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SOLITARY WAVE SOLUTIONS TO TIME-FRACTIONAL COUPLED DEGENERATE HAMILTONIAN EQUATIONS

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Abstract: The aim of this paper is to investigate the solution of time-fractional coupled degenerate Hamiltonian equations. We use the G'/G expansion method to determine soliton solutions of this system for different values of the fractional order.

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1. Introduction and Discussion

Fractional order partial differential equations, as generalizations of classical integer order partial differential equations, are increasingly used to model problems in fluid, finance, physical and biological processes and systems [1, 2, 3, 4]. Consequently, considerable attention has been given to the solution of fractional ordinary differential equations, integral equations and fractional partial differential equations. Since most fractional differential equations do not have exact analytic solutions, approximation and numerical techniques, therefore, are used extensively. Recently, He and Lee [5, 6, 7] suggested a fractional complex trans-

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form instrument that converts fractional derivatives into classical derivatives. They considered the following PDE:

$$f(u, u_t^{\alpha}, u_x^{\beta}, u_t^{2\alpha}, u_x^{2\beta}, ...) = 0, \quad (1)$$

where $u_t^{(\alpha)} = \frac{\partial^{\alpha} u(t,x)}{\partial t^{\alpha}}$ denotes the modified Riemann-Liouville derivative [8, 9, 10]; $0 < \alpha \leq 1$ and $0 < \beta \leq 1$. Then, the following transforms are introduced:

$$s = \frac{p\, t^{\alpha}}{\Gamma(1+\alpha)}, \quad (2)$$

$$X = \frac{q\, x^{\beta}}{\Gamma(1+\beta)}, \quad (3)$$

where p and q are constants.

Using the above transforms, we can convert fractional derivatives into classical derivatives:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = p\frac{\partial u}{\partial s}, \quad (4)$$

$$\frac{\partial^{\beta} u}{\partial x^{\beta}} = q\frac{\partial u}{\partial X}. \quad (5)$$

2. Preliminaries

Definitions for fractional derivatives: for more details see Ref. [8]. In our paper we use the modified Riemann-Liouville derivative which was defined by Jumarie [9, 10, 11]

$$\begin{aligned} D_x^{\alpha} f(x) &= \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-y)^{-1-\alpha}[f(y)-f(0)]dy, \ \alpha < 0 \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-y)^{-1-\alpha}[f(y)-f(0)]dy, \ 0 < \alpha < 1 \\ &= [f^{(\alpha-n)}(x)]^{(n)}, \ n \leq \alpha < n+1, \end{aligned} \quad (1)$$

which has merits over the original one, for example, the α-order derivative of a constant is zero. The main properties of the modified Riemann-Liouville derivative were summarized in [9] and three useful formulas of them are given as follows:

$$D_x^{\alpha} x^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \ \beta > 0$$

$$D_x^\alpha(u(x)v(x)) = u(x)D_x^\alpha(v(x)) + v(x)D_x^\alpha(u(x))$$
$$D_x^\alpha[f(u(x))] = \frac{df}{du}D_x^\alpha u(x) = (\frac{du}{dx})^\alpha D_x^\alpha f(u). \quad (2)$$

In the literature, many significant methods have been proposed for obtaining exact solutions of nonlinear partial differential equations such as the tanh method, trigonometric and hyperbolic function methods, the rational sine-cosine method, the extended tanh-function method, the Exp-function method, the Hirota's method , Hirota bilinear forms, the tanh-sech method , the first integral method and so on [12]-[28]. In this work, we implement the G'/G expansion method [29, 30, 31, 32, 33] with the help of symbolic computation to derive soliton solution of the Hamiltonian system that reads

$$D_t^\alpha u = u_x + 2v,$$
$$D_t^\alpha v = 2\,\epsilon\, uv, \quad \epsilon = \pm 1. \quad (3)$$

3. Survey of G'/G Method

Consider the following nonlinear partial differential equation:

$$P(u, u_t, u_x, u_{tt}, u_{xt}, ...) = 0, \quad (1)$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. By the wave variable $\zeta = x - ct$ the PDE (1) is then transformed to the ordinary differential equation (ODE)

$$P(u, -cu', u', c^2u'', -cu'', u'', ...) = 0, \quad (2)$$

where $u = u(\zeta)$. Suppose that the solution of ODE (2) can be expressed by a polynomial in G'/G as follows [29, 30, 31, 32, 33]

$$u(\zeta) = a_m(\frac{G'}{G})^m + ..., \quad (3)$$

where $G = G(\zeta)$ satisfies the second order differential equation in the form

$$G'' + \lambda G' + \mu G = 0, \quad (4)$$

$a_0, a_1, ..., a_m, \lambda$ and μ are constants to be determined later, provided that $a_m \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in the ODE (2).

