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SOME REGULAR ELEMENTS, IDEMPOTENTS AND RIGHT UNITS OF COMPLETE SEMIGROUPS OF BINARY RELATIONS DEFINED BY SEMILATTICES OF THE CLASS LOWER INCOMPLETE NETS

Yasha Diasamidze[1], Ali Erdogan[2 §], Neşet Aydın[3]

[1]Sh. Rustaveli State University
Batumi, GEORGIA

[2]Department of Mathematics
Hacettepe University
Beytepe, Ankara, TURKEY

[3]Department of Mathematics
Faculty of Arts and Sciences
Çanakkale Onsekiz Mart University
Çanakkale, TURKEY

Abstract: In this paper, we investigate such a regular elements α and idempotents of the complete semigroup of binary relations $B_X(D)$ defined by semilattices of the class lower incomplete nets, for which $V(D,\alpha) = Q$.

Also we investigate right units of the semigroup $B_X(Q)$. For the case where X is a finite set we derive formulas by means of which we can calculate the numbers of regular elements, idempotents and right units of the respective semigroup.

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[§]Correspondence author

1. Introduction

Let X be an arbitrary nonempty set. Let D be some nonempty set of subsets of the set X, closed with respect to the operation of set-theoretic union of elements of the set D, i.e., $\cup D' \in D$ for any nonempty subset D' of the set D. In that case, the set D is called complete $X-$semilettice of unions. The union of all elements of the set D is denoted by the symbol $\breve{D}$. Clearly, $\breve{D} \in D$ is the largest element.

Recall that a binary relation on the set X is a subset of the cartesian product $X \times X$. If α and β are binary relations on the set X with the elements $x, y, z \in X$ the condition $(x, y) \in \alpha$ is denoted as $x\alpha y$ and $x\alpha y\beta z$ means the conditions $x\alpha y$ and $y\beta z$ are satisfied simultaneously. The binary relation $\alpha^{-1} = \{(x, y) : y\alpha x\}$ is usually called the binary relation inverse to α. The empty binary relation which is empty subset of $X \times X$ is denoted by $\varnothing$. The binary relation $\delta = \alpha \circ \beta$ is called product of binary relations α and β. A pair (x, y) belongs to δ if only if there exists $y \in X$ such that $x\alpha y\beta z$. The binary operation $\circ$ is associative. So, B_X, the set of all binary relations on X, is therefore a semigroup with respect to the operation $\circ$. This semigroup is called the semigroup of all binary relations on the set X.

Let f be an arbitrary mapping from X into D. Then one can construct such a mapping f with a binary relation α_f on X provided by the condition below, $\alpha_f = \bigcup\limits_{x \in X} (\{x\} \times f(x))$. The set of all such binary relation is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the product operation of binary relations. This semigroup, $B_X(D)$, is called a complete semigroup of binary relations defined by an $X-$semilattice of unions D.

Further, let $x, y \in X$, $Y \subseteq X$, $\alpha \in B_X(D)$, $T \in D$, $\varnothing \neq D' \subseteq D$ and t$\in \breve{D}$. Then we have the following notation,

$$y\alpha = \{x \in X : y\alpha x\}, Y\alpha = \bigcup_{y \in Y} y\alpha, 2^X = \{Y : Y \subseteq X\}, \; X^* = 2^X \setminus \{\varnothing\}$$

$$V(D, \alpha) = \{Y\alpha : Y \in D\}, \; D_t = \left\{Z' : t \in Z'\right\},$$

$$D'_T = \left\{Z' \in D' : T \subseteq Z'\right\}, \; \ddot{D}'_T = \left\{Z' \in D' : Z' \subseteq T\right\}.$$

Now, let's take $\alpha \in B_X(D)$. If $\beta \circ \alpha = \beta$ for any $\beta \in B_X(D)$, then α is called a right unit of semigroup $B_X(D)$. If $\alpha \circ \alpha = \alpha$ then α is called an idempotent element of semigroup $B_X(D)$. And if $\alpha \circ \beta \circ \alpha = \alpha$ for some $\beta \in B_X(D)$, then a binary relation α is called a regular element of semigroup $B_X(D)$.

D is partially ordered with respect to the set–theoretic inclusion. Let $\varnothing \neq D' \subseteq D$ and $N(D, D') = \{Z \in D : Z \subseteq Z' \text{ for any } Z' \in D'\}$. It is clear that $N(D, D')$ is the set of lower bounds of a nonempty subset D' included in D. If $N(D, D') \neq \varnothing$ then $\cup N(D, D')$ belongs to D and it is the greatest lower bound of D' and is denoted by $\wedge(D, D') = \cup N(D, D')$.

Let $l(D', T) = \cup(D' \backslash D'_T)$. We say that a nonempty element T is a *non-limiting element* of D' if $T \backslash l(D', T) \neq \varnothing$. A nonempty element T is a *limiting element* of D' if $T \backslash l(D', T) = \varnothing$.

Now, we continue with some essential definitions and theorems given by the cited references.

Definition 1.1. [2, Definition 1] Let $\alpha \in B_X$, $T \in V(X^*, \alpha)$ and $Y_T^\alpha = \{y \in X : y\alpha = T\}$. Then a representation of a binary relation α of the form $\alpha = \bigcup\limits_{T \in V(X^*, \alpha)} (Y_T^\alpha \times T)$ is called quasinormal.

Note that, if $\alpha = \bigcup\limits_{T \in V(X^*, \alpha)} (Y_T^\alpha \times T)$ is a quasinormal representation of the binary relation α then the following are true,

1. $X = \bigcup\limits_{T \in V(X^*, \alpha)} Y_T^\alpha$
2. $Y_T^\alpha \cap Y_{T'}^\alpha = \varnothing$ for $T, T' \in V(X^*, \alpha)$ and $T \neq T'$.

Definition 1.2. [3, Definition 2] Let $\tilde{D}$ and D' be some nonempty subsets of the complete $X-$semilattices of unions. We say that a subset $\tilde{D}$ generates a set D' if any element from D' is a set-theoretic union of the elements from $\tilde{D}$.

Definition 1.3. [4, Definition 1.14.2] We say that a complete $X-$semilattice of unions D is an $XI-$semilattice of unions if it satisfies the following two conditions:

a) $\wedge(D, D_t) \in D$ for any $t \in \breve{D}$,

b) $Z = \bigcup\limits_{t \in Z} \wedge(D, D_t)$ for any nonempty element Z of D.

Theorem 1.1. *[4, Corollary 1.18.1] Let $Y = \{y_1, y_2, \ldots, y_k\}$ and $D_j = \{T_1, \ldots, T_j\}$ be some sets, where $k \geq 1$ and $j \geq 1$. Then the numbers $s(k, j)$ of all possible mappings of the sets Y on any subset D'_j of the set D_j and $T_j \in D'_j$ can be calculated by the formula $s(k, j) = j^k - (j-1)^k$.*

Lemma 1.1. *[1, Lemma 3.1] Let D complete $X-$semilattices of unions. If a binary relation ε having the form*

$$\varepsilon = \varepsilon(D, f) = \bigcup_{t \in \breve{D}} (\{t\} \times \wedge (D, D_t)) \cup \left(\left(X \backslash \breve{D}\right) \times \breve{D}\right)$$

is a right unit of the semigroup $B_x(D)$, then it is the largest right unit of this semigroup.

Theorem 1.2. *[1, Theorem 2.5] Let D' be a complete subsemilattice of the complete $X-$semilattice of unions D, $\breve{D}' = \cup D'$ and f be an arbitrary mapping of the set $X \backslash \breve{D}'$ in the semilattice D'. If D' is a complete $XI-$semilattice of unions then the binary relation*

$$\alpha = \alpha(D', f) = \bigcup_{t \in \breve{D}'} \left(\{t\} \times \wedge \left(D', D'_t\right)\right) \cup \bigcup_{t' \in X \backslash \breve{D}'} \left(\{t'\} \times f(t')\right)$$

is an idempotent element of semigroup $B_X(D)$ and $V(D', \alpha) = D'$.

Theorem 1.3. *[4, Theorem 4.1.3] A binary relation $\varepsilon \in B_X(D)$ is right units of this semigroup iff ε is idempotent and $V(D, \varepsilon) = D$.*

Definition 1.4. [6, Definition 7] A one-to-one mapping φ between the complete $X-$semilattices of unions D' and D'' is called a complete isomorphism if the condition $\varphi(\cup D_1) = \bigcup_{T' \in D_1} \varphi(T')$ is fulfilled for each nonempty subset D_1 of the semilattice D'.

