International Journal of Pure and Applied Mathematics
**Volume 93 No. 6 2014, 783-787**
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
**url:** http://www.ijpam.eu
**doi:** http://dx.doi.org/10.12732/ijpam.v93i6.3
ijpam.eu

# POISSON APPROXIMATION FOR RANDOM SUMS OF INDEPENDENT NEGATIVE BINOMIAL RANDOM VARIABLES

K. Teerapabolarn
Department of Mathematics
Faculty of Science
Burapha University
Chonburi, 20131, THAILAND

**Abstract:** In this paper, we focus on determining a bound for the total variation distance between the distribution of random sums of independent negative binomial random variables and an appropriate Poisson distribution. Two examples have been given to illustrate the result obtained.

**AMS Subject Classification:** 62E17, 60F05, 60G05
**Key Words:** negative binomial random variable, Poisson approximation, random sums, Stein's method

## 1. Introduction

Let $X_1, X_2, ...$ be a sequence of independent negative binomial random variables, each with probability $P(X_i = k) = \frac{\Gamma(r_i+k)}{\Gamma(r_i)k!} q_i^k p_i^{r_i}$, $k \in \mathbb{N} \cup \{0\}$, where $q_i = 1 - p_i$. Let $N$ be a non-negative integer-valued random variable and independent of the $X_i$'s and $\mathcal{P}_\lambda$ a Poisson random variable with mean $\lambda$. Let $S_N = \sum_{i=1}^{N} X_i$ be the random sums of independent negative binomial random variables. For

Received: December 25, 2013

$N = n \in \mathbb{N}$ is fixed, Vellaisamy and Upadhye [3] gave a bound for approximating the distribution of $S_n$ by a Poisson distribution with mean $\lambda_n = \sum_{i=1}^{n} r_i q_i$ in the form of

$$d_{TV}(S_n, \mathcal{P}_{\lambda_n}) \leq \sum_{i=1}^{n} \frac{r_i q_i^2}{p_i} \min\left\{1, \frac{1}{\sqrt{2\lambda e}}\right\}, \tag{1.1}$$

where $d_{TV}(S_n, \mathcal{P}_{\lambda_n}) = \sup_{A \subseteq \mathbb{N}\cup\{0\}} |P(S_n \in A) - P(\mathcal{P}_{\lambda_n} \in A)|$ is the total variation distance between the distribution of $S_n$ and the Poisson distribution. Let $\lambda_N = \sum_{i=1}^{N} r_i q_i$ and $\lambda = E(\lambda_N)$. In this study, we are interested to determine a bound for $d_{TV}(S_N, \mathcal{P}_\lambda)$, which is in Section 2. In Section 3, we give two examples to illustrate the desired result, and conclusion of this study is presented in the last section.

## 2. Main result

The following theorem presents a bound for $d_{TV}(S_N, \mathcal{P}_\lambda)$, which is the desired result.

**Theorem 2.1.** *We have*

$$d_{TV}(S_N, \mathcal{P}_\lambda) \leq \min\left\{E\left(\sum_{i=1}^{N} \frac{r_i q_i^2}{p_i}\right), E\left(\frac{\sum_{i=1}^{N} \frac{r_i q_i^2}{p_i}}{\sqrt{2\lambda_N e}}\right)\right\} + \min\left\{1, \sqrt{\frac{2}{\lambda e}}\right\} E|\lambda_N - \lambda|. \tag{2.1}$$

*Proof.* It follows the fact that

$$\begin{aligned}
d_{TV}(S_N, \mathcal{P}_\lambda) &\leq \sum_{n=0}^{\infty} P(N = n) d_{TV}(S_n, \mathcal{P}_\lambda) \\
&\leq \sum_{n=0}^{\infty} P(N = n) d_{TV}(S_n, \mathcal{P}_{\lambda_n}) + d_{TV}(\mathcal{P}_{\lambda_N}, \mathcal{P}_\lambda) \\
&\leq \sum_{n=0}^{\infty} P(N = n) \sum_{i=1}^{n} \frac{r_i q_i^2}{p_i} \min\left\{1, \frac{1}{\sqrt{2\lambda_n e}}\right\} \\
&\quad + d_{TV}(\mathcal{P}_{\lambda_N}, \mathcal{P}_\lambda) \qquad \text{(by (1.1))} \\
&\leq \min\left\{E\left(\sum_{i=1}^{N} \frac{r_i q_i^2}{p_i}\right), E\left(\frac{\sum_{i=1}^{N} \frac{r_i q_i^2}{p_i}}{\sqrt{2\lambda_N e}}\right)\right\}
\end{aligned}$$

$$+\min\left\{1,\sqrt{\frac{2}{\lambda e}}\right\}E|\lambda_N-\lambda| \qquad \text{(by [1])}.$$