Now, if we let

$$Y = Y(\zeta) = \frac{G'}{G}, \tag{5}$$

then by the help of (4) we get

$$Y' = \frac{GG'' - G'^2}{G^2} = \frac{G(-\lambda G' - \mu G) - G'^2}{G^2} = -\lambda Y - \mu - Y^2. \tag{6}$$

Or, equivalently

$$Y' = -Y^2 - \lambda Y - \mu. \tag{7}$$

By result (7) and implicit differentiation, one can derive the following two formulas

$$\begin{aligned} Y'' &= 2Y^3 + 3\lambda Y^2 + (2\mu + \lambda^2)Y + \lambda\mu, & (8)\\ Y''' &= -6Y^4 - 12\lambda Y^3 - (7\lambda^2 + 8\mu)Y^2 - (\lambda^3 + 8\lambda\mu)Y - (\lambda^2\mu + 2\mu^2). & (9) \end{aligned}$$

Combining equations (3), (5) and (7-9), then it results in a polynomial of powers of Y. Then, collecting all terms of same order of Y and equating to zero, yields a set of algebraic equations for $a_0, a_1, ..., a_m, \lambda$, and μ.

It is known that the solution of equation (4) is a linear combination of sinh and cosh or of sine and cosine, respectively, if $\triangle = \lambda^2 - 4\mu > 0$ or $\triangle < 0$. Without lost of generality, we consider the first case and therefore

$$G(\zeta) = \left(A \ \sinh(\frac{\sqrt{\lambda^2 - 4\mu}\ \zeta}{2}) + B \ \cosh(\frac{\sqrt{\lambda^2 - 4\mu}\ \zeta}{2}) \right) e^{-\frac{\lambda\zeta}{2}}. \tag{10}$$

4. Time-Fractional Hamiltonian System

In this section, we study the time-fractional Hamiltonian system:

$$\begin{aligned} D_t^\alpha u &= u_x + 2v, \\ D_t^\alpha v &= 2\,\epsilon\, uv, \quad \epsilon = \pm 1. \end{aligned} \tag{1}$$

The fractional transform $s = \frac{t^\alpha}{\Gamma(1+\alpha)}$ reduces (1) to

$$u_s = u_x + 2v,$$

$$v_s \quad = \quad 2\,\epsilon\, uv, \quad \epsilon = \pm 1. \tag{2}$$

where $u = u(s,x),\ v = v(s,x)$. The wave variable $\zeta = x - cs$ transforms the PDEs (2) into the following ODEs

$$\begin{aligned} -cu' &= u' + 2v, \\ -cv' &= 2\epsilon\, uv. \end{aligned} \tag{3}$$

where $u = u(\zeta),\ v = v(\zeta)$. From the first equation of (3) we have the relation

$$v = -\frac{1}{2}(1+c)u', \tag{4}$$

and from the second equation of (3) and the relation (4) we reach to the following ODE

$$\frac{1}{2}cu'' = -\epsilon u u'. \tag{5}$$

Integrating the obtained equation in (5) yields

$$cu' + \frac{\epsilon}{2}u^2 + R = 0, \tag{6}$$

where R is the constant of integration.