Definition 1.5. [6, Definition 8] Let α be some binary relation of the semigroup $B_X(D)$. We say that a complete isomorphism φ between $XI-$semilattice of unions Q and D' is a complete $\alpha-$isomorphism if

a) $Q = V(D, \alpha)$

b) $\varphi(\varnothing) = \varnothing$ for $\varnothing \in V(D, \alpha)$ and $\varphi(T)\alpha = T$ for any $T \in V(D, \alpha)$.

Theorem 1.4. *[4, Theorem 6.3.3] Let D be a finite $X-$semilattice of unions and $\alpha \circ \sigma \circ \alpha = \alpha$ for some elements $\alpha, \sigma \in B_X(D)$; $D(\alpha)$ the set those elements T of the semilattice $D = V(D, \alpha) \backslash \{\varnothing\}$ which are nonlimiting elements of the set $\ddot{D}(\alpha)_T$. Then binary relation α having a quasinormal representation of the form $\alpha = \bigcup_{T \in V(D, \alpha)} (Y_T^\alpha \times T)$ is a reguler element of the semigroup $B_X(D)$ iff the set $V(D, \alpha)$ is a $XI-$semilattice of unions and for some $\alpha-$isomorphism φ of the semilattice $V(D, \alpha)$ on some $X-$subsemilattice D' of the semilattice D the following conditions are fulfilled:*

i) $\bigcup_{T \in \ddot{D}(\alpha)_T} Y^{\alpha}_{T'} \supseteq \varphi(T)$ *for any* $T \in D(\alpha)$;

ii) $Y^{\alpha}_{T} \cap \varphi(T) \neq \varnothing$ *for any nonlimiting element* T *of the set* $\ddot{D}(\alpha)_T$.

Theorem 1.5. *[4, Theorem 6.3.7] A regular element* α *of the semigroup* $B_X(D)$ *is idempotent iff the mapping* φ *satisfying the condition* $\varphi(T) = T\alpha$ *for any* $T \in V(D, \alpha)$ *is an identity mapping of the semilattice* $V(D, \alpha)$.

Definition 1.6. [4, Definition 6.3.4] Let Q and D' be respectively some XI and $X-$subsemilattices of the complete $X-$semilattice of unions D. Then $R_\varphi(Q, D')$ is a subset of the semigroup $B_X(D)$ such that $\alpha \in R_\varphi(Q, D')$ only if the following conditions are fulfilled for the elements of α and φ,

a) The binary relation α be regular element of the semigroup $B_X(D)$,

b) $V(D, \alpha) = Q$,

c) φ is a $\alpha-$isomorphism between the complete semilattices of unions Q and D' satisfying the conditions $i)$ and $ii)$ of the Theorem 1.4.

Definition 1.7. [4, Definition 6.3.4] Let $\Phi(D, D')$ be the set of all complete isomorphism φ between $XI-$semilattice of unions D and D' such that $\varphi \in \Phi(D, D')$ only if φ is a $\alpha-$isomorphism for some $\alpha \in B_X(D)$ and $V(D, \alpha) = D$.

$\Omega(D)$ is the set of all $XI-$subsemilattices of the complete $X-$semilat-tice of unions D such that $D' \in \Omega(D)$ iff there exists a complete isomorphism between the semilattices D' and D.

Let us denote

$$R(D, D') = \bigcup_{\varphi \in \Phi(D, D')} R_\varphi(D, D') \text{ and } R(D') = \bigcup_{D' \in \Omega(D)} R(D, D').$$

Theorem 1.6. *[4, Theorem 6.3.5] Let* X *is a finite set. If* φ *is a fixed element of the set* $\Phi(D, D')$ *and* $\Omega(D) = m_0$ *and* q *is a number of all automorhisms of the semilattice* D, *then* $|R(D')| = m_0 \cdot q \cdot |R_\varphi(D, D')|$.

Theorem 1.7. *[6, Theorem 22] Let* $D = \left\{\breve{D}, T_1, T_2, \ldots, T_{m-1}\right\}$ *be some finite* $X-$*semilattice of unions and* $C(D) = \{P_0, P_1, P_2, \ldots, P_{m-1}\}$ *be the family of sets of pairwise disjoint subsets of the set* X. *If* φ *is a mapping of the semilattice* D *to the family sets* $C(D)$ *that satisfies the condition* $\varphi(\breve{D}) = P_0$

and $\varphi(T_i) = P_i$ $(i = 1, 2, \ldots, m-1)$ *and* $\hat{D}_Z = D \setminus \{T \in D : Z \subseteq T\}$ *then the following equalities are valid:*

$$\breve{D} = P_0 \cup P_1 \cup \cdots \cup P_{m-1}, \; T_i = P_0 \cup \bigcup_{T \in \hat{D}_{T_i}} \varphi(T). \tag{1.1}$$

In the sequel these equalities will be called formal.

It is proved that if the elements of the semilattice D are represented in the form (1.1), then among the parameters P_i $(i = 0, 1, 2, \ldots, m-1)$ there exist such parameters that cannot be empty sets for D. Such sets P_i $(0 < i \leq m-1)$ are called *basis sources*, whereas sets P_j $(0 \leq j \leq m-1)$ which can be empty sets too are called *completeness sources*.

It is proved that under the mapping φ the number of covering elements of the pre-image of a basis source is always equal to one, while under the mapping φ the number of covering elements of the pre-image of a completeness source either does not exist or is always greater than one.

Note that the set P_0 is always considered to be a source of completeness.

Lemma 1.2. *Let* D *and* $C(D) = \{P_0, P_1, \ldots, P_{n-1}\}$ *are the finite semilattice of unions and the family of sets of pairwise nonintersecting subsets of the set* X*;* φ *is a mapping of the semilattice* D *on the family of sets* $C(D)$*. If* $\varphi(T) = P \in C(D) \setminus \{P_0\}$ *for some* $T \in D$*, then* $D_t = D \setminus \ddot{D}_T$ *for all* $t \in P$*.*

Proof. Let t and Z' are any elements of the set P $(P \neq P_0)$ and of the semilattice D respectively. Then the equality $P \cap Z' = \varnothing$ (i.e., $Z' \notin D_t$ for any $t \in P$) is true if and only if $T \notin \hat{D}_{Z'}$ (if $T \in \hat{D}_{Z'}$, then $\varphi(T) \subseteq Z'$ by definition of the formal equalities of the semilattice D). Since $\hat{D}_{Z'} = D \setminus \{T' \in D : Z' \subseteq T'\}$ by definition of the set $\hat{D}_{Z'}$. Thus the condition $T \notin \hat{D}_{Z'}$ hold iff $T \in \{T' \in D : Z' \subseteq T'\}$. So, $Z' \subseteq T$ and $Z' \in \ddot{D}_T$ by definition of the set $\ddot{D}_T$.

Therefore, $\varphi(T) \cap Z' = \varnothing$ if and only if $Z' \in \ddot{D}_T$. Of this follows that the inclusion $\varphi(T) = P \subseteq Z'$ is true iff $D_t = D \setminus \ddot{D}_T$ for all $t \in \varphi(T) = P$. □

2. Results

Let $N_m = \{0,1,2,\ldots,m\}$ $(m \geq 1)$ be some subset of the set of all natural numbers. A subsemilattice

$$Q = \{T_{ij} \subseteq X : i \in N_s,\ j \in N_k\} \setminus \{T_{00}\}$$

of the complete $X-$semilattice of unions D is called lower incomplete net which contains two subsets $Q_1 = \{T_{10}, T_{20}, \ldots T_{s0}\}$, $Q_2 = \{T_{01}, T_{02}, \ldots, T_{0k}\}$ and satisfies the following conditions:

a) $T_{10} \subset T_{20} \subset \cdots \subset T_{s0}$ and $T_{01} \subset T_{02} \subset \cdots \subset T_{0k}$;

b) $Q_1 \cap Q_2 = \varnothing$;

c) $T_{pq} \neq T_{ij}$ if $(p,q) \neq (i,j)$

d) the elements of the sets Q_1 and Q_2 are pairwise noncomparable;

e) $T_{ij} \cup T_{i'j'} = T_{pq}$, if $p = \max\{i,i'\}$ and $q = \max\{j,j'\}$.

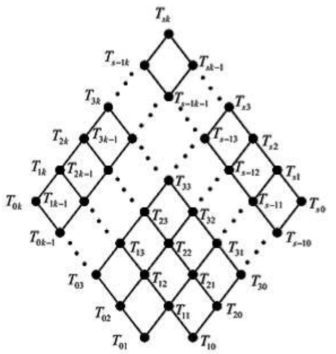

Fig. 2.1

Note that the diagram of the given $X-$semilattice of unions Q is shown in Fig. 2.1.

