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# ON SUBMANIFOLDS OF A RIEMANNIAN MANIFOLD WITH A SEMI-SYMMETRIC RECURRENT-METRIC CONNECTION

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**Abstract:** We study some properties of submanifolds of a Riemannian manifold with a semi-symmetric recurrent-metric connection. Among others, the Gauss equation, the Codazzi-Mainardi equation and the Ricci equation for such a connection have been derived.

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**Key Words:** submanifolds, semi-symmetric recurrent-metric connection, curvature with respect to semi-symmetric recurrent-metric connection, Gauss equation, Codazzi-Mainardi equation, Ricci equation, totally geodesic, totally umbilical, minimal, sectional curvature with respect to semi-symmetric recurrent-metric connection

## 1. Introduction

Let $M = (M, g)$ be a Riemannian manifold of dimension $n$ with a metric tensor $g$. A linear connection $\nabla$ on $M$ satisfies

$$(i)\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z, (ii)\nabla_X(fY) = (Xf)Y + f\nabla_X Y,$$

where $f, g$ are smooth functions on $M$ and $X, Y, Z$ are smooth vector fields on

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$M$. The torsion tensor $T$ of $\nabla$ is given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

If the torsion tensor $T$ vanishes, then $\nabla$ is called symmetric, otherwise it is nonsymmetric. If the metric tensor $g$ of $M$ satisfies $\nabla g = 0$, then $\nabla$ is called metric, otherwise it is nonmetric. It is well known that the Levi-Civita connection is the only linear connection which is both symmetric and metric. In particular, a nonsymmetric connection is said to be semi-symmetric if its torsion tensor $T$ is of the form

$$T(X,Y) = u(Y)X - u(X)Y,$$

where $u$ is a 1-form on $M$.

The idea of semi-symmetric metric connection on a Riemannian manifold was introduced by Yano and some of its properties were studied in [14]. A hypersurface of a Riemannian manifold with a semi-symmetric metric connection was studied by Imai [6]. Later in [9] Nakao investigated submanifolds of a Riemannian manifold with the semi-symmetric metric connection. On the other hand, Agashe and Chafle [1] introduced the idea of a semi-symmetric nonmetric connection on a Riemannian manifold and they [2] sequently studied submanifolds of a Riemannian manifold with the semi-symmetric nonmetric connection mentioned in [1]. Other types of semi-symmetric nonmetric connections were introduced by Sengupta, De and Binh [12]; Sengupta and De [11]. Later submanifolds of Riemannian manifolds with the semi-symmetric nonmetric connections defined in [11,12] were studied by Ozgur [10] and Dogru [5], respectively. On the other hand, a different kind of semi-symmetric nonmetric connection (namely, semi-symmetric recurrent-metric connection) was defined and extensively studied by Andonie and Smaranda [3]; Liang [8]. Considering these aspects, we are motivated to study submanifolds of a Riemannian manifold with the semi-symmetric recurrent-metric connection mentioned in [3,8]. More precisely, in Section 2, a general description of Riemannian manifold and its submanifold is given and then a semi-symmetric recurrent-metric connection is defined. In Section 3, we show that the induced connection on a submanifold of a Riemannian manifold with the semi-symmetric recurrent-metric connection is also a semi-symmetric recurrent-metric connection. And then the Gauss, the Codazzi-Mainardi and the Ricci equations for such a connection have been derived. We also consider the totally geodesic, totally umbilical and minimal submanifolds of a Riemannian manifold with the semi-symmetric recurrent-metric connection. Finally, a concrete example of submanifold of a Riemannian manifold with the semi-symmetric recurrent-metric connection is given.

