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IMPROVED DELAY-RANGE-DEPENDENT STABILITY CRITERIA FOR DISCRETE-TIME LINEAR SYSTEMS WITH INTERVAL TIME-VARYING DELAY AND NONLINEAR PERTURBATIONS

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Abstract: In this paper, we study the problem of stability analysis for discrete-time linear system with interval time-varying delay and nonlinear perturbations. By constructing a new Lyapunov-Krasovskii functional with triple summation terms, mixed model transformation, Jensen-type summation inequality and utilization of zero equation, new delay-range-dependent asymptotic stability criteria are obtained and formulated in terms of linear matrix inequalities (LMIs). Moreover, we obtain new delay-range-dependent asymptotic stability criteria of discrete-time linear system with interval time-varying delay. Numerical examples are given to demonstrate the effectiveness and less conservativeness of the proposed methods.

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1. Introduction

Time-delay systems exist in many fields such as electric systems, chemical processes systems, networked control systems, telecommunication systems and economical systems. During the past two decades, the problems of robust stability and stabilization analysis for dynamic systems with time delays have been widely investigated by many researchers [1]-[26]. It is well known that nonlinearities, as time delays, may cause instability and poor performance of practical systems. Commonly, stability criteria for dynamic systems with time delay are generally divided into two classes: a delay-independent one and a delay-dependent one. The delay-independent stability criteria tend to be more conservative, especially for a small size delay, such criteria do not give any information on the size of delay. On the other hand, delay-dependent stability criteria are concerned with the size of delay and usually provide a maximal delay size. Recently, a special type of time delay in practical engineering systems, that is interval time-varying delay, is investigated. The characteristic of interval time-varying delay is that time delay can vary in an interval in which the lower bound of delay is not restricted to zero. The typical examples of systems with interval time-varying delay are networked control systems, chemical process and flight systems [10].

Discrete-time systems with state delay have strong background in engineering applications, among which network based control has been well recognized to be a typical example. If the delay is constant in discrete systems, one can transform a delayed system in to a delay-free one by using state augmentation techniques. However, when the delay is large, the augmented system will become much complex and thus difficult to analyze and synthesize [5]. Hence, researchers have focussed on the delay-range-dependent stability and stabilization problems of discrete-time systems with interval time-varying delay and many existing results mainly focus on discrete-time linear delay systems [1]-[12], [14]-[15], [17]-[23], [25]-[26]. However, most real systems hold nonlinear dynamics. Therefore, researchers have been investigated the delay-range-dependent stability criteria for discrete-time linear systems with interval time-varying delay and nonlinear perturbations [13], [16], [24]. However, the existing results of delay-range-dependent stability criteria do not take into account the presence of nonlinear perturbations uncertainties in the discrete-time delay systems by model transformation.

This paper will focus on the delay-range-dependent stability analysis for discrete-time linear system with interval time-varying delay and nonlinear perturbations. By using the combination of mixed model transformation, Jensen-type summation inequality, utilization of zero equation and new Lyapunov-Krasovskii functional, new delay-range-dependent asymptotic stability criteria are obtained and formulated in terms of LMIs. Then, we can obtain new delay-range-dependent asymptotic stability criteria of discrete-time linear systems with interval time-varying delay. Finally, numerical examples will be given to show the effectiveness of the obtained results.

We introduce some notations, definition and lemmas that will be used throughout the paper. Z^+ denotes the set of all real non-negative numbers; R^n denotes the n-dimensional space with the vector norm $\|\cdot\|$; $\|x\|$ denotes the Euclidean vector norm of $x \in R^n$; $R^{n\times r}$ denotes the set of $n \times r$ real constant matrices; A^T denotes the transpose of the matrix A; A is symmetric if $A = A^T$; I denotes the identity matrix; Matrix A is called semi-positive definite ($A \geq 0$) if $x^T Ax \geq 0$, for all $x \in R^n$; A is positive definite ($A > 0$) if $x^T Ax > 0$ for all $x \neq 0$; Matrix B is called semi-negative definite ($B \leq 0$) if $x^T Bx \leq 0$, for all $x \in R^n$; B is negative definite ($B < 0$) if $x^T Bx < 0$ for all $x \neq 0$; $A > B$ means $A - B > 0$ ($B - A < 0$); $A \geq B$ means $A - B \geq 0$ ($B - A \leq 0$).

2. Problem Formulation and Preliminaries

Consider the discrete-time linear system with interval time-varying delay and nonlinear perturbations of the form

$$\begin{aligned} x(k+1) &= Ax(k) + Bx(k-h(k)) + f(k,x(k)) \\ &\quad + g(k,x(k-h(k))), \qquad (1) \\ x(s) &= \phi(s), \quad s \in \{-h_2, -h_2+1, \ldots, 0,\}, \qquad (2) \end{aligned}$$

where $k \in Z^+$, $x(k) \in R^n$ is the state variable and $\phi(s)$ is a initial value at s. A and $B \in R^{n\times n}$ are known real constant matrices. $f(k,x(k))$ and $g(k,x(k-h(k)))$ are the nonlinear perturbations with respect to current state $x(k)$ and discrete delay state $x(k-h(k))$, respectively, and are bounded in magnitude:

$$\|f(k,x(k))\| \leq \alpha\|x(k)\|, \qquad (3)$$

$$\|g(k,x(k-h(k)))\| \leq \beta\|x(k-h(k))\|, \qquad (4)$$

where α and β are given positive real constants. In addition, we assume that the time-varying delay $h(k)$ is upper and lower bounded. It satisfies the following assumption of the form

$$0 < h_1 \leq h(k) \leq h_2,$$

where h_1 and h_2 are known positive real constants. Rewrite the system (1) in the following system:

$$\begin{aligned} x(k+1) &= x(k) + y(k), \qquad (5) \\ y(k) &= \Big[A + B - I\Big]x(k) - B \sum_{i=k-h(k)}^{k-1} y(i) \\ &\quad + f(k, x(k)) + g(k, x(k - h(k))). \qquad (6) \end{aligned}$$

By utilizing the following zero equation, we have

$$Jx(k) - Jx(k - h(k)) - J \sum_{i=k-h(k)}^{k-1} y(i) = 0, \qquad (7)$$

where $J \in R^{n\times n}$ will be chosen to guarantee the asymptotic stability of the system (1)-(2). By (7), system (5)-(6) can be represented by the form

$$\begin{aligned} x(k+1) &= x(k) + y(k) + Jx(k) - Jx(k - h(k)) \\ &\quad -J \sum_{i=k-h(k)}^{k-1} y(i), \qquad (8) \\ y(k) &= \Big[A + B - I\Big]x(k) - B \sum_{i=k-h(k)}^{k-1} y(i) \\ &\quad + f(k, x(k)) + g(k, x(k - h(k))). \qquad (9) \end{aligned}$$

Definition 2.1. **[18]** The system (1)-(2) is said to be asymptotically stable if there exists a positive definite function $V(k, x(k)) : Z^+ \times R^n \to R$ such that

$$\Delta V(k, x(k)) = V(k + 1, x(k + 1)) - V(k, x(k)) < 0,$$

along any trajectory of solution for system (1)-(2).

Lemma 2.2. *[6]* ***[Schur complement lemma]*** *Given constant symmetric matrices X, Y and Z of appropriate dimensions with $Y > 0$, then $X + Z^T Y^{-1} Z < 0$ if and only if*

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0.$$

Lemma 2.3. *[8] For any constant matrix $W \in R^{n\times n}, W = W^T > 0$, two integers r_M and r_m satisfying $r_M \geq r_m$ and vector function $x : [r_m, r_M] \to R^n$, the following inequality holds:*

$$\Big(\sum_{i=r_m}^{r_M} x(i)\Big)^T W\Big(\sum_{i=r_m}^{r_M} x(i)\Big) \quad \leq \quad \delta \sum_{i=r_m}^{r_M} x^T(i)Wx(i),$$

where $\delta = r_M - r_m + 1$.

Lemma 2.4. *[19] Let $M \in R^{n\times n}$ be a positive-definite matrix, $X_i \in R^n, i = 1, 2, ...$ If the sums concerned are well defined, then*

$$\begin{aligned}&\Big(\sum_{i=k-M}^{k-N-1}\sum_{j=i}^{k-N-1} X_j\Big)^T M\Big(\sum_{i=k-M}^{k-N-1}\sum_{j=i}^{k-N-1} X_j\Big)\\ &\leq \frac{(M-N)^2}{2}\sum_{i=k-M}^{k-N-1}\sum_{j=i}^{k-N-1} X_j^T M X_j.\end{aligned}$$

We introduce the following notations for later use.

$$\prod = \big(\ \Sigma_{k,l}\ \big)_{13\times 13}\,, \tag{10}$$

where $\quad \Sigma_{k,l} = \Sigma_{l,k}^T, \quad k, l = 1, 2, 3, ..., 13,$

$$\begin{aligned}\Sigma_{1,1} &= L_1^T A + L_1^T B - L_1^T + A^T L_1 + B^T L_1 - L_1\\ &\quad +Q + R + T + h_1 M + N + N^T + K + L\\ &\quad +L^T + h_1^2 F + C_1^T + C_1 + D_1^T + D_1\\ &\quad +E_1^T + E_1 + G_1^T + G_1 + \epsilon_1\alpha^2 I + 2\hat{P},\\ \Sigma_{1,2} &= P - L_1^T + A^T L_2 + B^T L_2 - L_2 + C_2 + D_2\\ &\quad +E_2 + G_2 + \hat{P},\\ \Sigma_{1,3} &= -N + C_3 - C_1^T + D_3 + E_3 + G_3,\\ \Sigma_{1,4} &= L + C_4 + D_4 - D_1^T + E_4 + G_4 - G_1^T,\end{aligned}$$

$$\begin{aligned}\Sigma_{1,5} &= C_5 + D_5 + E_5 - E_1^T + G_5 - \hat{P},\\ \Sigma_{1,6} &= h_1 F + C_6 - C_1^T + D_6 + E_6 + G_6,\\ \Sigma_{1,7} &= C_7 + D_7 - D_1^T + E_7 + G_7,\\ \Sigma_{1,8} &= -L_1^T B + A^T L_3 + B^T L_3 - L_3 + C_8\\ &\quad +D_8 + E_8 - E_1^T + G_8 - \hat{P},\end{aligned}$$