Let $C(Q) = \{P_{01}, P_{10}, P_{02}, \ldots, P_{s-1k}, P_{ks-1}, P_{sk}\}$ is a family sets, where P_{01}, P_{10}, P_{02}, $\ldots$, P_{ks-1}, P_{sk} are pairwise disjoint subsets of the set X and

$$\varphi = \begin{pmatrix} T_{01} & T_{10} & T_{02} & T_{11} & T_{20} & \ldots & T_{s-1k} & T_{ks-1} & T_{sk} \\ P_{01} & P_{10} & P_{02} & P_{11} & P_{20} & \ldots & P_{s-1k} & P_{ks-1} & P_{sk} \end{pmatrix}$$

is a mapping of the semilattice Q onto the family sets $C(Q)$. Then for the formal equalities of Q we have a form (see Theorem 1.7):

$$T_{sk} = P_{sk} \cup \bigcup_{T_{rt} \in \hat{D}_{sk}} \varphi(T_{rt}), \tag{2.1}$$

$$T_{s-1k} = P_{sk} \cup \bigcup_{T_{rt} \in \hat{D}_{s-1k}} \varphi(T_{rt}), T_{sk-1} = P_{sk} \cup \bigcup_{T_{rt} \in \hat{D}_{sk-1}} \varphi(T_{rt}),$$

$$\ldots$$

$$T_{20} = P_{sk} \cup \bigcup_{T_{rt} \in \hat{D}_{20}} \varphi(T_{rt}),$$

$$T_{11} = P_{sk} \cup \bigcup_{T_{rt} \in \hat{D}_{11}} \varphi(T_{rt}), T_{02} = P_{sk} \cup \bigcup_{T_{rt} \in \hat{D}_{02}} \varphi(T_{rt}),$$

$$T_{10} = P_{sk} \cup \bigcup_{T_{rt}\in \hat{D}_{10}} \varphi(T_{rt}) = T_{01} \cup T_{02} \cup \cdots \cup T_{0k}$$
$$T_{01} = P_{sk} \cup \bigcup_{T_{rt}\in \hat{D}_{01}} \varphi(T_{rt}) = T_{10} \cup T_{20} \cup \cdots \cup T_{s0}.$$

Here $P_{0k},\ P_{1k}, \ldots,\ P_{s-1k},\ P_{s0},\ P_{s1},\ \ldots,\ P_{sk-1}$ are basis sources, the elements of the set $C(Q)\backslash\{P_{0k},\ P_{1k},\ P_{2k}, \ldots,\ P_{s0},\ P_{s1},\ \ldots,\ P_{sk-1}\}$ are sources of completeness of the semilattice Q (see Theorem 1.7).

Lemma 2.1. *Let Q be a lower incomplete net. Then Q is $XI-$semilattice of unions iff it satisfies the condition $T_{0k} \cap T_{s0} = \varnothing$.*

Proof. Let $t \in \breve{Q}$, $Q_t = \{Z \in Q : t \in Z\}$ and $\wedge(Q, Q_t)$ is the exact lower bound of the set Q_t in Q. Then fromLemma 1.2 and from the formal equalities (2.1) we have:

$$\begin{array}{c} t \in P_{sk},\ Q_t = Q \\ t \in P_{s-1k},\ Q_t = Q\backslash \ddot{Q}_{T_{s-1k}}, \\ t \in P_{s-2k},\ Q_t = Q\backslash \ddot{Q}_{T_{s-2k}}, \\ t \in P_{sk-1},\ Q_t = Q\backslash \ddot{Q}_{T_{sk-1}}, \\ t \in P_{sk-2},\ Q_t = Q\backslash \ddot{Q}_{T_{sk-2}}, \\ \cdots \\ t \in P_{0k},\ Q_t = Q\backslash \ddot{Q}_{T_{0k}}, \\ t \in P_{s0},\ Q_t = Q\backslash \ddot{Q}_{T_{s0}}, \end{array} \qquad \wedge(Q, Q_t) = \begin{cases} T_{s0}, & \text{if } t \in P_{s-1k}, \\ T_{s-10}, & \text{if } t \in P_{s-2k} \\ T_{0k}, & \text{if } t \in P_{sk-1}, \\ T_{0k-1}, & \text{if } t \in P_{sk-2}, \\ \cdots & \cdots \\ T_{10}, & \text{if } t \in P_{0k}, \\ T_{01}, & \text{if } t \in P_{s0}. \end{cases}$$

If $T_{ij} \subseteq T_{s-1k-1}$, $\varphi(T_{ij}) = P_{ij}$ and $t \in P_{ij}$, then $Q_t = Q\backslash \ddot{Q}_{T_{ij}}$ and $\wedge(Q, Q_t) \notin Q$.

We have $Q^{\wedge} = \{\wedge(Q, Q_t) : \wedge(Q, Q_t) \in Q\} = \{T_{10}, \ldots, T_{s0},\ T_{01}, \ldots, T_{0k}\}$ and $\wedge(Q, Q_t) \notin Q$, when $t \notin \cup\{P_{0k}, P_{1k}, \ldots, P_{s-1k},\ P_{s0}, P_{s1}, \ldots, P_{sk-1}\}$, i.e., $\wedge(Q, Q_t) \notin Q$, when $t \in T_{0k} \cap T_{s0}$. Therefore the semilattice Q is not XI-semilattice of unions, if $T_{0k} \cap T_{s0} \neq \varnothing$.

$(*)$ If $T_{0k} \cap T_{s0} = \varnothing$, i.e., $t \in \cup\{P_{0k}, P_{1k}, \ldots, P_{s-1k},\ P_{s0}, \ldots\ P_{sk-1}\}$ then $\wedge(Q, Q_t) \in Q$ for all elements t of the set $\breve{D}$ and $T_{ij} = \bigcup_{t\in T_{ij}} \wedge(Q, Q_t)$. Therefore the semilattice Q is XI-semilattice of unions, if $T_{0k} \cap T_{s0} = \varnothing$.

If the equality $T_{0k} \cap T_{s0} = \varnothing$ is true, then by $(*)$ follows that Q is XI-semilattice of unions. □

Lemma 2.2. *Let Q be a $XI-$lower incomplete net. Then following equalities are true:*

$$P_{s-1k} = T_{s0}\backslash T_{s-1k},\ P_{s-2k} = T_{s-10}\backslash T_{s-2k},\ \ldots,\ P_{0k} = T_{10}\backslash T_{0k}$$

$$P_{sk-1} = T_{0k} \backslash T_{sk-1},\ P_{sk-2} = T_{0k-1} \backslash T_{sk-2},\ \ldots,\ P_{s0} = T_{01} \backslash T_{s0}.$$

Proof. The given Lemma immediately follows from the formal equalities (2.1) of the semilattice Q. For the largest right unit ε of the semigroup $B_X(D)$ we have:

$$\begin{aligned}\varepsilon &= (P_{s-1k} \times T_{s0}) \cup (P_{s-2k} \times T_{s-10}) \cup (P_{sk-1} \times T_{0k}) \\ &\cup ((T_{0k-1} \backslash Y_{sk-2}) \times T_{0k-1}) \cup \cdots \cup (P_{0k} \times T_{10}) \cup (P_{s0} \times T_{01}) \\ &= ((T_{s0} \backslash T_{s-1k}) \times T_{s0}) \cup ((T_{s-10} \backslash T_{s-2k}) \times T_{s-10}) \cup ((T_{0k} \backslash T_{sk-1}) \times T_{0k}) \\ &\cup ((T_{0k-1} \backslash T_{sk-2}) \times T_{0k-1}) \cup \cdots \cup ((T_{10} \backslash T_{0k}) \times T_{10}) \cup ((T_{01} \backslash T_{s0}) \times T_{01})\end{aligned}$$