## 2. Preliminaries

Let $M$ be an $n$-dimensional submanifold of an $(n+l)$-dimensional Riemannian manifold $\widetilde{M}$. From now on, $g$ refers to the Riemannian metric tensor on $\widetilde{M}$ as well as the induced one on $M$. Also, in the sequel, $\widetilde{X}$,$\widetilde{Y}$,$\widetilde{Z}$, $\widetilde{W}$ denote the vector fields on $\widetilde{M}$; $X, Y, Z, W$ denote the vector fields tangent to $M$. The formulas of Gauss and Weingarten are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.1}$$

and

$$\widetilde{\nabla}_X \eta = -A_\eta X + \nabla_X{}^{\perp}\eta \tag{2.2}$$

respectively, where $\eta$ is a normal vector field of $M$ in $\widetilde{M}$ and $\nabla$ is the induced Riemannian connection on $M$ from the Riemannian connection $\widetilde{\nabla}$ on $\widetilde{M}$ and $\nabla^{\perp}$ is a (metric) connection in the normal bundle $T(M)^{\perp}$ with respect to the fibre metric induced from $g$ [7]. Note that the second fundamental form $h$ is related to the shape operator $A_\eta$ by

$$g(h(X, Y), \eta) = g(A_\eta X, Y) = g(X, A_\eta Y). \tag{2.3}$$

If $h = 0$, then $M$ is said to be totally geodesic. The mean curvature vector $H$ of $M$ is given by

$$H = \frac{1}{n} trace h. \tag{2.4}$$

$M$ is said to be minimal if $H$=0 ; $M$ is said to be totally umbilical if $h(X, Y) = g(X, Y)H$. The covariant derivative of $h$ is defined by

$$(\widehat{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

The connection $\widehat{\nabla}$ is called the van der Waerden-Bortolotti connection of $M$ [4]. In [13], a linear connection $\acute{\widetilde{\nabla}}$ on a Riemannian manifold $\widetilde{M}$ is given by

$$\acute{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} + u_1(\widetilde{Y})\widetilde{X} - g(\widetilde{X}, \widetilde{Y})U_1 - u_2(\widetilde{X})\widetilde{Y} - u_2(\widetilde{Y})\widetilde{X} + g(\widetilde{X}, \widetilde{Y})U_2, \tag{2.5}$$

where $u_1$ and $u_2$ are 1-forms associated with the vector fields $U_1$ and $U_2$ on $\widetilde{M}$ by $u_1(\widetilde{X}) = g(U_1, \widetilde{X})$ and $u_2(\widetilde{X}) = g(U_2, \widetilde{X})$, respectively. Using (2.5), the torsion tensor $\acute{\widetilde{T}}$ of $\acute{\widetilde{\nabla}}$ is given by

$$\acute{\widetilde{T}}(\widetilde{X}, \widetilde{Y}) = u_1(\widetilde{Y})\widetilde{X} - u_1(\widetilde{X})\widetilde{Y}. \tag{2.6}$$

Furthermore, using (2.5), we get

$$(\acute{\tilde{\nabla}}_{\widetilde{X}} g)(\widetilde{Y}, \widetilde{Z}) = 2u_2(\widetilde{X}) g(\widetilde{Y}, \widetilde{Z}). \tag{2.7}$$

Therefore, the linear connection $\acute{\tilde{\nabla}}$ defined by (2.5) is adequate to be called a semi-symmetric recurrent-metric connection [13]. From now on, for the sake of simplicity, a semi-symmetric recurrent-metric connection is briefly denoted by a SSRM connection. We define the curvature tensor of type (1,3) of a SSRM connection $\acute{\tilde{\nabla}}$ by

$$\acute{\tilde{R}}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = \acute{\tilde{\nabla}}_{\widetilde{X}} \acute{\tilde{\nabla}}_{\widetilde{Y}} \widetilde{Z} - \acute{\tilde{\nabla}}_{\widetilde{Y}} \acute{\tilde{\nabla}}_{\widetilde{X}} \widetilde{Z} - \acute{\tilde{\nabla}}_{[\widetilde{X}, \widetilde{Y}]} \widetilde{Z}. \tag{2.8}$$

Also, the curvature tensor of type (0,4) of the SSRM connection $\acute{\tilde{\nabla}}$ is defined by

$$\acute{\tilde{R}}(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{W}) = g(\acute{\tilde{R}}(\widetilde{X}, \widetilde{Y})\widetilde{Z}, \widetilde{W}). \tag{2.9}$$