Hence, the inequality (2.1) is obtained. □

**Corollary 2.1.** *For $r_1=r_2=\cdots=r_n=1$, then $\lambda_N=\sum_{i=1}^N q_i$ and*

$$d_{TV}(S_N,\mathcal{P}_\lambda)\le\min\left\{E\left(\sum_{i=1}^N\frac{q_i^2}{p_i}\right),E\left(\frac{\sum_{i=1}^N\frac{q_i^2}{p_i}}{\sqrt{2\lambda_N e}}\right)\right\}$$
$$+\min\left\{1,\sqrt{\frac{2}{\lambda e}}\right\}E|\lambda_N-\lambda|. \tag{2.2}$$

The result (2.2) is a Poisson approximation for the random sums of independent geometric random variables, which is the same result as in [2].

If $X_i$'s are identically distributed, then the following corollary is an immediately consequence of the Theorem 2.1.

**Corollary 2.2.** *If $p_1=p_2=\cdots=p$ and $r_1=r_2=\cdots=r$, then we have the following:*

$$d_{TV}(S_N,\mathcal{P}_\lambda)\le\min\left\{\frac{rq^2}{p}E(N),\frac{q}{p}\sqrt{\frac{rq}{2}}E\left(\sqrt{N}\right)\right\}$$
$$+\min\left\{1,\sqrt{\frac{2}{rqeE(N)}}\right\}rqE|N-E(N)|, \tag{2.3}$$

*and for $r=1$,*

$$d_{TV}(S_N,\mathcal{P}_\lambda)\le\min\left\{\frac{q^2}{p}E(N),\frac{q}{p}\sqrt{\frac{q}{2}}E\left(\sqrt{N}\right)\right\}$$
$$+\min\left\{1,\sqrt{\frac{2}{qeE(N)}}\right\}qE|N-E(N)|. \tag{2.4}$$

## 3. Examples

This section gives two examples to illustrate the result in the case of $X_i$'s to be identically distributed.

**Example 3.1.** For $n$ ($n \in \mathbb{N}$) is fixed, let $N$ be a random variable with probability function

$$P(N=k)=\begin{cases}\frac{1}{2} & , k=n,\\ \frac{1}{2} & , k=2n,\\ 0 & , \text{otherwise.}\end{cases}$$

Therefore, $E(N)=\frac{3n}{2}$, $E\left(\sqrt{N}\right)=1.366\sqrt{n}$ and $E|N-E(N)|=\frac{n}{2}$. Let $p_1=p_2=\cdots=p$ and $r_1=r_2=\cdots=r$, then $\lambda=\frac{3nrq}{2}$ and we have

$$d_{TV}(S_N,\mathcal{P}_\lambda)\le\min\left\{\frac{1.5nrq^2}{p},\frac{0.9659q\sqrt{rqn}}{p}\right\}+\min\left\{1,\sqrt{\frac{4}{3nrqe}}\right\}\frac{nrq}{2}.$$

**Example 3.2.** Let $N$ be a random variable with probability function

$$P(N=n)=\frac{1}{2^n},\ n=1,2,...,$$

then we have $E(N)=2$ and $E|N-E(N)|=1$. If $p_1=p_2=\cdots=p$ and $r_1=r_2=\cdots=r$, then $\lambda=2rq$ and we can obtain

$$d_{TV}(S_N,\mathcal{P}_\lambda)\le\frac{4rq^2}{p}+\min\left\{1,\sqrt{\frac{1}{rqe}}\right\}rq.$$

## 4. Conclusion

In this study, a bound for the total variation distance between the distribution of the random sums of independent native binomial random variables and an appropriate Poisson distribution with mean $E\left(\sum_{i=1}^{N}r_iq_i\right)$ was obtained. With this bound, it can be seen that the result gives a good Poisson approximation when all $r_iq_i$ are sufficiently small.

## References

[1] A.D. Barbour, L. Holst, S. Janson, *Poisson approximation*, Oxford Studies in Probability 2, Clarendon Press, Oxford, 1992.

[2] K. Teerapabolarn, Poisson approximation for random sums of geometric random variables, *Int. J. Pure Appl. Math.*, **89** (2013), 35–39.

[3] P. Vellaisamy and N. S. Upadhye, Compound negative binomial approximations for sums of random variables, *Probab. Math. Statist.*, **29** (2009), 205–226.