(see, Lemma 1.1). □

Theorem 2.1. *Let Q be a XI-lower incomplete net. Then a binary relation α of the semigroup $B_X(D)$ having a quasinormal representation of the form $\alpha = \bigcup_{T_{ij} \in Q} \left(Y_{ij}^{\alpha} \times T_{ij}\right)$ such that $Q = V(D, \alpha)$, is a regular element of the semigroup $B_X(D)$ iff for some $\alpha-$isomorphism φ of the semilattice Q on some subsemilattice D' of the semilattice D the following conditions are fulfilled:*

$$\begin{gathered} Y_{00}^{\alpha} \supseteq \varphi(T_{0k}) \cap \varphi(T_{s0}),\ Y_{00}^{\alpha} \cup Y_{01}^{\alpha} \supseteq \varphi(T_{01}),\ Y_{00}^{\alpha} \cup Y_{01}^{\alpha} \cup Y_{02}^{\alpha} \supseteq \varphi(T_{02}), \ldots, \\ Y_{00}^{\alpha} \cup Y_{01}^{\alpha} \cup Y_{02}^{\alpha} \cup \cdots \cup Y_{0k}^{\alpha} \supseteq \varphi(T_{0k}), \\ Y_{00}^{\alpha} \cup Y_{10}^{\alpha} \supseteq \varphi(T_{10}), Y_{00}^{\alpha} \cup Y_{10}^{\alpha} \cup Y_{20}^{\alpha} \supseteq \varphi(T_{20}), \\ \ldots,\ Y_{00}^{\alpha} \cup Y_{10}^{\alpha} \cup Y_{20}^{\alpha} \cup \cdots \cup Y_{s0}^{\alpha} \supseteq \varphi(T_{s0}),\ Y_{ij}^{\alpha} \cap \varphi(T_{ij}) \neq \varnothing \end{gathered}$$

for any $T_{ij} \in (Q_1 \cup Q_2) \setminus \{\varnothing\}$.

Proof. It is easy to see that the set $Q\hat{} = \{T_{10}, T_{20}, \ldots, T_{s0}, T_{01}, \ldots, T_{0k}\}$ is an irreducible generating set of the semilattice Q. Moreover, all elements of the set $Q\hat{} = Q_1 \cup Q_2$ are nonlimiting. Now by Theorem 1.4 we obtain

a) $\bigcup_{T_{ij} \in \ddot{Q}\hat{}_{T_{pq}}} \supseteq \varphi(T_{pq})$ for any $T_{pq} \in Q\hat{}$.

b) $Y_{ij}^{\alpha} \cap \varphi(T_{ij}) \neq \varnothing$ for any $T_{ij} \in Q\hat{}$.

From the condition *a*) of this theorem we immediately have the validity of the follo-wing inclusions:

$$\begin{aligned} &Y_{10}^{\alpha} \supseteq \varphi(T_{10}),\ Y_{10}^{\alpha} \cup Y_{20}^{\alpha} \supseteq \varphi(T_{20}),\ \ldots,\ Y_{10}^{\alpha} \cup Y_{20}^{\alpha} \cup \cdots \cup Y_{s0}^{\alpha} \supseteq \varphi(T_{s0}) \\ &Y_{01}^{\alpha} \supseteq \varphi(T_{01}),\ Y_{01}^{\alpha} \cup Y_{02}^{\alpha} \supseteq \varphi(T_{02}),\ \ldots,\ Y_{01}^{\alpha} \cup Y_{02}^{\alpha} \cup \cdots \cup Y_{0k}^{\alpha} \supseteq \varphi(T_{0k}) \\ &Y_{20}^{\alpha} \cap \varphi(T_{20}) \neq \varnothing,\ \ldots,\ Y_{s0}^{\alpha} \cap \varphi(T_{s0}) \neq \varnothing, \\ &Y_{02}^{\alpha} \cap \varphi(T_{02}) \neq \varnothing,\ \ldots, Y_{0k}^{\alpha} \cap \varphi(T_{0k}) \neq \varnothing, \end{aligned}$$

□

Theorem 2.2. *Let Q be a $XI-$lower incomplete net. Then a binary relation α of the semigroup $B_X(Q)$, which has a quasinormal representation of the form $\alpha = \bigcup_{T_{ij}\in Q}\left(Y_{ij}^{\alpha}\times T_{ij}\right)$ such that $Q = V(D,\alpha)$, is an idempotent element of the semigroup $B_X(D)$ iff the following conditions are fulfilled:*

$$\begin{gathered} Y_{10}^{\alpha}\supseteq T_{10},\ Y_{10}^{\alpha}\cup Y_{20}^{\alpha}\supseteq T_{20},\ \ldots,\ Y_{10}^{\alpha}\cup Y_{20}^{\alpha}\cup\cdots\cup Y_{s0}^{\alpha}\supseteq T_{s0},\\ Y_{01}^{\alpha}\supseteq T_{01},\ Y_{01}^{\alpha}\cup Y_{02}^{\alpha}\supseteq T_{02},\ \ldots,\ Y_{01}^{\alpha}\cup Y_{02}^{\alpha}\cup\cdots\cup Y_{0k}^{\alpha}\supseteq T_{0k},\\ Y_{20}^{\alpha}\cap T_{20}\neq\varnothing,\ \ldots,\ Y_{s0}^{\alpha}\cap T_{s0}\neq\varnothing,\ Y_{02}^{\alpha}\cap T_{02}\neq\varnothing,\ \ldots,Y_{0k}^{\alpha}\cap T_{0k}\neq\varnothing, \end{gathered}$$

for any $T_{ij}\in Q\hat{}$.

Proof. This theorem immediately follows from Lemma 2.1, from the Theorem 2.1 and Theorem 1.5. □

Theorem 2.3. *Let Q be a $XI-$lower incomplete net. Then a binary relation α of the semigroup $B_X(Q)$ that has a qusinormal representation of the form $\alpha = \bigcup_{T_{ij}\in Q}\left(Y_{ij}^{\alpha}\times T_{ij}\right)$ such that $Q = V(D,\alpha)$, is a right unit of the semigroup $B_X(Q)$ iff the following conditions are fulfilled:*

$$\begin{gathered} Y_{10}^{\alpha}\supseteq T_{10},\ Y_{10}^{\alpha}\cup Y_{20}^{\alpha}\supseteq T_{20},\ \ldots,\ Y_{10}^{\alpha}\cup Y_{20}^{\alpha}\cup\cdots\cup Y_{s0}^{\alpha}\supseteq T_{s0},\\ Y_{01}^{\alpha}\supseteq T_{01},\ Y_{01}^{\alpha}\cup Y_{02}^{\alpha}\supseteq T_{02},\ \ldots,\ Y_{01}^{\alpha}\cup Y_{02}^{\alpha}\cup\cdots\cup Y_{0k}^{\alpha}\supseteq T_{0k},\\ Y_{20}^{\alpha}\cap T_{20}\neq\varnothing,\ \ldots,\ Y_{s0}^{\alpha}\cap T_{s0}\neq\varnothing,\ Y_{02}^{\alpha}\cap T_{02}\neq\varnothing,\ \ldots,Y_{0k}^{\alpha}\cap T_{0k}\neq\varnothing, \end{gathered}$$

for any $T_{ij}\in Q\hat{}$.

Proof. This theorem immediately follows from Theorem 1.3. □

Theorem 2.4. *Let Q be a $XI-$lower incomplete net. If the semilattice Q and $D' = \left\{\bar{T}_{01},\bar{T}_{10},\ldots,\bar{T}_{sk}\right\}$ (see Fig. 2.2) are $\alpha-$isomorphic and $|\Omega(Q)| = m_0$, then the following equalities are valid:*

a)

$$\begin{aligned} \left|R(D')\right| = m_0\cdot &\left(j^{|T_{02}\setminus T_{s1}|}-1\right)\cdot\left(3^{|T_{03}\setminus T_{s2}|}-2^{|T_{03}\setminus T_{s2}|}\right)\\ &\cdots\left((k-1)^{|T_{0k-1}\setminus T_{sk-2}|}-(k-2)^{|T_{0k-1}\setminus T_{sk-2}|}\right)\cdot\\ &\left(k^{|T_{0k}\setminus T_{sk-1}|}-(k-1)^{|T_{0k}\setminus T_{sk-1}|}\right)\cdot\left(2^{|T_{20}\setminus T_{1k}|}-1\right)\cdot\\ &\left(3^{|T_{30}\setminus T_{2k}|}-2^{|T_{30}\setminus T_{2k}|}\right)\ldots\left((s-1)^{|T_{s-10}\setminus T_{s-2k}|}-(i-1)^{|T_{s-10}\setminus T_{s-2k}|}\right) \end{aligned}$$

$$\cdot\left(s^{|T_{s0}\setminus T_{s-1k}|}-(s-1)^{|T_{s0}\setminus T_{s-1k}|}\right)\cdot|Q|^{\left|X\setminus\bar{T}_{sk}\right|},$$

if $s\neq k$ *or*

b)

$$\begin{aligned}\left|R(D')\right|=2\cdot m_0\cdot&\left(j^{|T_{02}\setminus T_{s1}|}-1\right)\cdot\left(3^{|T_{03}\setminus T_{s2}|}-2^{|T_{03}\setminus T_{s2}|}\right)\\&\cdots\left((k-1)^{|T_{0k-1}\setminus T_{sk-2}|}-(k-2)^{|T_{0k-1}\setminus T_{sk-2}|}\right)\cdot\\&\left(k^{|T_{0k}\setminus T_{sk-1}|}-(k-1)^{|T_{0k}\setminus T_{sk-1}|}\right)\cdot\left(2^{|T_{20}\setminus T_{1k}|}-1\right)\cdot\\&\left(3^{|T_{30}\setminus T_{2k}|}-2^{|T_{30}\setminus T_{2k}|}\right)\cdots\left((s-1)^{|T_{s-10}\setminus T_{s-2k}|}-(i-1)^{|T_{s-10}\setminus T_{s-2k}|}\right)\\&\cdot\left(s^{|T_{s0}\setminus T_{s-1k}|}-(s-1)^{|T_{s0}\setminus T_{s-1k}|}\right)\cdot|Q|^{\left|X\setminus\bar{T}_{sk}\right|},\end{aligned}$$

if $s=k$.