## 3. Submanifolds of a Riemannian Manifold with a Semi-Symmetric Recurrent-Metric Connection

Let $\acute{\nabla}$ be the induced connection on $M$ from the SSRM connection $\acute{\tilde{\nabla}}$ on $\widetilde{M}$ by the equation which may be called the formula of Gauss with respect to the SSRM connection

$$\acute{\tilde{\nabla}}_X Y = \acute{\nabla}_X Y + \acute{h}(X, Y), \tag{3.1}$$

where $\acute{h}$ is a normal bundle valued tensor of type (0,2). Taking account of (2.1), (2.5) and (3.10), we have

$$\acute{\nabla}_X Y + \acute{h}(X, Y) = \nabla_X Y + h(X, Y) + u_1(Y)X - g(X, Y)U_1^{\top} - g(X, Y)U_1^{\perp}$$
$$-u_2(X)Y - u_2(Y)X + g(X, Y)U_2^{\top} + g(X, Y)U_2^{\perp},$$

which yields

$$\acute{\nabla}_X Y = \nabla_X Y + u_1(Y)X - g(X, Y)U_1^{\top} - u_2(X)Y - u_2(Y)X + g(X, Y)U_2^{\top} \tag{3.2}$$

and

$$\acute{h}(X, Y) = h(X, Y) - g(X, Y)U_1^{\perp} + g(X, Y)U_2^{\perp}. \tag{3.3}$$

Here we denote by $U_i^{\top}$ and $U_i^{\perp}$ the tangential and normal components of $U_i$ ($i = 1, 2$), respectively. It follows from (3.11) and the properties of a Riemannian connection $\nabla$ that

$$\acute{T}(X, Y) = \acute{\nabla}_X Y - \acute{\nabla}_Y X - [X, Y] = u_1(Y)X - u_1(X)Y \tag{3.4}$$

and

$$(\acute{\nabla}_X g)(Y,Z) = X(g(Y,Z)) - g(\acute{\nabla}_X Y, Z) - g(Y, \acute{\nabla}_X Z) = 2u_2(X)g(Y,Z). \quad (3.5)$$

Therefore, we obtain the following:

**Theorem 3.1.** *The induced connection $\acute{\nabla}$ on a submanifold $M$ of a Riemannian manifold $\widetilde{M}$ with the SSRM connection $\tilde{\acute{\nabla}}$ is also a SSRM connection.*

Analogous to the definition of the mean curvature vector $H$ of $M$, we define the mean curvature vector $\acute{H}$ of $M$ with respect to the SSRM connection $\acute{\tilde{\nabla}}$ by

$$\acute{H} = \frac{1}{n} trace \acute{h}. \quad (3.6)$$

$M$ is said to be minimal with respect to the SSRM connection $\acute{\tilde{\nabla}}$ if $\acute{H} = 0$ ; $M$ is said to be totally geodesic with respect to the SSRM connection $\acute{\tilde{\nabla}}$ if $\acute{h} = 0$ ; $M$ is said to be totally umbilical with respect to the SSRM connection $\acute{\tilde{\nabla}}$ if $\acute{h}(X,Y) = g(X,Y)\acute{H}$. Now we can state the following:

**Theorem 3.2.** *Let $M$ be a submanifold of a Riemannian manifold $\widetilde{M}$ with the SSRM connection $\tilde{\acute{\nabla}}$. Then $M$ is totally umbilical if and only if $M$ is totally umbilical with respect to the SSRM connection $\acute{\tilde{\nabla}}$. Furthermore, if the associated vector fields $U_1$ and $U_2$ are tangent to $M$, then:*

*(i) the second fundamental form $h$ of $M$ and the second fundamental form $\acute{h}$ of $M$ with respect to the SSRM connection $\acute{\tilde{\nabla}}$ coincide;*