Assume the solution of (6) is

$$u(\zeta) = a_m\left(\frac{G'}{G}\right)^m + ... \tag{7}$$

Then, recalling (3) and (7) we have

$$u^2(\zeta) = a_m^2\left(\frac{G'}{G}\right)^{2m} + ..., \tag{8}$$

and

$$u'(\zeta) = m a_m\left(\frac{G'}{G}\right)^{m+1} + ... \tag{9}$$

Balancing the orders in (8) and (9), we require that $m+1 = 2m$. Thus, $m = 1$, and therefore (7) can be rewritten as

$$u(\zeta) = \ a_1\left(\frac{G'}{G}\right) + a_0 \ = \ a_1 Y + a_0. \tag{10}$$

Now, we substitute equations (10) and (7) in (6) to get the following algebraic system:

$$0 \quad = \quad R - \frac{a_0^2\epsilon}{2} - c\mu,$$

$$\begin{aligned} 0 &= a_0 a_1 \epsilon + c\lambda, \\ 0 &= c + \frac{a_1^2 \epsilon}{2}. \end{aligned} \tag{11}$$

Solving the above system yields

$$\begin{aligned} \lambda &= \frac{2a_0}{a_1}, \\ \mu &= \frac{a_0^2\epsilon - 2R}{a_1^2\epsilon}, \\ c &= -\frac{a_1^2\epsilon}{2}. \end{aligned} \tag{12}$$

By the assumption given in (10) the function $u(x,t)$ is given by

$$u(x,t) = \sqrt{\frac{2R}{\epsilon}} \left(\frac{A + B \ \tanh\left(\frac{1}{a_1}\sqrt{\frac{2R}{\epsilon}}(x + \frac{a_1^2\epsilon t^\alpha}{2\Gamma(\alpha)})\right)}{B + A \ \tanh\left(\frac{1}{a_1}\sqrt{\frac{2R}{\epsilon}}(x + \frac{a_1^2\epsilon t^\alpha}{2\Gamma(\alpha)})\right)} \right), \tag{13}$$

provided that $A^2 \neq B^2$.

Now, by (4) the function $v(x,t)$ is given by

$$v(x,t) = \frac{R(2 - a_1^2\epsilon)(A^2 - B^2)}{2a_1\epsilon \left(B \ \cosh\left(\frac{1}{a_1}\sqrt{\frac{2R}{\epsilon}}(x + \frac{a_1^2\epsilon t^\alpha}{2\Gamma(\alpha)})\right) + A \ \sinh\left(\frac{1}{a_1}\sqrt{\frac{2R}{\epsilon}}(x + \frac{a_1^2\epsilon t^\alpha}{2\Gamma(\alpha)})\right)\right)^2}. \tag{14}$$

5. Conclusion

In general, there exist no method that yields solitary solutions for fractional solitary wave equations. But, a fractional wave transform is adopted in this paper to convert such equations into classical partial differential equations. We succeeded in extracting solitary solutions for time-fractional Hamiltonian system.

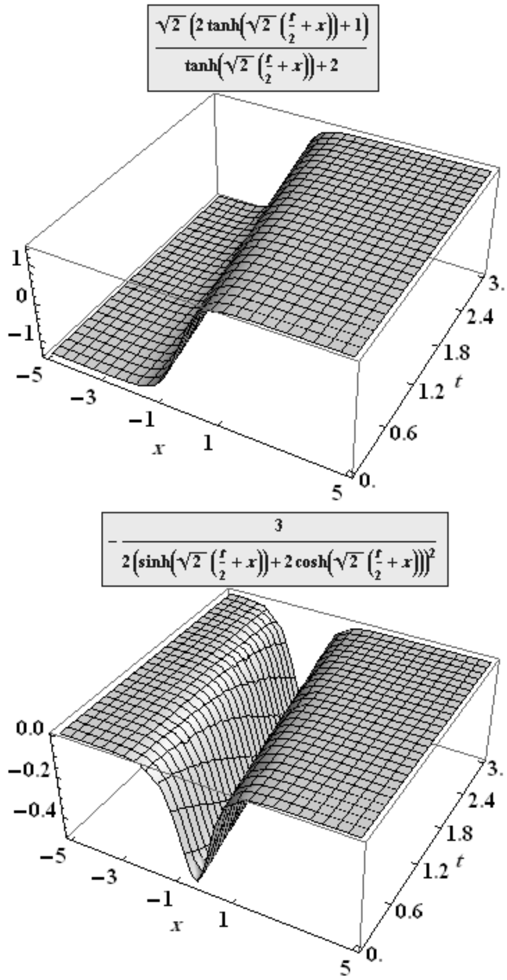

Figure 1: Plots of $u(x,t)$ on the left and $v(x,t)$ on the right, where $a_0 = 1,\ a_1 = 1,\ A = 1,\ B = 1,\ R = 1,\ \epsilon = 1,\ \alpha = 1$