Proof. In the first place, we note that the given semilattice Q has one automorphisms (i.e., $|\Phi(Q,Q)|=1$) if $s\neq k$ and two automorphism (i.e., $|\Phi(Q,Q)|=2$) if $s=k$ (see [4, Theorem 11.7.1]). Hence, we have$|\Phi(Q,D')|=1$ or $|\Phi(Q,D\prime)|=2$ (see [4, Lemma 6.3.2]).

Next, assume that $\alpha\in\bar{R}(Q,D')$ and a quasinormal representation of a regular binary relation α has the form

$$\alpha=\bigcup_{T_{ij}\in Q}\left(Y_{ij}^{\alpha}\times T_{ij}\right)\tag{2.2}$$

Then, according to Theorem 2.1, we have

$$\begin{gathered}Y_{10}^{\alpha}\supseteq\bar{T}_{10},\ Y_{10}^{\alpha}\cup Y_{20}^{\alpha}\supseteq\bar{T}_{20},\ \ldots,\ Y_{10}^{\alpha}\cup Y_{20}^{\alpha}\cup\cdots\cup Y_{s0}^{\alpha}\supseteq\bar{T}_{s0},\\Y_{01}^{\alpha}\supseteq\bar{T}_{01},\ Y_{01}^{\alpha}\cup Y_{02}^{\alpha}\supseteq\bar{T}_{02},\ \ldots,\ Y_{01}^{\alpha}\cup Y_{02}^{\alpha}\cup\cdots\cup Y_{0k}^{\alpha}\supseteq\bar{T}_{0k},\\Y_{20}^{\alpha}\cap\bar{T}_{20}\neq\varnothing,\ \ldots,\ Y_{s0}^{\alpha}\cap\bar{T}_{s0}\neq\varnothing,\ Y_{02}^{\alpha}\cap\bar{T}_{02}\neq\varnothing,\ \ldots,Y_{0k}^{\alpha}\cap\bar{T}_{0k}\neq\varnothing,\end{gathered}\tag{2.3}$$

for any $\bar{T}_{ij}\in\left\{\bar{T}_{01},\bar{T}_{02},\ldots,\bar{T}_{0k},\bar{T}_{10},\bar{T}_{20},\ldots,\bar{T}_{s0}\right\}$.

Further, let f_α be a mapping the set X in the semilattice D satisfying the conditions $f_\alpha(t)=t\alpha$ for all $t\in X$.

$$f_{01\alpha},f_{02\alpha},\ldots,f_{0k-1\alpha},f_{0k\alpha},$$

$$f_{10\alpha},f_{20\alpha},\ldots,f_{s-10\alpha},f_{s0\alpha}$$

and $f_{sk\alpha}$, are the restrictions of the mapping f_α, on the sets

$$\bar{T}_{01},\bar{T}_{02}\setminus\bar{T}_{s1},\ \ldots,\bar{T}_{0k-1}\setminus\bar{T}_{sk-2},\ \bar{T}_{0k}\setminus\bar{T}_{sk-1},$$

$$\bar{T}_{10},\ \bar{T}_{20}\backslash\bar{T}_{1k},\ \ \ldots,\ \bar{T}_{s-10}\backslash\bar{T}_{s-2k},\ \bar{T}_{s0}\backslash\bar{T}_{s-1k},\ X\backslash\bar{T}_{sk}$$

respectively. It is clear that the intersection disjoint elements of the set

$$\left\{\begin{array}{c}\bar{T}_{01},\bar{T}_{02}\backslash\bar{T}_{s1},\ \ \ldots,\bar{T}_{0k-1}\backslash\bar{T}_{sk-2},\ \bar{T}_{0k}\backslash\bar{T}_{sk-1},\\ \bar{T}_{10},\ \bar{T}_{20}\backslash\bar{T}_{1k},\ \ \ldots,\ \bar{T}_{s-10}\backslash\bar{T}_{s-2k},\ \bar{T}_{s0}\backslash\bar{T}_{s-1k},\ X\backslash\bar{T}_{sk}\end{array}\right\}$$

are empty set, and

$$\bar{T}_{01}\cup\left(\bar{T}_{02}\backslash\bar{T}_{s1}\right)\cup\cdots\cup\left(\bar{T}_{0k-1}\backslash\bar{T}_{sk-2}\right)\cup\left(\bar{T}_{0k}\backslash\bar{T}_{sk-1}\right)\cup$$
$$\bar{T}_{10}\cup\left(\bar{T}_{20}\backslash\bar{T}_{1k}\right)\cup\cdots\cup\left(\bar{T}_{s-10}\backslash\bar{T}_{s-2k}\right)\cup\left(\bar{T}_{s0}\backslash\bar{T}_{s-1k}\right)\cup\left(X\backslash\bar{T}_{sk}\right)=X.$$

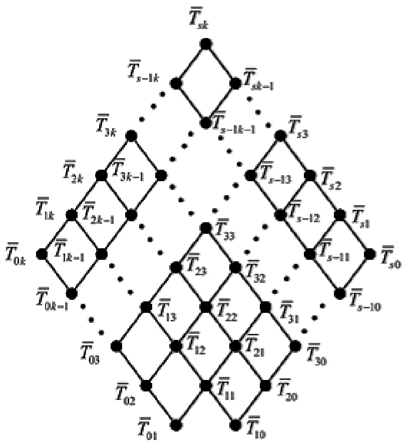
Fig. 2.2

We are going to find properties of the maps $f_{01\alpha}$, $f_{02\alpha}$, $\ldots$, $f_{0k-1\alpha}$, $f_{0k\alpha}$, $f_{10\alpha}$, $f_{20\alpha}$, $\ldots$, $f_{s-10\alpha}$, $f_{s0\alpha}$ and $f_{sk\alpha}$,.

1) $t\in\bar{T}_{01}$. Then by properties (2.3) we have $Y_{01}^{\alpha}\supseteq\bar{T}_{01}$, i.e., $t\in Y_{01}^{\alpha}$ and $t\alpha=T_{01}$ by definition of the set Y_{01}^{α}. Therefore $f_{01\alpha}(t)=T_{01}$ for all $t\in\bar{T}_{01}$.

2) $t\in\bar{T}_{0j}\backslash\bar{T}_{sj-1}$ $(j=2,\ldots,k-1,k)$. Then by properties (2.3) we have $\bar{T}_{0j}\backslash\bar{T}_{sj-1}\subseteq\bar{T}_{0j}\subseteq Y_{01}^{\alpha}\cup Y_{01}^{\alpha}\cup\cdots\cup Y_{0j}^{\alpha}$, i.e., $t\in Y_{01}^{\alpha}\cup Y_{01}^{\alpha}\cup\cdots\cup Y_{0j}^{\alpha}$ and $t\alpha\in\{T_{01},T_{02},\ldots,T_{0j}\}$ by definition of the sets $Y_{01}^{\alpha},Y_{01}^{\alpha},\ldots,Y_{0j}^{\alpha}$. Therefore $f_{0j\alpha}(t)\in\{T_{01},T_{02},\ldots,T_{0j}\}$ for all $t\in\bar{T}_{0j}\backslash\bar{T}_{sj-1}$.