*(ii) the mean curvature vector $H$ of $M$ and the mean curvature vector $\acute{H}$ of $M$ with respect to the SSRM connection $\acute{\tilde{\nabla}}$ coincide;*

*(iii) $M$ is totally geodesic if and only if $M$ is totally geodesic with respect to the SSRM connection $\acute{\tilde{\nabla}}$;*

*(iv) $M$ is minimal if and only if $M$ is minimal with respect to the SSRM connection $\acute{\tilde{\nabla}}$.*

*Proof.* In view of (3.12) and (3.15), we have

$$\acute{H} = H - U_1^{\perp} + U_2^{\perp}.$$

Comparing the above relation with (3.12), we conclude that a totally umbilical $M$ is also totally umbilical with respect to $\acute{\tilde{\nabla}}$, and vice versa. On the other hand, let us assume that $U_1$ and $U_2$ are tangent to $M$. Then we have from (3.12)

$$\acute{h}(X,Y) = h(X,Y).$$

Taking account of the above identity and (3.15), we obtain that (i), (ii), (iii) and (iv) hold true. This completes the proof of theorem 3.2. □

Let $\eta$ be a normal vector field of $M$ in $\widetilde{M}$. Taking account of (2.5), we get

$$\acute{\tilde{\nabla}}_X\eta = \widetilde{\nabla}_X\eta + u_1(\eta)X - u_2(X)\eta - u_2(\eta)X, \tag{3.7}$$

which yields from (2.2)

$$\acute{\tilde{\nabla}}_X\eta = -A_\eta X + \nabla^{\perp}_X\eta + u_1(\eta)X - u_2(X)\eta - u_2(\eta)X. \tag{3.8}$$

From (3.17), we can define a tensor of type (1,1) on $M$ as follows:

$$\acute{A}_\eta = A_\eta - u_1(\eta)Id + u_2(\eta)Id. \tag{3.9}$$

Taking (3.17) and (3.18) into account, we have

$$\acute{\tilde{\nabla}}_X\eta = -\acute{A}_\eta X + \nabla^{\perp}_X\eta - u_2(X)\eta, \tag{3.10}$$

which may be called the formula of Weingarten with respect to the SSRM connection $\acute{\tilde{\nabla}}$. Now we obtain the following:

**Theorem 3.3.** *If $M$ is a submanifold of a Riemannian manifold $\widetilde{M}$ with the SSRM connection $\tilde{\nabla}$, then for the unit normal vector field $\eta$ of $M$ in $\widetilde{M}$, the principal directions and the principal directions with respect to the SSRM connection $\acute{\tilde{\nabla}}$ coincide. Moreover, if the associated vector field $U_1$ and $U_2$ are tangent to $M$, then the principal curvatures are equal to the principal curvatures with respect to the SSRM connection $\acute{\tilde{\nabla}}$.*

*Proof.* Taking account of (3.18), we conclude that the principal directions and the principal directions with respect to the SSRM connection $\acute{\tilde{\nabla}}$ coincide. On the other hand, let us assume that $U_1$ and $U_2$ are tangent to $M$. Then we have from (3.18)

$$\acute{A}_\eta = A_\eta,$$

which implies that the principal curvatures are equal to the principal curvatures with respect to $\acute{\tilde{\nabla}}$. This completes the proof of theorem 3.3. □

**Theorem 3.4.** *Let $M$ be a submanifold of a Riemannian manifold $\widetilde{M}$ with the SSRM connection $\tilde{\nabla}$. Then the shape operators are simultaneously diagonalizable if and only if the shape operators with respect to the SSRM connection $\acute{\tilde{\nabla}}$ are simultaneously diagonalizable.*