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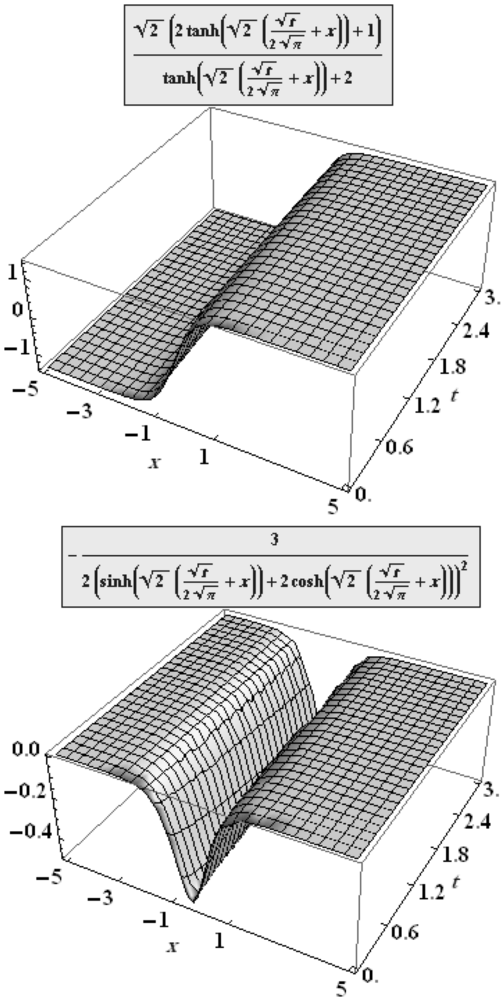

Figure 2: Plots of $u(x,t)$ on the left and $v(x,t)$ on the right, where $a_0 = 1,\ a_1 = 1,\ A = 1,\ B = 1,\ R = 1,\ \epsilon = 1,\ \alpha = \frac{1}{2}$

References

[1] Debnath L., Bhatta D., Solutions to few linear fractional inhomogeneous partial differential equations in fluid mechanics. *Fractional Calculus and Applied Analysis*, Volume (7) (2004) 153-192.

[2] Oldham K.B., Spanier J., The Fractional Calculus. *Academic Press, New*

York, (1974).

[3] Podlubny I., Fractional Differential Equations. *Academic Press, New York*, (1999).

[4] Podlubny I., Geometric and physical interpretation of fractional integration and fractional differentiation. *Fractional Calculus and Applied Analysis*, Volume (5) (2002) 367-386.

[5] Li, Z. B., He, J. H., Fractional complex transform for fractional differential equations. *Mathematical and Computational Applications*, Volume (15)(5) (2010) 970-973.

[6] Li, Z. B., An extended fractional complex transform. *Journal of Nonlinear Science and Numerical Simulation*, Volume (11)(2010) 335-337.

[7] He, J. H., Li, Z. B., Converting fractional differential equations into partial differential equations. *Thermal Science*, Volume (16)(2) (2012) 331-334.

[8] Oldham K B and Spanier J, The Fractional Calculus (New York: Academic Press) 1974

[9] Jumarie, G., Cauchy's integral formula via modified Riemann-Liouville derivative for analytical functions of fractional order. *Applied Mathematics Letters*, Volume (23)(12) (2010) 1444-1450.

[10] Jumarie, G., Fractional partial differential equations and modified Riemann-Liouville derivative: New method for solution. *Journal of Applied Mathematics and Computing*, Volume (24)(1) (2007) 31-48.

[11] Jumarie, G., Modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions: Further results. *Computers and Mathematics with Applications*, Volume (51)(9-10) (2007) 1367-1376.

[12] Alquran M., Bright and dark soliton solutions to the Ostrovsky-Benjamin-Bona-Mahony (OS-BBM) equation. *J. Math. Comput. Sci.*, Volume 2(1) (2012) pp. 15-22.

[13] Alquran M., Al-khaled K., Mathematical methods for a reliable treatment of the (2+1)-dimensional Zoomeron equation. *Mathematical Sciences*, Volume 6(12) (2012).

