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POISSON APPROXIMATION FOR INDEPENDENT NEGATIVE BINOMIAL RANDOM VARIABLES

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Abstract: We give a bound for the total variation distance between the distribution of a sum of independent negative binomial random variables and an appropriate Poisson distribution with mean $\sum_{i=1}^{n} \frac{r_i q_i}{p_i}$, where r_i and $p_i = 1 - q_i$ are parameters of each negative binomial distribution. It is indicated that the distribution of the sum can be approximated by the Poisson distribution with this mean when each $r_i q_i$ is small.

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1. Introduction

Let $X_1, ..., X_n$ be n independently distributed negative binomial random variables, each with probability $P(X_i = k) = \frac{\Gamma(r_i+k)}{\Gamma(r_i)k!} q_i^k p_i^{r_i}$, $k \in \mathbb{N} \cup \{0\}$, mean $\mu_i = \frac{r_i q_i}{p_i}$ and variance $\sigma_i^2 = \frac{r_i q_i}{p_i^2}$ where $q_i = 1 - p_i$. Let $\mathbf{S}_n = \sum_{i=1}^{n} X_i$ and $\mathbf{P}_\lambda$ denote the the Poisson random variable with mean λ. Note that, if all $r_i q_i$ are small, then the distribution of $\mathbf{S}_n$ is approximately a Poisson distribution with mean λ. For $\lambda = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} r_i q_i$,Vellaisamy and Upadhye [3] gave a

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bound in the form of

$$d_{TV}(\mathbf{S}_n, \mathbf{P}_\lambda) \leq \sum_{i=1}^{n} \frac{r_i q_i^2}{p_i} \min\left\{1, \frac{1}{\sqrt{2\lambda e}}\right\}, \tag{1.1}$$

where $d_{TV}(\mathbf{S}_n, \mathbf{P}_\lambda) = \sup_{A \subseteq \mathbb{N}\cup\{0\}} |P(\mathbf{S}_n \in A) - P(\mathbf{P}_\lambda \in A)|$ is the total variation distance between the distribution of $\mathbf{S}_n$ and the Poisson distribution. In this paper, we give a bound on $d_{TV}(\mathbf{S}_n, \mathbf{P}_\lambda)$ for a different Poisson mean $\lambda = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \frac{r_i q_i}{p_i}$, which is derived in Section 2. In Section 3, the conclusion of this study is presented.

2. Result

The following lemma is also need to prove the main result.

Lemma 2.1. *For $1 \leq i \leq n$, let w_i be the w-function associated with the negative binomial random variable X_i, then we have the following:*

$$w_i(k) = p_i\left(1 + \frac{k}{r_i}\right), \; k \in \mathbb{N}\cup\{0\} \quad [1]. \tag{2.1}$$

Theorem 2.1. *Let $\lambda = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \mu_i$ and $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$. Then the following inequality holds:*

$$d_{TV}(\mathbf{S}_n, \mathbf{P}_\lambda) \leq \frac{1 - e^{-\lambda}}{\lambda} \sum_{i=1}^{n} \frac{r_i q_i^2}{p_i^2}. \tag{2.2}$$

Proof. Since $\lambda_i - \sigma_i^2 w_i(k) = \frac{r_i q_i}{p_i} - \frac{r_i q_i}{p_i^2} p_i \left(1 + \frac{k}{r_i}\right) = -\frac{k q_i}{p_i} \leq 0$ for every $k \geq 0$, it follows from [2] that

$$\begin{aligned} d_{TV}(\mathbf{S}_n, \mathbf{P}_\lambda) &\leq \frac{1 - e^{-\lambda}}{\lambda} \left|\lambda - \sigma^2\right| \\ &= \frac{1 - e^{-\lambda}}{\lambda} (\sigma^2 - \lambda) \\ &= \frac{1 - e^{-\lambda}}{\lambda} \sum_{i=1}^{n} \left(\frac{r_i q_i}{p_i^2} - \frac{r_i q_i}{p_i}\right) \\ &= \frac{1 - e^{-\lambda}}{\lambda} \sum_{i=1}^{n} \frac{r_i q_i^2}{p_i^2}. \end{aligned}$$

Hence (2.2) holds. □

Corollary 2.1. *For* $r_1 = r_2 = \cdots = r_n = 1$*, then* $\lambda = \sum_{i=1}^{n} \frac{q_i}{p_1}$ *and*

$$d_{TV}(\mathbf{S}_n, \mathbf{P}_\lambda) \leq \frac{1-e^{-\lambda}}{\lambda} \sum_{i=1}^{n} \frac{q_i^2}{p_i^2}. \tag{2.3}$$

The result (2.3) is a Poisson approximation for a sum of independent geometric random variables, which is the same result as in [2].

When all X_i are identically distributed random variables, thus immediately from the Theorem 2.1, we have the following Corollary.

Corollary 2.2. *If* $r_1 = r_2 = \cdots = r_n = r$ *and* $p_1 = p_2 = \cdots = p_n = p$*, then* $\lambda = \frac{nrq}{p}$ *and the following inequality holds:*

$$d_{TV}(\mathbf{S}_n, \mathbf{P}_\lambda) \leq (1-e^{-\lambda})\frac{q}{p}. \tag{2.4}$$

3. Conclusion

In this study, a bound on the total variation distance between the distribution of a sum of independent negative binomial random variables and an appropriate Poisson distribution with a different mean λ was obtained. With this bound, it is seen that the distribution of the summands can be approximated by the Poisson distribution with mean $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \frac{r_i q_i}{p_i}$ when $r_i q_i$ is small for every $i \in \{1, ..., n\}$.

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