$$\Sigma_{1,9} = C_9 + D_9 + E_9 + G_9 - G_1^T,$$
$$\Sigma_{1,10} = C_{10} + D_{10} + E_{10} + G_{10},$$
$$\Sigma_{1,11} = C_{11} + D_{11} + E_{11} + G_{11},$$
$$\Sigma_{1,12} = C_{12} + D_{12} + E_{12} + G_{12} + L_1^T,$$
$$\Sigma_{1,13} = C_{13} + D_{13} + E_{13} + G_{13} + L_1^T,$$
$$\Sigma_{2,2} = P - L_2^T - L_2 + h_1^2 U + h_2^2 V + h_2^2 W + \rho^2 X + h_1 Y + h_2 Z + \frac{1}{4} h_1^4 F + \frac{1}{4}\rho^2 H,$$
$$\Sigma_{2,3} = -C_2^T,$$
$$\Sigma_{2,4} = -D_2^T - G_2^T,$$
$$\Sigma_{2,5} = -E_2^T - \hat{P},$$
$$\Sigma_{2,6} = -C_2^T,$$
$$\Sigma_{2,7} = -D_2^T,$$
$$\Sigma_{2,8} = -L_2^T B - L_3 - E_2^T - \hat{P},$$
$$\Sigma_{2,9} = -G_2^T,$$
$$\Sigma_{2,10} = \Sigma_{2,11} = 0,$$
$$\Sigma_{2,12} = L_2^T,$$
$$\Sigma_{2,13} = L_2^T,$$
$$\Sigma_{3,3} = -Q + S - H - C_3^T - C_3,$$
$$\Sigma_{3,4} = -C_4 - D_3^T - G_3^T,$$
$$\Sigma_{3,5} = -C_5 - E_3^T,$$
$$\Sigma_{3,6} = -C_6 - C_3^T,$$
$$\Sigma_{3,7} = -C_7 - D_3^T,$$
$$\Sigma_{3,8} = -C_8 - E_3^T,$$
$$\Sigma_{3,9} = -C_9 - G_3^T,$$
$$\Sigma_{3,10} = -C_{10},$$

$$\Sigma_{3,11} = H - C_{11},$$
$$\Sigma_{3,12} = -C_{12},$$
$$\Sigma_{3,13} = -C_{13},$$
$$\Sigma_{4,4} = -R - S - D_4^T - D_4 - G_4^T - G_4,$$
$$\Sigma_{4,5} = -D_5 - E_4^T - G_5,$$
$$\Sigma_{4,6} = -C_4^T - D_6 - G_6,$$

$$\Sigma_{4,7} = -D_7 - D_4^T - G_7,$$
$$\Sigma_{4,8} = -D_8 - E_4^T - G_8,$$
$$\Sigma_{4,9} = -D_9 - G_9 - G_4^T,$$
$$\Sigma_{4,10} = -D_{10} - G_{10},$$
$$\Sigma_{4,11} = -D_{11} - G_{11},$$
$$\Sigma_{4,12} = -D_{12} - G_{12},$$
$$\Sigma_{4,13} = -D_{13} - G_{13},$$
$$\Sigma_{5,5} = -T - H - E_5^T - E_5 + \epsilon_1 \beta^2 I,$$
$$\Sigma_{5,6} = -C_5^T - E_6,$$
$$\Sigma_{5,7} = -D_5^T - E_7,$$
$$\Sigma_{5,8} = -E_8 - E_5^T,$$
$$\Sigma_{5,9} = -E_9 - G_5^T,$$
$$\Sigma_{5,10} = H - E_{10},$$
$$\Sigma_{5,11} = -E_{11},$$
$$\Sigma_{5,12} = -E_{12},$$
$$\Sigma_{5,13} = -E_{13},$$
$$\Sigma_{6,6} = -U - F - C_6^T - C_6,$$
$$\Sigma_{6,7} = -C^7 - D_6^T,$$
$$\Sigma_{6,8} = -C^8 - E_6^T,$$
$$\Sigma_{6,9} = -C_9 - G_6^T,$$
$$\Sigma_{6,10} = -C_{10},$$
$$\Sigma_{6,11} = -C_{11},$$
$$\Sigma_{6,12} = -C_{12},$$
$$\Sigma_{6,13} = -C_{13},$$
$$\Sigma_{7,7} = -V - D_7^T - D_7,$$

$$\Sigma_{7,8} = -D_8 - E_7^T,$$
$$\Sigma_{7,9} = -D_9 - G_7^T,$$
$$\Sigma_{7,10} = -D_{10},$$
$$\Sigma_{7,11} = -D_{11},$$
$$\Sigma_{7,12} = -D_{12},$$
$$\Sigma_{7,13} = -D_{13},$$
$$\Sigma_{8,8} = -L_3^T B - B^T L_3 - W - E_8^T - E_8,$$