*Proof.* For the unit normal vector fields $\eta$, $\mu$ of $M$ in $\widetilde{M}$, we have from (3.18)

$$g(\acute{A}_\eta X, Y) = g(X, \acute{A}_\eta Y)$$

and

$$g([\acute{A}_\eta, \acute{A}_\mu]X, Y) = g((\acute{A}_\eta \acute{A}_\mu - \acute{A}_\mu \acute{A}_\eta)X, Y)$$
$$= g((A_\eta A_\mu - A_\mu A_\eta)X, Y) = g([A_\eta, A_\mu]X, Y),$$

which gives the required result. □

Let $\acute{R}$ be the curvature tensor with respect to the induced SSRM connection $\acute{\nabla}$ on a submanifold $M$. More precisely,

$$\acute{R}(X,Y)Z = \acute{\nabla}_X \acute{\nabla}_Y Z - \acute{\nabla}_Y \acute{\nabla}_X Z - \acute{\nabla}_{[X,Y]} Z.$$

Taking (3.10) and (3.19) into account, we have

$$\acute{\tilde{R}}(X,Y)Z = \acute{R}(X,Y)Z + \acute{h}(X, \acute{\nabla}_Y Z) - \acute{h}(Y, \acute{\nabla}_X Z)$$
$$-\acute{h}([X,Y], Z) - \acute{A}_{\acute{h}(Y,Z)}(X) + \acute{A}_{\acute{h}(X,Z)}(Y) + \nabla^{\perp}_X \acute{h}(Y,Z)$$
$$-\nabla^{\perp}_Y \acute{h}(X,Z) - u_2(X)\acute{h}(Y,Z) + u_2(Y)\acute{h}(X,Z). \tag{3.11}$$

From (2.3), (3.12) and (3.18), it follows that

$$\acute{\tilde{R}}(X,Y,Z,W) = \acute{R}(X,Y,Z,W) - g(h(X,W), h(Y,Z))$$
$$+u_1(h(X,W))g(Y,Z) - u_2(h(X,W))g(Y,Z) + g(h(X,Z), h(Y,W))$$
$$-u_1(h(Y,W))g(X,Z) + u_2(h(Y,W))g(X,Z) + u_1(h(Y,Z))g(X,W)$$
$$-u_1({U_1}^{\perp})g(X,W)g(Y,Z) + u_1({U_2}^{\perp})g(X,W)g(Y,Z) - u_1(h(X,Z))g(Y,W)$$
$$+u_1({U_1}^{\perp})g(X,Z)g(Y,W) - u_1({U_2}^{\perp})g(X,Z)g(Y,W) - u_2(h(Y,Z))g(X,W)$$
$$+u_2({U_1}^{\perp})g(X,W)g(Y,Z) - u_2({U_2}^{\perp})g(X,W)g(Y,Z) + u_2(h(X,Z))g(Y,W)$$
$$-u_2({U_1}^{\perp})g(X,Z)g(Y,W) + u_2({U_2}^{\perp})g(X,Z)g(Y,W), \tag{3.12}$$

which may be called the Gauss equation with respect to the SSRM connection. The manifold $\widetilde{M}$ (resp. $M$) with a SSRM connection $\acute{\tilde{\nabla}}$ (resp. $\acute{\nabla}$) is said to be $\acute{\tilde{\nabla}}$-flat (resp. $\acute{\nabla}$-flat) if the curvature tensor $\acute{\tilde{R}}$ (resp.$\acute{R}$) of $\acute{\tilde{\nabla}}$ (resp. $\acute{\nabla}$) vanishes. Now we can state the following:

**Theorem 3.5.** *Let $\widetilde{M}$ be a $\acute{\tilde{\nabla}}$-flat manifold. If $M$ is totally geodesic and the associated vector fields $U_1$, $U_2$ are tangent to $M$, then $M$ is a $\acute{\nabla}$-flat manifold.*

*Proof.* From $h = 0$, $U_i^{\perp} = 0$ $(i = 1, 2)$ and (3.21), it follows that the theorem holds. □

Concerning the sectional curvature with respect to the SSRM connection, we have the following:

**Theorem 3.6.** *Let $M$ be a submanifold of a Riemannian $\widetilde{M}$ with the SSRM connection $\acute{\tilde{\nabla}}$.*