[14] Alquran M., Al-Khaled K., Ananbeh H., New Soliton Solutions for Systems of Nonlinear Evolution Equations by the Rational Sine-Cosine Method. *Studies in Mathematical Sciences.*, Volume 3(1) (2011) pp. 1-9.

[15] Alquran M., Ali M., Al-Khaled K., Solitary wave solutions to shallow water waves arising in fluid dynamics. *Nonlinear Studies*, Volume 19(4) (2012) pp. 555-562.

[16] Alquran M., Qawasmeh A., Classifications of solutions to some generalized nonlinear evolution equations and systems by the sine-cosine method. *Nonlinear Studies*, Volume 20(2) (2013) pp. 261-270.

[17] Bekir A., Unsal O., Exact Solutions for a Class of Nonlinear Wave Equations By Using First Integral Method. *International Journal of Nonlinear Science.*, Volume 15(2) (2013) pp.99-110.

[18] Palacios S.L., Two simple ansatze for obtaining exact solutions of high dispersive nonlinear schrodinger equations. *Chaos, Solitons and Fractals.*, Volume 19 (2004) pp. 203-207.

[19] Nakkeeran K., Bright and dark optical solitons in fiber media with higher order effects. *Chaos, Solitons and Fractals.*, Volume 13 (2002) pp.673-679.

[20] Li X., Wang M., A sub-ODE method for finding exact solutions of a generalized KdV-mKdV equation with high order nonlinear terms. *Physics Letters A.*, Volume 361 (2007) pp. 115-118.

[21] Wazwaz A.M., The extended Tanh method for Abundant solitary wave solutions of nonlinear wave equations. *Applied Mathematics and Computation*, Volume 187 (2007) pp. 1131-1142.

[22] Alquran M., Al-Khaled K., Sinc and solitary wave solutions to the generalized Benjamin-Bona-Mahony-Burgers equations. *Physica Scripta*, Volume 83 (2011).

[23] Alquran M., Al-Khaled K., The tanh and sine-cosine methods for higher order equations of Korteweg-de Vries type. *Physica Scripta*, Volume 84, (2011).

[24] Alquran M., Solitons and periodic solutions to nonlinear partial differential equations by the Sine-Cosine method . *Applied Mathematics and Information Sciences* AMIS, Volume 6(1), (2012) pp. 85-88.

[25] Bekir A., Unsal O., Exact Solutions for a Class of Nonlinear Wave Equations By Using First Integral. *International Journal of Nonlinear Science*. Volume 15(2) (2013) pp. 99-110.

[26] Wazwaz A.M., Nonlinear variants of KdV and KP equations with compactons, solitons and periodic solutions. *Communications in Nonlinear Science and Numerical Simulation*. Volume 10 (2005) pp. 451-463.

[27] Wazwaz A.M., The extended Tanh method for Abundant solitary wave solutions of nonlinear wave equations. *Applied Mathematics and Computation*. Volume 187 (2007) pp. 1131-1142.

[28] Qawasmeh A., Soliton solutions of (2+1)-Zoomeron equation and duffing equation and SRLW equation. *J. Math. Comput. Sci.* Volume 3(6) (2013) pp. 1475-1480.

[29] Chand F., Malik A.K., Exact Traveling Wave Solutions of Some Nonlinear Equations Using G'/G-expansion Method. *International Journal of Nonlinear Science*. Volume 14(4) (2012) pp. 416-424.

[30] Kkeiri H., Moghaddam M.R. and Vafafi V., Application of the G'/G-expansion method for the Burgers, Burgers-Huxley and modified Burgers-KdV equations. *Pramana-journal of physics*. Volume 76(6) (2011) pp. 831-842.

[31] Liu X., Zhang W., Li Z., Application of improved G'/G-expansion method to traveling wave solutions of two nonlinear evolution equations. *Adv. Appl. Math. Mech.* Volume 4(1) (2012) pp. 122-130.

[32] Zhang H., New application of the G'/G-expansion method. *Commun Nonlinear Sci Numer Simulat*. Volume 14 (2009) pp. 3220-3225.

[33] Gao H., Zhao R.X., New application of the G'/G-expansion method to high-order nonlinear equations. *Appl Math Computation*. Volume 215 (2009) pp. 2781-2786.