$$\Sigma_{8,9} = -E_9 - G_8^T,$$
$$\Sigma_{8,10} = -E_{10},$$
$$\Sigma_{8,11} = -E_{11},$$
$$\Sigma_{8,12} = -E_{12} + L_3^T,$$
$$\Sigma_{8,13} = -E_{13} + L_3^T,$$
$$\Sigma_{9,9} = -X - G_9^T - G_9,$$
$$\Sigma_{9,10} = -G_{10},$$
$$\Sigma_{9,11} = -G_{11},$$
$$\Sigma_{9,12} = -G_{12},$$
$$\Sigma_{9,13} = -G_{13},$$
$$\Sigma_{10,10} = -H,$$
$$\Sigma_{11,11} = -H,$$
$$\Sigma_{12,12} = -\epsilon_1 I,$$
$$\Sigma_{13,13} = -\epsilon_2 I,$$
$$\rho = h_2 - h_1,$$
$$\alpha(k) = h(k) - h_1,$$
$$\beta(k) = h_2 - h(k),$$
$$\psi(k) = \frac{1}{\alpha(k)}\left[\sum_{i=k-h(k)}^{k-h_1-1} x(i)\right],$$
$$\phi(k) = \frac{1}{\beta(k)}\left[\sum_{i=k-h_2}^{k-h(k)-1} x(i)\right],$$
$$\hat{P} = PJ.$$

3. Delay-Range-Dependent Stability Criteria

Theorem 3.1. *The system (1)-(2) is asymptotically stable, if there exist positive definite symmetric matrices* $P, Q, R, S, T, U, V,$ W, X, Y, Z*, any appropriate dimensional matrices* $L_w, C_j, D_j, E_j, G_j,$ M, N, K, L, J, $w = 1, 2, 3,$ $j = 1, 2, \ldots, 13$ *and positive real constants* ϵ_1 *and* ϵ_2 *such that the following symmetric linear matrix inequalities hold:*

$$\prod < 0, \tag{11}$$

$$\begin{pmatrix} M & N \\ N^T & Y \end{pmatrix} \geq 0, \tag{12}$$

$$\begin{pmatrix} K & L \\ L^T & Z \end{pmatrix} \geq 0. \tag{13}$$

Proof. Consider the following Lyapunov-Krasovskii function for system (8)-(9) of the form

$$V(k) = \sum_{i=1}^{6} V_i(k), \tag{14}$$

where

$$\begin{aligned}
V_1(k) &= x^T(k)Px(k), \\
V_2(k) &= \sum_{i=k-h_1}^{k-1} x^T(i)Qx(i) + \sum_{i=k-h_2}^{k-1} x^T(i)Rx(i) \\
&\quad + \sum_{i=k-h_2}^{k-h_1-1} x^T(i)Sx(i) + \sum_{i=k-h(k)}^{k-1} x^T(i)Tx(i), \\
V_3(k) &= h_1 \sum_{j=-h_1}^{-1} \sum_{i=k+j}^{k-1} y^T(i)Uy(i) \\
&\quad +h_2 \sum_{j=-h_2}^{-1} \sum_{i=k+j}^{k-1} y^T(i)Vy(i) \\
&\quad +h_2 \sum_{j=-h_2}^{-1} \sum_{i=k+j}^{k-1} y^T(i)Wy(i), \\
V_4(k) &= \rho \sum_{j=-h_2}^{-h_1-1} \sum_{i=k+j}^{k-1} y^T(i)Xy(i),
\end{aligned}$$

$$\begin{aligned}
V_5(k) &= \sum_{j=-h_1+1}^{0} \sum_{i=k-1+j}^{k-1} y^T(i)Yy(i) \\
&\quad + \sum_{j=-h_2+1}^{0} \sum_{i=k-1+j}^{k-1} y^T(i)Zy(i),
\end{aligned}$$

$$\begin{aligned} V_6(k) &= \frac{h_1^2}{2}\sum_{i=-h_1}^{-1}\sum_{j=i}^{0}\sum_{l=k+j}^{k-1} y^T(l)Fy(l) \\ &+\frac{1}{2}\sum_{i=k-h_2}^{k-h_1-1}\sum_{j=i}^{k-h_1-1}\sum_{l=j}^{k-h_1-1} y^T(l)Hy(l). \end{aligned}$$

Evaluating the forward deference of $V(k)$, it is defined as

$$\Delta V(k) = \sum_{i=1}^{6}\Delta V_i(k). \tag{15}$$

Let us define for $i = 1, 2, ..., 6$,

$$\Delta V_i(k) = V_i(k+1) - V_i(k). \tag{16}$$

Taking the forward deference of $V_1(k)$ and $V_2(k)$, the increments of $V_1(k)$ and $V_2(k)$ are

$$\begin{aligned} \Delta V_1(k) &= x^T(k+1)Px(k+1) - x^T(k)Px(k) \\ &= \Big[x^T(k) + y^T(k)\Big]P\Big[x(k) + y(k) + Jx(k) \\ &\quad -Jx(k-h(k)) - J\sum_{i=k-h(k)}^{k-1} y(i)\Big] \\ &\quad +\Big[x^T(k)J^T - x^T(k-h(k))J^T \\ &\quad -\sum_{i=k-h(k)}^{k-1} y^T(i)J^T\Big]P\Big[x(k) + y(k)\Big] \\ &\quad +2x^T(k)L_1^T\Big[-y(k) + \Big[A+B-I\Big]x(k) \\ &\quad -B\sum_{i=k-h(k)}^{k-1} y(i) + f(k,x(k)) + g(k,x(k-h(k)))\Big] \\ &\quad +2y^T(k)L_2^T\Big[-y(k) + [A+B-I\Big]x(k) \\ &\quad -B\sum_{i=k-h(k)}^{k-1} y(i) + f(k,x(k)) + g(k,x(k-h(k)))\Big] \end{aligned}$$