By suppose we have that $Y_{0j}^{\alpha}\cap\bar{T}_{0j}\neq\varnothing$, i.e., $t_{0j}\alpha=T_{0j}$ for some $t_{0j}\in\bar{T}_{0j}$. If $t_{oj}\in\bar{T}_{sj-1}$, then $\bar{T}_{sj-1}=\bar{T}_{s0}\cup\bar{T}_{0j-1}$ by definition of the non complete net, and

$$t_{0j}\in\bar{T}_{sj-1}=\bar{T}_{s0}\cup\bar{T}_{0j-1}\subseteq\left(Y_{10}^{\alpha}\cup Y_{20}^{\alpha}\cup\cdots\cup Y_{s0}^{\alpha}\right)\cup\left(Y_{01}^{\alpha}\cup\cdots\cup Y_{0j-1}^{\alpha}\right).$$

So,

$$t_{0j}\alpha \in \{T_{10}, T_{20}, \ldots, T_{s0}, T_{01}, T_{02}, \ldots, T_{0j-1}\}$$

by definition of the sets Y_{10}^{α}, Y_{20}^{α}, $\ldots$, Y_{s0}^{α}, Y_{01}^{α}, Y_{02}^{α}, $\ldots$, Y_{0j-1}^{α}. The condition $t_{0j}\alpha \in \{T_{10},\ T_{20},\ \ldots,\ T_{s0},\ T_{01},\ T_{02},\ \ldots,\ T_{0j-1}\}$ contradict of the equality $t_{0j}\alpha \in T_{0j}$, while $T_{0j} \notin \{T_{10}, T_{20}, \ldots, T_{s0},\ T_{01}, \ldots, T_{0j-1}\}$. Therefore, $f_{0j\alpha}(t_{0j}) = T_{0j}$ for some $t_{0j} \in \bar{T}_{0j} \backslash \bar{T}_{sj-1}$.

3) $t \in \bar{T}_{10}$. Then by properties (2.3) we have $Y_{10}^{\alpha} \supseteq \bar{T}_{10}$, i.e., $t \in Y_{10}^{\alpha}$ and $t\alpha = T_{10}$ by definition of the set Y_{10}^{α}. Therefore $f_{10\alpha}(t) = T_{10}$ for all $t \in \bar{T}_{10}$.

4) $t \in \bar{T}_{i0} \backslash \bar{T}_{i-1k}$ $(j = 2, \ldots, s-1, s)$. Then by properties (2.3) we have $\bar{T}_{i0} \backslash \bar{T}_{i-10} \subseteq \bar{T}_{i0} \subseteq Y_{10}^{\alpha} \cup Y_{20}^{\alpha} \cup \cdots \cup Y_{i0}^{\alpha}$, i.e., $t \in Y_{10}^{\alpha} \cup Y_{20}^{\alpha} \cup \cdots \cup Y_{i0}^{\alpha}$ and $t\alpha \in \{T_{10}, T_{20}, \ldots, T_{i0}\}$ by definition of the sets $Y_{10}^{\alpha}, Y_{20}^{\alpha}, \ldots, Y_{i0}^{\alpha}$. Therefore $f_{i0\alpha}(t) \in \{T_{10}, T_{20}, \ldots, T_{i0}\}$ for all $t \in \bar{T}_{i0} \backslash \bar{T}_{i-1k}$.

By suppose we have that $Y_{i0}^{\alpha} \cap \bar{T}_{i0} \neq \varnothing$, i.e., $t_{i0\alpha}\alpha = T_{i0}$ for some $t_{i0} \in \bar{T}_{i0}$. If $t_{i0} \in \bar{T}_{i-1k}$, then $\bar{T}_{i-1k} = \bar{T}_{i-10} \cup \bar{T}_{0k}$ by definition of the non complete net, and

$$t_{i0} \in \bar{T}_{i-1k} = \bar{T}_{i-10} \cup \bar{T}_{0k} \subseteq \left(Y_{10}^{\alpha} \cup Y_{20}^{\alpha} \cup \cdots \cup Y_{i-10}^{\alpha}\right) \cup \left(Y_{01}^{\alpha} \cup \cdots \cup Y_{0k}^{\alpha}\right).$$

So,

$$t_{i0}\alpha \in \{T_{10}, T_{20}, \ldots, T_{i-10}, T_{01}, T_{02}, \ldots, T_{0k}\}$$

by definition of the sets Y_{10}^{α}, Y_{20}^{α}, $\ldots$, Y_{i-10}^{α}, Y_{01}^{α}, Y_{02}^{α}, $\ldots$, Y_{0k}^{α}. The condition $t_{i0\alpha}\alpha \in \{T_{10},\ T_{20},\ \ldots,\ T_{i-10},\ T_{01},\ T_{02},\ \ldots,\ T_{0k}\}$ contradict of the equality $t_{i0\alpha}\alpha = T_{i0}$, while $T_{i0} \notin \{T_{10}, T_{20}, \ldots, T_{i-10}, T_{01}, \ldots,\ T_{0k}\}$. Therefore, $f_{i0\alpha}(t_{i0\alpha}) = T_{i0}$ for some $t_{i0} \in \bar{T}_{i0} \backslash \bar{T}_{i-1k}$.

5) $t \in X \backslash \bar{T}_{sk}$. Then by definition quasinormal representation binary relation α and by property (2.3) we have $t \in X \backslash \bar{T}_{sk} \subseteq X = \bigcup\limits_{i \in N_s, j \in N_k} Y_{ij}\ \ ((i,j) \notin \{(0,0)\})$, i.e., $t\alpha \in Q$ by definition of the sets Y_{ij}^{α}. Therefore $f_{sk\alpha}(t) \in Q$ for all $t \in X \backslash \bar{T}_{sk}$.

Therefore, for every binary relation $\alpha \in \bar{R}(Q, D')$ there exists an ordered system

$$\left(f_{01\alpha}, f_{02\alpha}, \ldots, f_{0k-1\alpha}, f_{0k\alpha}, f_{10\alpha}, f_{20\alpha}, \ldots, f_{s-10\alpha}, f_{s0\alpha}, f_{sk\alpha}\right). \tag{2.4}$$

Further, let

$$\begin{aligned}
&f_{01} : \bar{T}_{01} \to \{T_{01}\},\ f_{10} : \bar{T}_{10} \to \{T_{10}\},\\
&f_{0j} : \bar{T}_{0j} \backslash \bar{T}_{sj-1} \to \{T_{00}, T_{01}, T_{02}, \ldots, T_{0j}\},\ (j = 2, \ldots, k-1, k),\\
&f_{i0} : \bar{T}_{i0} \backslash \bar{T}_{i-1k} \to \{T_{00}, T_{10}, T_{20}, \ldots, T_{i0}\},\ (i = 2, \ldots, s-1, s),\\
&f_{sk} : X \backslash \bar{T}_{sk} \to \{T_{ij} : i \in N_s, j \in N_k\},
\end{aligned}$$

are such mappings, which satisfying the conditions:

6) $f_{01}(t) = T_{01}$ for all $t \in \bar{T}_{01}$;

7) $f_{10}(t) = T_{10}$ for all $t \in \bar{T}_{10}$;

8) $f_{0j}(t) \in \{T_{01}, T_{02}, \ldots, T_{0j}\}$ for all $t \in \bar{T}_{0j} \backslash \bar{T}_{sj-1}$ and $f_{0j}(t_{0j}) = T_{0j}$ for some $t_{0j} \in \bar{T}_{0j} \backslash \bar{T}_{sj-1}$;

9) $f_{i0}(t) \in \{T_{10}, T_{20}, \ldots, T_{i0}\}$ for all $t \in \bar{T}_{i0} \backslash \bar{T}_{i-1k}$ and $f_{i0}(t_{i0}) = T_{i0}$ for some $t_{i0} \in \bar{T}_{i0} \backslash \bar{T}_{i-1k}$;

10) $f_{sk}(t) \in Q$ for all $t \in X \backslash \bar{T}_{sk}$.

