a $\mu$-open set $U$ containing $x$ such that $c^*U \cap f^{-1}(Y - c^*V) = \emptyset$. This implies $f(c^*U) \subseteq c^*V$. Thus $f$ is $\mathcal{H}(\theta)$-continuous. □

**Theorem 5.5.** *Let $f : X \to Y$ be a mapping on $\mathcal{H}$-GTS's $(X, \mu, \mathcal{H}_1)$ and $(Y, \nu, \mathcal{H}_2)$. If $f$ is $(\mu, \nu)$-continuous and $\mathcal{H}$-preserving, then it is $\mathcal{H}(\theta)$-continuous.*

*Proof.* It follows from Theorem 4.4 and (3) of Theorem 5.4. □

**Theorem 5.6.** *Let $f : X \to Y$ be a mapping on $\mathcal{H}$-GTS's $(X, \mu, \mathcal{H}_1)$ and $(Y, \nu, \mathcal{H}_2)$. Then the following are equivalent:*
*(1) $f$ is $\mathcal{H}(\theta)$-continuous.*
*(2) For every $\mathcal{H}(\theta)$-open set $A \subseteq Y$, $f^{-1}(A)$ is $\mathcal{H}(\theta)$-open in $X$.*

*Proof.* It follows from Theorem 3.6, Theorem 3.8 and Theorem 5.4. □

**Remark 5.7.** If a mapping $f : (X, \mu, \mathcal{H}_1) \to (Y, \nu, \mathcal{H}_2)$ is strong $\mathcal{H}(\theta)$-continuous, then obviously $f$ is $\mathcal{H}(\theta)$-continuous. But the converse is not true in general.

**Example 5.8.** Let $X = \{a, b, c\}$, a generalized topology $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and a hereditary class $\mathcal{H} = \{\emptyset, \{a\}, \{b\}\}$.
Note that $\{a\}^* = \emptyset$; $\{b\}^* = \emptyset$; $\{a, b\}^* = \{c, d\}$; $X^* = \emptyset$.
$c^*\{a\} = \{a\}$; $c^*\{b\} = \{b\}$; $c^*\{a, b\} = X$; $c^*X = X$;
$ic^*\{a\} = \{a\}$; $ic^*\{b\} = \{b\}$; $ic^*\{a, b\} = X$; $ic^*X = X$.
Furthermore, $\mathcal{H}(\theta) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$.
Consider the identity mapping $f : (X, \mu, \mathcal{H}) \to (X, \mu, \mathcal{H})$; then obviously the mapping is $\mathcal{H}(\theta)$-continuous. For a $\mathcal{H}(\theta)$-open set $\{a, b\}$, $f^{-1}(\{a, b\})$ is not $\mathcal{H}(r)$-open and so $f$ is strong $\mathcal{H}(\theta)$-continuous.

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