*(i) If the associated vector fields $U_1$, $U_2$ are tangent to $M$, then for orthonormal tangent vector fields $X, Y$ on $M$, we have*

$$\acute{\tilde{K}}(X,Y) = \acute{K}(X,Y) + g(h(X,Y),h(X,Y)) - g(h(X,X),h(Y,Y)).$$

*(ii) Moreover, if $\alpha$ is a geodesic curve of $\widetilde{M}$ which lies in $M$ and $X$ is the unit tangent vector field of $\alpha$ in $M$, then we have*

$$\acute{\tilde{K}}(X,Y) \geq \acute{K}(X,Y).$$

*(iii) Furthermore, if $Y$ is parallel along $\alpha$ in $\widetilde{M}$, then we have*

$$\acute{\tilde{K}}(X,Y) = \acute{K}(X,Y).$$

*Proof.* Suppose that $X = W$, $Y = Z$ are orthonormal tangent vector fields on $M$. Then we have from (3.21)

$$\acute{\tilde{K}}(X,Y) = \acute{K}(X,Y) - g(h(X,X),h(Y,Y)) + u_1(h(X,X))$$
$$-u_2(h(X,X)) + g(h(X,Y),h(X,Y)) + u_1(h(Y,Y)) - u_1(U_1^{\perp})$$
$$+u_1(U_2^{\perp}) - u_2(h(Y,Y)) + u_2(U_1^{\perp}) - u_2(U_2^{\perp}).$$

Since $U_1$, $U_2$ are tangent to $M$, the last relation reduces to

$$\acute{\tilde{K}}(X,Y) = \acute{K}(X,Y) + g(h(X,Y),h(X,Y)) - g(h(X,X),h(Y,Y)). \quad (3.13)$$

Let $\alpha$ be a geodesic in $\widetilde{M}$ which lies in $M$ and $X$ be a unit tangent vector field of $\alpha$ in $M$. Then we have from (2.1)

$$h(X,X) = 0.$$

Therefore, from (3.22) it follows that

$$\acute{\tilde{K}}(X,Y)=\acute{K}(X,Y)+g(h(X,Y),h(X,Y)). \tag{3.14}$$

If we assume $\widetilde{\nabla}_X Y=0$, then we have from (2.1)

$$\nabla_X Y=h(X,Y)=0,$$

which yields from (3.23)

$$\acute{\tilde{K}}(X,Y)=\acute{K}(X,Y).$$

This completes the proof of theorem 3.6. □

In view of (3.13), (3.18) and (3.20), the normal component of $\acute{\tilde{R}}(X,Y)Z$ is obtained as follows:

$$\begin{aligned}
(\acute{\tilde{R}}(X,Y)Z)^{\perp} &= \acute{h}(X,\acute{\nabla}_Y Z)-\acute{h}(Y,\acute{\nabla}_X Z)\\
&-\acute{h}([X,Y],Z)+\nabla^{\perp}_X\acute{h}(Y,Z)-\nabla^{\perp}_Y\acute{h}(X,Z)\\
&-u_2(X)\acute{h}(Y,Z)+u_2(Y)\acute{h}(X,Z)\\
&=(\acute{\hat{\nabla}}_X\acute{h})(Y,Z)-(\acute{\hat{\nabla}}_Y\acute{h})(X,Z)-u_1(X)\acute{h}(Y,Z)\\
&+u_1(Y)\acute{h}(X,Z)-u_2(X)\acute{h}(Y,Z)+u_2(Y)\acute{h}(X,Z),
\end{aligned}$$

which may be called the Codazzi-Mainardi equation with respect to the SSRM connection. Here the connection $\acute{\hat{\nabla}}$ is the van der Waerden-Bortolotti connection with respect to SSRM connection defined by

$$(\acute{\hat{\nabla}}_X\acute{h})(Y,Z)=\nabla_X{}^{\perp}(\acute{h}(Y,Z))-\acute{h}(\acute{\nabla}_X Y,Z)-\acute{h}(Y,\acute{\nabla}_X Z).$$