Now, we define a map f of a set X in the semilattice D, which satisfies the condition:

$$f(t) = \begin{cases} f_{01}(t), & \text{if } t \in \bar{T}_{01} \\ f_{oj}(t), & \text{if } t \in \bar{T}_{0j} \backslash \bar{T}_{sj-1} \quad (j = 2, \ldots, k-1, k) \\ f_{10}(t), & \text{if } t \in \bar{T}_{10} \\ f_{i0}(t), & \text{if } t \in \bar{T}_{i0} \backslash \bar{T}_{i-1k} \quad (i = 2, \ldots, s-1, s) \\ f_{sk}(t), & \text{if } t \in X \backslash \bar{T}_{sk}. \end{cases}$$

Further, let $\beta = \bigcup\limits_{x \in X} (\{x\} \times f(x))$, and $Y_{ij}^{\beta} = \{t : t\beta = T_{ij}\}$ $(T_{ij} \in Q)$. Then binary relation β may be representation by form $\beta = \bigcup\limits_{T_{ij} \in Q} \left(Y_{ij}^{\beta} \times T_{ij}\right)$ and satisfying the conditions:

$$\begin{gathered} Y_{10}^{\alpha} \supseteq \bar{T}_{10},\ Y_{10}^{\alpha} \cup Y_{20}^{\alpha} \supseteq \bar{T}_{20},\ \ldots,\ Y_{10}^{\alpha} \cup Y_{20}^{\alpha} \cup \cdots \cup Y_{s0}^{\alpha} \supseteq \bar{T}_{s0}, \\ Y_{01}^{\alpha} \supseteq \bar{T}_{01},\ Y_{01}^{\alpha} \cup Y_{02}^{\alpha} \supseteq \bar{T}_{02},\ \ldots,\ Y_{01}^{\alpha} \cup Y_{02}^{\alpha} \cup \cdots \cup Y_{0k}^{\alpha} \supseteq \bar{T}_{0k}, \\ Y_{20}^{\alpha} \cap \bar{T}_{20} \neq \varnothing,\ \ldots,\ Y_{s0}^{\alpha} \cap \bar{T}_{s0} \neq \varnothing,\ Y_{02}^{\alpha} \cap \bar{T}_{02} \neq \varnothing,\ \ldots, Y_{0k}^{\alpha} \cap \bar{T}_{0k} \neq \varnothing, \end{gathered}$$

(By suppose $f_{i0}(t_{i0}) = T_{i0}$ for some $t_{i0} \in \bar{T}_{i0} \backslash \bar{T}_{i-1k}$ and $f_{0j}(t_{0j}) = T_{0j}$ for some $t_{0j} \in \bar{T}_{0j} \backslash \bar{T}_{sj-1}$). From this and by Theorem 2.1 we have that $\beta \in \bar{R}(Q, D')$. Therefore for every binary relation $\alpha \in \bar{R}(Q, D')$ and ordered system (2.4) exist one to one mapping.

By the Theorem 1.1 the number of the mappings $f_{01\alpha}$, $f_{02\alpha}$, $\ldots$, $f_{0k-1\alpha}$, $f_{0k\alpha}$, $f_{10\alpha}$, $f_{20\alpha}$, $\ldots$, $f_{s-10\alpha}$, $f_{s0\alpha}$, $f_{sk\alpha}$ are respectively:

$$1, j^{|T_{02} \backslash T_{s1}|} - 1, 3^{|T_{03} \backslash T_{s2}|} - 2^{|T_{03} \backslash T_{s2}|}, \ldots, (k-1)^{|T_{0k-1} \backslash T_{sk-2}|} - (k-2)^{|T_{0k-1} \backslash T_{sk-2}|},$$
$$k^{|T_{0k} \backslash T_{sk-1}|} - (k-1)^{|T_{0k} \backslash T_{sk-1}|}, 2^{|T_{20} \backslash T_{1k}|} - 1,\ 3^{|T_{30} \backslash T_{2k}|} - 2^{|T_{30} \backslash T_{2k}|},\ \ldots,$$
$$(s-1)^{|T_{s-10} \backslash T_{s-2k}|} - (i-1)^{|T_{s-10} \backslash T_{s-2k}|}, s^{|T_{s0} \backslash T_{s-1k}|} - (s-1)^{|T_{s0} \backslash T_{s-1k}|}, |D|^{\left|X \backslash \bar{T}_{sk}\right|}.$$

Therefore the equality

$$\left|\bar{R}(Q, D')\right| = \left(2^{|T_{02} \backslash T_{s1}|} - 1\right) \cdot \left(3^{|T_{03} \backslash T_{s2}|} - 2^{|T_{03} \backslash T_{s2}|}\right) \cdots$$

$$\left((k-1)^{|T_{0k-1}\setminus T_{sk-2}|}-(k-2)^{|T_{0k-1}\setminus T_{sk-2}|}\right)\cdot$$
$$\left(k^{|T_{0k}\setminus T_{sk-1}|}-(k-1)^{|T_{0k}\setminus T_{sk-1}|}\right)\cdot\left(2^{|T_{20}\setminus T_{1k}|}-1\right)\cdot$$
$$\left(3^{|T_{30}\setminus T_{2k}|}-2^{|T_{30}\setminus T_{2k}|}\right)\dots$$
$$\left((s-1)^{|T_{s-10}\setminus T_{s-2k}|}-(i-1)^{|T_{s-10}\setminus T_{s-2k}|}\right)\cdot$$
$$\left(s^{|T_{s0}\setminus T_{s-1k}|}-(s-1)^{|T_{s0}\setminus T_{s-1k}|}\right)\cdot|Q|^{\left|X\setminus\bar{T}_{sk}\right|},$$

is valid. Now, using the equalities $|\Omega(Q)|=m_0$ we Obtain

$$\left|R(D')\right|=m_0\cdot\left(2^{|T_{02}\setminus T_{s1}|}-1\right)\cdot\left(3^{|T_{03}\setminus T_{s2}|}-2^{|T_{03}\setminus T_{s2}|}\right)\dots$$
$$\left((k-1)^{|T_{0k-1}\setminus T_{sk-2}|}-(k-2)^{|T_{0k-1}\setminus T_{sk-2}|}\right)\cdot$$
$$\left(k^{|T_{0k}\setminus T_{sk-1}|}-(k-1)^{|T_{0k}\setminus T_{sk-1}|}\right)\cdot\left(2^{|T_{20}\setminus T_{1k}|}-1\right)\cdot$$
$$\left(3^{|T_{30}\setminus T_{2k}|}-2^{|T_{30}\setminus T_{2k}|}\right)\dots$$
$$\left((s-1)^{|T_{s-10}\setminus T_{s-2k}|}-(i-1)^{|T_{s-10}\setminus T_{s-2k}|}\right)\cdot$$
$$\left(s^{|T_{s0}\setminus T_{s-1k}|}-(s-1)^{|T_{s0}\setminus T_{s-1k}|}\right)\cdot|Q|^{\left|X\setminus\bar{T}_{sk}\right|},$$

if $s\neq k$ or

$$\left|R(D')\right|=2\cdot m_0\cdot\left(2^{|T_{02}\setminus T_{s1}|}-1\right)\cdot\left(3^{|T_{03}\setminus T_{s2}|}-2^{|T_{03}\setminus T_{s2}|}\right)\dots$$
$$\left((k-1)^{|T_{0k-1}\setminus T_{sk-2}|}-(k-2)^{|T_{0k-1}\setminus T_{sk-2}|}\right)\cdot$$
$$\left(k^{|T_{0k}\setminus T_{sk-1}|}-(k-1)^{|T_{0k}\setminus T_{sk-1}|}\right)\cdot\left(2^{|T_{20}\setminus T_{1k}|}-1\right)\cdot$$
$$\left(3^{|T_{30}\setminus T_{2k}|}-2^{|T_{30}\setminus T_{2k}|}\right)\dots\left((s-1)^{|T_{s-10}\setminus T_{s-2k}|}-(i-1)^{|T_{s-10}\setminus T_{s-2k}|}\right)\cdot$$
$$\left(s^{|T_{s0}\setminus T_{s-1k}|}-(s-1)^{|T_{s0}\setminus T_{s-1k}|}\right)\cdot|Q|^{\left|X\setminus\bar{T}_{sk}\right|},$$

if $s=k$ (see Theorem 1.6). □

Corollary 2.1. *Let Q be a $XI-$lower incomplete net and $E_X^{(r)}(Q)$ be the set of all right units of the semigroup $B_x(Q)$. If X is a finite set, then the following formula is true*

$$\left|E_X^{(r)}(Q)\right|=\left(2^{|T_{02}\setminus T_{s1}|}-1\right)\cdot\left(3^{|T_{03}\setminus T_{s2}|}-2^{|T_{03}\setminus T_{s2}|}\right)\dots$$