$$
\begin{aligned}
& +2 \sum_{i=k-h(k)}^{k-1} y(i)^T L_3^T \Big[-y(k) + [A+B-I]x(k) \\
& -B \sum_{i=k-h(k)}^{k-1} y(i) + f(k,x(k)) + g(k,x(k-h(k))) \Big] \\
& -x^T(k)Px(k), \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
\Delta V_2(k) \;=\; & x^T(k)Qx(k) - x^T(k-h_1)Qx(k-h_1) \\
& +x^T(k)Rx(k) - x^T(k-h_2)Rx(k-h_2) \\
& +x^T(k-h_1)Sx(k-h_1) - x^T(k-h_2)Sx(k-h_2) \\
& +x^T(k)Tx(k) - x^T(k-h(k))Tx(k-h(k)). \qquad (18)
\end{aligned}
$$

By Lemma 2.3, the increments of $V_3(k)$ and $V_4(k)$ are easily computed as

$$
\begin{aligned}
\Delta V_3(k) \;=\; & h_1^2 y^T(k)Uy(k) - h_1 \sum_{i=k-h_1}^{k-1} y^T(i)Uy(i) \\
& +h_2^2 y^T(k)Vy(k) - h_2 \sum_{i=k-h_2}^{k-1} y^T(i)Vy(i) \\
& +h_2^2 y^T(k)Wy(k) - h_2 \sum_{i=k-h_2}^{k-1} y^T(i)Wy(i) \\
\le\; & h_1^2 y^T(k)Uy(k) - \Big(\sum_{i=k-h_1}^{k-1} y(i)\Big)^T U \Big(\sum_{i=k-h_1}^{k-1} y(i)\Big) \\
& +h_2^2 y^T(k)Vy(k) - \Big(\sum_{i=k-h_2}^{k-1} y(i)\Big)^T V \Big(\sum_{i=k-h_2}^{k-1} y(i)\Big) \\
& +h_2^2 y^T(k)Wy(k) - \Big(\sum_{i=k-h(k)}^{k-1} y(i)\Big)^T W \Big(\sum_{i=k-h(k)}^{k-1} y(i)\Big), \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
\Delta V_4(k) \;=\; & \rho^2 y^T(k)Xy(k) - \rho \sum_{i=k-h_2}^{k-h_1-1} y^T(i)Xy(i) \\
\le\; & \rho^2 y^T(k)Xy(k) - \Big(\sum_{i=k-h_2}^{k-h_1-1} y(i)\Big)^T X \Big(\sum_{i=k-h_2}^{k-h_1-1} y(i)\Big), \qquad (20)
\end{aligned}
$$

Taking the forward deference of $V_5(k)$ yields

$$\begin{aligned}\Delta V_5(k) &= h_1 y^T(k)Yy(k) - \sum_{i=-h_1+1}^{0} y^T(k-1+i)Yy(k-1+i)\\ &\quad + h_2 y^T(k)Zy(k) - \sum_{i=-h_2+1}^{0} y^T(k-1+i)Zy(k-1+i). \quad (21)\end{aligned}$$

By (12) and (13), it is easy to see that

$$\begin{aligned}&2x^T(k)N\sum_{i=-h_1+1}^{0} y(k-1+i)\\ &+\sum_{i=-h_1+1}^{0} y^T(k-1+i)Yy(k-1+i)\\ &+h_1x^T(k)Mx(k) = \sum_{i=-h_1+1}^{0}\begin{pmatrix} x(k)\\ y(k-1+i)\end{pmatrix}^T\\ &\begin{pmatrix} M & N\\ N^T & Y\end{pmatrix}\begin{pmatrix} x(k)\\ y(k-1+i)\end{pmatrix} \geq 0, \quad (22)\\ &2x^T(k)L\sum_{i=-h_1+1}^{0} y(k-1+i)\\ &+\sum_{i=-h_2+1}^{0} y^T(k-1+i)Zy(k-1+i)\\ &+h_2x^T(k)Kx(k) = \sum_{i=-h_2+1}^{0}\begin{pmatrix} x(k)\\ y(k-1+i)\end{pmatrix}^T\\ &\begin{pmatrix} K & L\\ L^T & Z\end{pmatrix}\begin{pmatrix} x(k)\\ y(k-1+i)\end{pmatrix} \geq 0. \quad (23)\end{aligned}$$