For unit normal vector fields $\eta$, $\mu$ of $M$ in $\widetilde{M}$, we have from (3.10), (3.16) and (3.19)

$$\begin{aligned}
\acute{\tilde{\nabla}}_X\acute{\tilde{\nabla}}_Y\eta &= -\acute{\nabla}_X(\acute{A}_\eta Y)-\acute{h}(X,\acute{A}_\eta Y)-\acute{A}_{\nabla^{\perp}_Y\eta}X+\nabla^{\perp}_X\nabla^{\perp}_Y\eta-u_2(X)\nabla^{\perp}_Y\eta\\
&-g(\widetilde{\nabla}_X Y,U_2^{\top})\eta-g(Y,\widetilde{\nabla}_X U_2^{\top})\eta-u_2(Y)\widetilde{\nabla}_X\eta-u_2(Y)u_1(\eta)X\\
&+u_2(X)u_2(Y)\eta+u_2(Y)u_2(\eta)X, \qquad (3.15)\\
\acute{\tilde{\nabla}}_Y\acute{\tilde{\nabla}}_X\eta &= -\acute{\nabla}_Y(\acute{A}_\eta X)-\acute{h}(Y,\acute{A}_\eta X)-\acute{A}_{\nabla^{\perp}_X\eta}Y+\nabla^{\perp}_Y\nabla^{\perp}_X\eta-u_2(Y)\nabla^{\perp}_X\eta
\end{aligned}$$

$$-g(\widetilde{\nabla}_Y X, U_2^\top)\eta - g(X, \widetilde{\nabla}_Y U_2^\top)\eta - u_2(X)\widetilde{\nabla}_Y \eta - u_2(X)u_1(\eta)Y$$
$$+u_2(Y)u_2(X)\eta + u_2(X)u_2(\eta)Y, \tag{3.16}$$

and

$$\acute{\tilde{\nabla}}_{[X,Y]}\eta = -\acute{A}_\eta[X,Y] + \nabla^\perp_{[X,Y]}\eta - u_2([X,Y])\eta. \tag{3.17}$$

From (3.24), (3.25) and (3.26), it follows that

$$\acute{\tilde{R}}(X,Y,\eta,\mu) = R^\perp(X,Y,\eta,\mu) - g(\acute{h}(X,\acute{A}_\eta Y),\mu) + g(\acute{h}(Y,\acute{A}_\eta X),\mu)$$
$$-u_2(X)g(\nabla^\perp_Y\eta,\mu) + u_2(Y)g(\nabla^\perp_X\eta,\mu) - g(Y,\widetilde{\nabla}_X U_2^\top)g(\eta,\mu)$$
$$+g(X,\widetilde{\nabla}_Y U_2^\top)g(\eta,\mu) - u_2(Y)g(\widetilde{\nabla}_X\eta,\mu) + u_2(X)g(\widetilde{\nabla}_Y\eta,\mu),$$

which yields from (2.1), (2.2), (2.3), (3.12) and (3.18)

$$\acute{\tilde{R}}(X,Y,\eta,\mu) = R^\perp(X,Y,\eta,\mu) - g(h(X,A_\eta Y),\mu) + g(h(Y,A_\eta X),\mu)$$
$$+g(X,\nabla_Y U_2^\top)g(\eta,\mu) - g(Y,\nabla_X U_2^\top)g(\eta,\mu)$$
$$= R^\perp(X,Y,\eta,\mu) + g((A_\mu A_\eta - A_\eta A_\mu)X,Y)$$
$$+g(X,\nabla_Y U_2^\top)g(\eta,\mu) - g(Y,\nabla_X U_2^\top)g(\eta,\mu)$$
$$= R^\perp(X,Y,\eta,\mu) + g([A_\mu,A_\eta]X,Y)$$
$$+g(X,\nabla_Y U_2^\top)g(\eta,\mu) - g(Y,\nabla_X U_2^\top)g(\eta,\mu),$$

which may be called the Ricci equation with respect to the SSRM connection.