Fig. 2.3

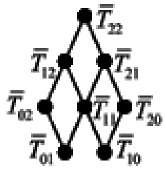

Fig. 2.4

$$\left((k-1)^{|T_{0k-1}\setminus T_{sk-2}|}-(k-2)^{|T_{0k-1}\setminus T_{sk-2}|}\right)\cdot$$
$$\left(k^{|T_{0k}\setminus T_{sk-1}|}-(k-1)^{|T_{0k}\setminus T_{sk-1}|}\right)\cdot\left(2^{|T_{20}\setminus T_{1k}|}-1\right)\cdot$$
$$\left(3^{|T_{30}\setminus T_{2k}|}-2^{|T_{30}\setminus T_{2k}|}\right)\dots\left((s-1)^{|T_{s-10}\setminus T_{s-2k}|}-(i-1)^{|T_{s-10}\setminus T_{s-2k}|}\right)$$
$$\cdot\left(s^{|T_{s0}\setminus T_{s-1k}|}-(s-1)^{|T_{s0}\setminus T_{s-1k}|}\right)\cdot|Q|^{\left|X\setminus\bar{T}_{sk}\right|}.$$

Proof. By virtue of Theorem [4, Theorem 6.3.11] 6.3.11 (see [4]) we have $E_X^{(r)}(Q)=R_{\varepsilon_Q}(Q,Q)$, where ε_Q is the identity mapping of the net Q. Now, taking into account Theorem 1.5 and Theorem 2.4, we obtain the validity of corollary. □

Corollary 2.2. *Let $Q=\{T_{01},T_{10},T_{11}\}$ be a $XI-$subsemilattice of the semilattice D (see Fig. 2.3). If the semilattices Q and $D'=\left\{\bar{T}_{01},\bar{T}_{10},\bar{T}_{11}\right\}$ are $\alpha-$isomorphic, $|\Omega(Q)|=m_0$, then the following equality is valid: $|R(D')|=2\cdot m_0\cdot 3^{\left|X\setminus\bar{T}_{11}\right|}$.*

Proof. It is obvious that in this case $|\Phi(Q,D')|=2$. Therefore the corollary immediately follows from Theorem 2.4. □

Corollary 2.3. *Let $Q=\{T_{01},T_{10},T_{02},T_{11},T_{20},T_{12},T_{21},T_{22}\}$ be a $XI-$subsemilattice of the semilattice D (see Fig. 2.4). If the semilatices Q and*

$$D'=\left\{\bar{T}_{01},\bar{T}_{10},\bar{T}_{02},\bar{T}_{11},\bar{T}_{20},\bar{T}_{12},\bar{T}_{21},\bar{T}_{22}\right\}$$

are $\alpha-$isomorphic, $|\Omega(Q)|=m_0$, then the following equality is valid:

$$\left|R(D')\right|=2\cdot m_0\cdot\left(2^{|\bar{T}_{20}\setminus\bar{T}_{12}|}-1\right)\cdot\left(2^{|\bar{T}_{02}\setminus\bar{T}_{21}|}-1\right)\cdot 8^{|X\setminus\bar{T}_{22}|}.$$

Proof. It is obvious that in this case $|\Phi(Q,D')|=2$. Therefore the corollary immediately follows from Theorem 2.4. □

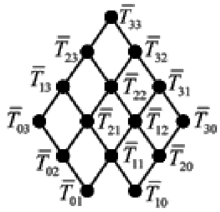

Fig. 2.5

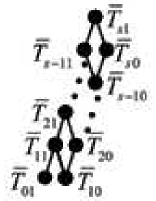

Fig. 2.6

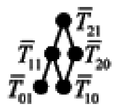

Fig. 2.7

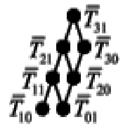

Fig. 2.8

Corollary 2.4. *Let $Q = \{T_{01}, T_{10}, T_{02}, T_{11}, T_{20}, \ldots, T_{33}\}$ be a $XI-$sub-semilattice of the semilattice D (see Fig. 2.5). If the semilatices Q and*

$$D' = \left\{\bar{T}_{01}, \bar{T}_{10}, \bar{T}_{02}, \bar{T}_{11}, \bar{T}_{20}, \ldots, \bar{T}_{33}\right\}$$

are $\alpha-$isomorphic, $|\Omega(Q)| = m_0$, then the following equality is valid:

$$\left|R(D')\right| = 2 \cdot m_0 \cdot \left(2^{\left|\bar{T}_{20} \setminus \bar{T}_{12}\right|} - 1\right) \cdot \left(3^{\left|\bar{T}_{30} \setminus \bar{T}_{23}\right|} - 2^{\left|\bar{T}_{30} \setminus \bar{T}_{23}\right|}\right)$$
$$\cdot \left(2^{\left|\bar{T}_{02} \setminus \bar{T}_{21}\right|} - 1\right) \cdot \left(3^{\left|\bar{T}_{03} \setminus \bar{T}_{32}\right|} - 2^{\left|\bar{T}_{03} \setminus \bar{T}_{32}\right|}\right) \cdot 15^{\left|X \setminus \bar{T}_{33}\right|}.$$

Proof. It is obvious that in this case $|\Phi(Q, D')| = 2$. Therefore the corollary immediately follows from Theorem 2.4. □

Corollary 2.5. *Let $Q = \{T_{01}, T_{10}, T_{02}, T_{11}, T_{20}, \ldots, T_{s-11}, T_{s0}, T_{s1}\}$ be a $XI-$sub-semilattice of the semilattice D (see Fig. 2.6). If the semilattices Q and*

$$D' = \left\{\bar{T}_{01}, \bar{T}_{10}, \bar{T}_{02}, \bar{T}_{11}, \bar{T}_{20}, \ldots, \bar{T}_{s-11}, \bar{T}_{s0}, \bar{T}_{s1}\right\}$$

are $\alpha-$isomorphic, $|\Omega(Q)| = m_0$, then the following equality is valid:

$$\left|R(D')\right| = m_0 \cdot \left(2^{\left|\bar{T}_{20} \setminus \bar{T}_{11}\right|} - 1\right) \cdot \left(3^{\left|\bar{T}_{30} \setminus \bar{T}_{21}\right|} - 2^{\left|\bar{T}_{30} \setminus \bar{T}_{21}\right|}\right) \cdots$$
$$\left((s-1)^{\left|\bar{T}_{s-10} \setminus \bar{T}_{s-21}\right|} - (s-2)^{\left|\bar{T}_{s-10} \setminus \bar{T}_{s-21}\right|}\right)$$
$$\cdot \left(s^{\left|\bar{T}_{s0} \setminus \bar{T}_{s-11}\right|} - (s-1)^{\left|\bar{T}_{s0} \setminus \bar{T}_{s-11}\right|}\right) \cdot (2 \cdot (s+1) - 1)^{\left|X \setminus \bar{T}_{s1}\right|}.$$

Proof. It is obvious that in this case $|\Phi(Q, D')| = 1$. Therefore the corollary immediately follows from Theorem 2.4. □

Corollary 2.6. *Let $Q = \{T_{01}, T_{10}, T_{11}, T_{20}, T_{21}\}$ be a $XI-$subsemilattice of the semilattice D (see Fig. 2.7). If the semilattices Q and*

$$D' = \left\{\bar{T}_{01}, \bar{T}_{10}, \bar{T}_{11}, \bar{T}_{20}, \bar{T}_{21}\right\}$$

are $\alpha-$isomorphic, $|\Omega(Q)| = m_0$, then the following equality is valid: $|R(D')| = m_0 \cdot \left(2^{|\bar{T}_{20}\setminus\bar{T}_{11}|} - 1\right) \cdot 5^{|X\setminus\bar{T}_{21}|}$.

Proof. It is obvious that in this case $|\Phi(Q, D')| = 1$. Therefore the corollary immediately follows from Theorem 2.4. □

Corollary 2.7. *Let $Q = \{T_{01}, T_{10}, T_{11}, T_{20}, T_{21}, T_{30}, T_{31}\}$ be a $XI-$sub-semilattice of the semilattice D (see Fig. 2.8). If the semilattices Q and*

$$D' = \left\{\bar{T}_{01}, \bar{T}_{10}, \bar{T}_{11}, \bar{T}_{20}, \bar{T}_{21}, \bar{T}_{30}, \bar{T}_{31}\right\}$$

are $\alpha-$isomorphic, $|\Omega(Q)| = m_0$, then the following equality is valid: $|R(D')| = m_0 \cdot \left(2^{|\bar{T}_{20}\setminus\bar{T}_{11}|} - 1\right) \cdot \left(3^{|\bar{T}_{30}\setminus\bar{T}_{21}|} - 2^{|\bar{T}_{30}\setminus\bar{T}_{21}|}\right) \cdot 7^{|X\setminus\bar{T}_{31}|}$.

Proof. It is obvious that in this case $|\Phi(Q, D')| = 1$. Therefore the corollary immediately follows from Theorem 2.4. □

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