From (22) and (23), we can obtain

$$\begin{aligned}&-\sum_{i=-h_1+1}^{0} y^T(k-1+i)Yy(k-1+i)\\ \leq\;& h_1x^T(k)Mx(k) + 2x^T(k)N\sum_{i=-h_1+1}^{0} y(k-1+i)\end{aligned}$$

$$
\begin{aligned}
&= h_1 x^T(k) M x(k) + 2x^T(k) N x(k) \\
&\quad -2x^T(k) N x(k-h_1),
\end{aligned} \tag{24}
$$

and we have

$$
\begin{aligned}
& -\sum_{i=-h_2+1}^{0} y^T(k-1+i) Z y(k-1+i) \\
\leq\ & h_2 x^T(k) K x(k) + 2x^T(k) L \sum_{i=-h_2+1}^{0} y(k-1+i) \\
=\ & h_2 x^T(k) K x(k) + 2x^T(k) L x(k) \\
& -2x^T(k) L x(k-h_2).
\end{aligned} \tag{25}
$$

Therefore, we conclude that

$$
\begin{aligned}
\Delta V_5(k) \leq\ & h_1 y^T(k) Y y(k) + h_1 x^T(k) M x(k) \\
& +2x^T(k) N x(k) - 2x^T(k) N x(k-h_1) \\
& +h_2 y^T(k) Z y(k) + h_2 x^T(k) K x(k) \\
& +2x^T(k) L x(k) - 2x^T(k) L x(k-h_2).
\end{aligned} \tag{26}
$$

We take the forward difference of $V_6(k)$ as

$$
\begin{aligned}
\Delta V_6(k) =\ & y^T(k) \left[\frac{h_1^4}{4} F + \frac{(h_2-h_1)^2}{4} H \right] y(k) \\
& -\frac{h_1^2}{2} \sum_{i=-h_1}^{-1} \sum_{j=k+i}^{k-1} y^T(j) F y(j) \\
& \frac{-1}{2} \sum_{i=k-h_2}^{k-h_1-1} \sum_{j=i}^{k-h_1-1} y^T(j) H y(j).
\end{aligned} \tag{27}
$$

Furthermore, it follows from Lemma 2.4 that

$$
\begin{aligned}
& -\frac{h_1^2}{2} \sum_{i=-h_1}^{-1} \sum_{j=k+i}^{k-1} y^T(j) F y(j) \\
\leq\ & -\left[h_1 x(k) - \sum_{i=k-h_1}^{k-1} x(i) \right]^T F \left[h_1 x(k) - \sum_{i=k-h_1}^{k-1} x(i) \right],
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
& -\frac{1}{2}\sum_{i=k-h_2}^{k-h_1-1}\sum_{j=i}^{k-h_1-1} y^T(j)Hy(j) \\
= \quad & -1/2 \sum_{i=k-h(k)}^{k-h_1-1}\sum_{j=i}^{k-h_1-1} y^T(j)Hy(j) \\
& -1/2 \sum_{i=k-h_2}^{k-h(k)-1}\sum_{j=i}^{k-h(k)-1} y^T(j)Hy(j) \\
& -1/2 \sum_{i=k-h_2}^{k-h(k)-1}\sum_{j=k-h(k)}^{k-h_1-1} y^T(j)Hy(j) \\
\leq \quad & -\left[\sum_{i=k-h(k)}^{k-h_1-1}\sum_{j=i}^{k-h_1-1} y(j)\right]^T \frac{1}{\alpha^2(k)}H\left[\sum_{i=k-h(k)}^{k-h_1-1}\sum_{j=i}^{k-h_1-1} y(j)\right] \\
& -\left[\sum_{i=k-h_2}^{k-h(k)-1}\sum_{j=i}^{k-h(k)-1} y(j)\right]^T \frac{1}{\beta^2(k)}H\left[\sum_{i=k-h_2}^{k-h(k)-1}\sum_{j=i}^{k-h(k)-1} y(j)\right] \\
= \quad & -\Big[x(k-h_1)-\psi(k)\Big]^T H\Big[x(k-h_1)-\psi(k)\Big] \\
& -\Big[x(k-h(k))-\phi(k)\Big]^T H\Big[x(k-h(k))-\phi(k)\Big]. \qquad (29)
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
& \Delta V_6(k) \leq y^T(k)\left[\frac{h_1^4}{4}F + \frac{(h_2-h_1)^2}{4}H\right]y(k) \\
& -\left[h_1x(k) - \sum_{i=k-h_1}^{k-1} x(i)\right]^T F\left[h_1x(k) - \sum_{i=k-h_1}^{k-1} x(i)\right] \\
& -\left[x(k-h_1)-\psi(k)\right]^T H\left[x(k-h_1)-\psi(k)\right] \\
& -\left[x(k-h(k))-\phi(k)\right]^T H\left[x(k-h(k))-\phi(k)\right]. \qquad (30)
\end{aligned}
$$

It is obvious that

$$
\begin{aligned}
\Upsilon \quad &\equiv \quad x(k)-x(k-h_1)-\sum_{i=k-h_1}^{k-1} y(i) = 0, \\
\Phi \quad &\equiv \quad x(k)-x(k-h_2)-\sum_{i=k-h_2}^{k-1} y(i) = 0,
\end{aligned}
$$

$$\begin{aligned}\Psi &\equiv x(k) - x(k-h(k)) - \sum_{i=k-h(k)}^{k-1} y(i) = 0,\\ \Omega &\equiv x(k-h_1) - x(k-h_2) - \sum_{i=k-h_2}^{k-h_1-1} y(i) = 0.\end{aligned}$$