**Example.** Let $T^n$ be a torus embedded in $R^{2n}$ as follows:

$$T^n = \{(cost_1, sint_1, cost_2, sint_2, \ldots, cost_n, sint_n) | t_i \in R\}.$$

Then we have for each point $p = (cost_1, sint_1, cost_2, sint_2, \ldots, cost_n, sint_n) \in T^n$

$$T_p(T^n) = span\{e_1, e_2, \ldots, e_n\}$$

and

$$T_p(T^n)^\perp = span\{e_{n+1}, e_{n+2}, \ldots, e_{2n}\},$$

where

$$e_i = (0, \ldots, 0, -sint_i, cost_i, 0, \ldots, 0)$$

and

$$e_{i+n} = (0, \ldots, 0, cost_i, sint_i, 0, \ldots, 0)$$

for $i = 1, \ldots, n$.

Taking a differentiation of the above ones with respect to the Riemannian connection $\widetilde{\nabla}$, we have for $i = 1, \ldots, n$

$$\widetilde{\nabla}_{e_i} e_i = -e_{i+n}$$

and

$$\widetilde{\nabla}_{e_i} e_{i+n} = e_i$$

and the other cases are zero.

From the formula of Gauss and the above relations, it follows that

$$\widetilde{\nabla}_{e_i} e_i = -e_{i+n} = \nabla_{e_i} e_i + h(e_i, e_i),$$

which yields

$$\nabla_{e_i} e_i = 0$$

and

$$h(e_i, e_i) = -e_{i+n} \tag{3.18}$$

for $i = 1, \ldots, n$.

Similarly, we have

$$\widetilde{\nabla}_{e_i} e_j = 0 = \nabla_{e_i} e_j + h(e_i, e_j),$$

which implies

$$\nabla_{e_i} e_j = 0$$

and

$$h(e_i, e_j) = 0 \tag{3.19}$$

for $i, j = 1, \ldots, n$ and $i \neq j$.

Now we define the vector fields $U_1$ and $U_2$ in $R^{2n}$ as follows:

For each point $p = (x_1, x_2, \ldots, x_{2n-1}, x_{2n}) \in R^{2n}$,

$$U_1 = (x_1, x_2, \ldots, x_{2n-1}, x_{2n})$$

and

$$U_2 = (-x_2, x_1, \ldots, -x_{2n}, x_{2n-1}).$$

It is easy to see that on $T^n$, the tangential and normal components of $U_1$ and $U_2$ are

$$U_1^\top = 0, U_1^\perp = U_1$$

and

$$U_2^{\top} = U_2, U_2^{\perp} = 0 \tag{3.20}$$

, respectively.

Therefore, we have from (3.11), (3.27), (3.28) and (3.29)

$$\acute{\nabla}_{e_i} e_i = u_1(e_i)e_i - 2u_2(e_i)e_i + U_2$$

and

$$\acute{\nabla}_{e_i} e_j = u_1(e_j)e_i - u_2(e_i)e_j - u_2(e_j)e_i$$

for $i, j = 1, \ldots, n$ and $i \neq j$.

Furthermore, taking account of (3.13), (3.14) and the last identities, we get for $i, j = 1, \ldots, n$ and $i \neq j$

$$\acute{T}(e_i, e_j) = u_1(e_j)e_i - u_1(e_i)e_j$$

and

$$(\acute{\nabla}_{e_i} g)(e_i, e_i) = 2u_2(e_i)$$

and

$$(\acute{\nabla}_{e_j} g)(e_i, e_i) = 2u_2(e_j)$$

and the other cases are zero.

Also, from (3.12), (3.27), (3.28) and (3.29) we have for $i = 1, \ldots, n$

$$\acute{h}(e_i, e_i) = -e_{i+n} - U_1$$

and the other cases are zero.

Therefore, from (3.15), (3.27), (3.29) and the last identities, it follows that

$$\acute{H} = -\frac{1}{n}\sum_{i=1}^{n} e_{i+n} - U_1.$$


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