The following equations are true for any matrices with appropriate dimensions:

$$\begin{aligned}&\Big[2x^T(k)C_1^T + 2y^T(k)C_2^T + 2x^T(k-h_1)C_3^T\\ &+2x^T(k-h_2)C_4^T + 2x^T(k-h(k))C_5^T\\ &+2\Big(\sum_{i=k-h_1}^{k-1} y(i)\Big)^T C_6^T + 2\Big(\sum_{i=k-h_2}^{k-1} y(i)\Big)^T C_7^T\\ &+2\Big(\sum_{i=k-h(k)}^{k-1} y(i)\Big)^T C_8^T + 2\Big(\sum_{i=k-h_2}^{k-h_1-1} y(i)\Big)^T C_9^T\\ &+2\phi^T(k)C_{10}^T + 2\psi^T(k)C_{11}^T + 2f^T(k,x(k))C_{12}^T\\ &+2g^T(k,x(k-h(k)))C_{13}^T\Big] \times \Upsilon = 0,\end{aligned} \tag{31}$$

$$\begin{aligned}&\Big[2x^T(k)D_1^T + 2y^T(k)D_2^T + 2x^T(k-h_1)D_3^T\\ &+2x^T(k-h_2)D_4^T + 2x^T(k-h(k))D_5^T\\ &+2\Big(\sum_{i=k-h_1}^{k-1} y(i)\Big)^T D_6^T + 2\Big(\sum_{i=k-h_2}^{k-1} y(i)\Big)^T D_7^T\\ &+2\Big(\sum_{i=k-h(k)}^{k-1} y(i)\Big)^T D_8^T + 2\Big(\sum_{i=k-h_2}^{k-h_1-1} y(i)\Big)^T D_9^T\\ &+2\phi^T(k)D_{10}^T + 2\psi^T(k)D_{11}^T + 2f^T(k,x(k))D_{12}^T\\ &+2g^T(k,x(k-h(k)))D_{13}^T\Big] \times \Phi = 0,\end{aligned} \tag{32}$$

$$\begin{aligned}&\Big[2x^T(k)E_1^T + 2y^T(k)E_2^T + 2x^T(k-h_1)E_3^T\\ &+2x^T(k-h_2)E_4^T + 2x^T(k-h(k))E_5^T\\ &+2\Big(\sum_{i=k-h_1}^{k-1} y(i)\Big)^T E_6^T + 2\Big(\sum_{i=k-h_2}^{k-1} y(i)\Big)^T E_7^T\end{aligned}$$

$$
\begin{aligned}
&+2\Big(\sum_{i=k-h(k)}^{k-1} y(i)\Big)^T E_8^T + 2\Big(\sum_{i=k-h_2}^{k-h_1-1} y(i)\Big)^T E_9^T \\
&+2\phi^T(k)E_{10}^T + 2\psi^T(k)E_{11}^T + 2f^T(k,x(k))E_{12}^T \\
&+2g^T(k,x(k-h(k)))E_{13}^T\Big] \times \Psi = 0, \qquad (33)\\
&\Big[2x^T(k)G_1^T + 2y^T(k)G_2^T + 2x^T(k-h_1)G_3^T \\
&+2x^T(k-h_2)G_4^T + 2x^T(k-h(k))G_5^T \\
&+2\Big(\sum_{i=k-h_1}^{k-1} y(i)\Big)^T G_6^T + 2\Big(\sum_{i=k-h_2}^{k-1} y(i)\Big)^T G_7^T \\
&+2\Big(\sum_{i=k-h(k)}^{k-1} y(i)\Big)^T G_8^T + 2\Big(\sum_{i=k-h_2}^{k-h_1-1} y(i)\Big)^T G_9^T \\
&+2\phi^T(k)G_{10}^T + 2\psi^T(k)G_{11}^T + 2f^T(k,x(k))G_{12}^T \\
&+2g^T(k,x(k-h(k)))G_{13}^T\Big] \times \Omega = 0. \qquad (34)
\end{aligned}
$$

From (3) and (4), we obtain for any scalars $\epsilon_1, \epsilon_2 > 0$,

$$
\begin{aligned}
&\epsilon_1\Big(\alpha^2 x^T(k)x(k) - f^T(k,x(k))f(k,x(k))\Big) \geq 0, \qquad (35)\\
&\epsilon_2\Big(\beta^2 x^T(k-h(k))x(k-h(k)) \\
&-g^T(k,x(k-h(k)))g(k,x(k-h(k)))\Big) \geq 0. \qquad (36)
\end{aligned}
$$

It follows form (15)-(36) that

$$
\Delta V(k) \;\; \leq \;\; \xi^T(k)\prod\xi(k), \qquad (37)
$$

where $\xi^T(k) = \Big[x(k)^T \quad y(k)^T \quad x(k-h_1)^T \quad x(k-h_2)^T$
$x(k-h(k))^T \quad \sum_{i=k-h_1}^{k-1} y^T(i) \quad \sum_{i=k-h_2}^{k-1} y^T(i)$
$\sum_{i=k-h(k)}^{k-1} y(i)^T \quad \sum_{i=k-h_2}^{k-h_1-1} y^T(i) \quad \phi(k) \quad \psi(k)$
$f(k,x(k)) \quad g(k,x(k-h(k)))\Big]$ and $\prod$ is defined in (10). By (37), if conditions (11)-(13) are true, then

$$
\Delta V(k) < -\omega\|x\|^2, \qquad (38)
$$

where $\omega > 0$. This means that system (1)-(2) is asymptotically stable. The proof of theorem is complete.

If $f(k, x(k)) = g(k, x(k-h(k))) = 0$ then system (1)-(2) reduces to the following system:

$$x(k+1) = Ax(k) + Bx(k-h(k)), \tag{39}$$
$$x(s) = \phi(s), \quad s \in \{-h_2, \ldots, -1, 0\}. \tag{40}$$

According to Theorem 3.1, we have the following Corollary 3.2 for the delay-range-dependent asymptotic stability criteria of system (39)-(40). We introduce the following notations for later use.

$$\coprod = \left(\ \Omega_{k,l} \ \right)_{11\times 11}, \tag{41}$$

where $\Omega_{k,l} = \Omega_{l,k}^T = \Sigma_{k,l}$, $k, l = 1, 2, 3, ..., 11$, except $\Omega_{1,1} = \Sigma_{1,1} - \epsilon_1\alpha^2 I$, $\Omega_{5,5} = \Sigma_{5,5} - \epsilon_2\beta^2 I$.

Corollary 3.2. *The system (39)-(40) is asymptotically stable, if there exist positive definite symmetric matrices* $P, Q, R, S, T, U, V, W, X, Y, Z$, *any appropriate dimensional matrices* $L^w, C^j, D^j, E^j, G^j, M, N, K, L, J$, $w = 1, 2, 3$, $j = 1, 2, \ldots, 11$ *such that the following symmetric linear matrix inequalities hold:*

$$\coprod < 0, \tag{42}$$
$$\begin{pmatrix} M & N \\ * & Y \end{pmatrix} \geq 0, \tag{43}$$
$$\begin{pmatrix} K & L \\ * & Z \end{pmatrix} \geq 0. \tag{44}$$

4. Numerical Examples

Example 4.1 Consider the system (1)-(2) with the following parameters which is considered in [16] and [24] :

$$A = \begin{pmatrix} 0.80 & 0 \\ 0.05 & 0.90 \end{pmatrix}, B = \begin{pmatrix} -0.10 & 0 \\ -0.20 & -0.10 \end{pmatrix}, \tag{45}$$
$$\alpha \geq 0, \quad \beta \geq 0. \tag{46}$$

By using the LMI Toolbox in MATLAB (with accuracy 0.01) for Theorem 3.1 to system (1)-(2) with (45)-(46), one can obtain the maximum upper bounds of the time delay under different values of h_1 as shown in Table 1. We can see

Table 1: Upper bounds of time delay h_2 for different conditions for Example 4.1.

Method	h_1	2	6	10	14
$\alpha = 0.1, \beta = 0$					
Zhang et al. (2011) [24]	h_2	10	12	14	17
Ramakrishnan and Ray (2013) [16]	h_2	13	14	16	18
Corollary 3.1	h_2	21	23	23	24
$\alpha = 0, \beta = 0.1$					
Ramakrishnan and Ray (2013) [16]	h_2	11	13	15	17
Corollary 3.1	h_2	19	24	23	22
$\alpha = 0.1, \beta = 0.1$					
Ramakrishnan and Ray (2013) [16]	h_2	10	11	13	15
Corollary 3.1	h_2	16	19	22	21

that our results in Theorem 3.1 are much less conservative than in [16] and [24].

Example 4.2 Consider the system (39)-(40) with the following parameters which is considered in [3], [5], [22] and [23] :

$$A = \begin{pmatrix} 0.80 & 0 \\ 0.05 & 0.90 \end{pmatrix}, \qquad B = \begin{pmatrix} -0.10 & 0 \\ -0.20 & -0.10 \end{pmatrix}. \tag{47}$$

By using the LMI Toolbox in MATLAB (with accuracy 0.01) for Corollary 3.2 to system (39)-(40) with (47), the maximum upper bounds h_2 for asymptotic stability of Example 4.2 is listed in the comparison in Table 2, for different values of h_1. We can see that our results in Corollary 3.2 are much less conservative than in [3], [5], [22] and [23].

Table 2: Upper bounds of time delay h_2 for different conditions for Example 4.2.

Method	h_1	2	6	10	14
Fridman and Shaked (2005) [3]	h_2	13	15	19	23
Gao and Chen (2007) [5]	h_2	14	15	18	22
Zhang et al. (2010) [22]	h_2	15	17	21	25
Zhang et al. (2011) [23]	h_2	15	17	21	25
Corollary 3.2	h_2	> 16	> 18	> 22	> 26

5. Conclusion

In this paper, the problem of asymptotic stability analysis for discrete-time linear system with interval time-varying delay and nonlinear perturbations has been presented. The method combining augmented Lyapunov-Krasovskii functional, mixed model transformation, Jensen-type summation inequality and utilization of zero equation have been studied. New delay-range-dependent asymptotic stability criteria have been obtained and formulated in terms of LMIs. By comparing the proposed results with the results available in the existing literature, it is shown that the derived criteria are less conservative.

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