ON THE TYPICAL RANK OF REAL CLOSED FIELDS

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Abstract: In this note we introduce the notion of typical rank for any real closed field $R$, mimicking the case $R = \mathbb{R}$. We show that the typical ranks are the same if we take a larger real closed field and in particular that for every format $(n_1, \ldots, n_s)$ the typical ranks of tensors of format $(n_1, \ldots, n_s)$ are the same over $R$ and over $\mathbb{R}$.

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Let $R$ be a real closed field. In this note we introduce the notion of typical rank for $R$, mimicking the case $R = \mathbb{R}$. We show that the typical ranks are the same if we take a larger real closed field and in particular that for every format $(n_1, \ldots, n_s)$ the typical ranks of tensors of format $(n_1, \ldots, n_s)$ are the same over $R$ and over $\mathbb{R}$.

For any field $K$ let $\overline{K}$ denote its algebraic closure. For any field $K$, any non-degenerate and geometrically integral variety $X \subset \mathbb{P}^r$ defined over $K$ and any $P \in \mathbb{P}^r(K)$ let $r_{X,K}(P)$ (the $K$-X-rank of $P$) be the minimal cardinality of a set $S \subseteq X(K)$ such that $P \in \langle S \rangle_K$ (where $\langle \cdot \rangle_K$ denote the linear span over $K$), with the convention $r_{X,K}(P) = +\infty$ if there is no such set $S$. Notice that if $L \supset K$ is any field, then $r_{X,K}(P) \geq r_{X,L}(P)$, and that $n\langle S \rangle_L \cap \mathbb{P}^r(K) \cap n\langle S \rangle_K$
for any set $S \subseteq X(K)$. For each integer $b \geq 0$ let $\sigma_b(X)$ denote the secant variety of order $b$ of $X$, i.e. the geometrically integral variety whose $K$-points are the closure in $\mathbb{P}^r(K)$ of the union of all sets $(S)_{K}$ with $S \subseteq X(K)$ and $\sharp(S) = b$ (it is defined over $K$). The \textit{generic $X$-rank} is the first integer $b$ such that $\sigma_b(X) = \mathbb{P}^r$. This integer is the only integer such that

Let $U_b(X)(R) \subseteq X(R)^b$ be the set of all $b$-ples of distinct points of $X(R)$. Set $V_b(X)(R) = \{a \in U_b(X)(R) : \dim((a)_R) = b - 1\}$. For any integer $b$ set $S(X, R, b) := \{P \in \mathbb{P}^r(R) : r_{X,R}(P) = r, S(X, R, \leq b) := \bigcup_{i=1}^b S(X, R, i) \}$ and $S(X, R, \geq b) = \bigcup_{i \geq b} S(X, R, i)$. There is a regular map $\rho_{b,R} : V_b(X)(R) \times \mathbb{P}^{b-1}(R) \rightarrow \mathbb{P}^r(R)$. The image $\text{Im}(\rho_{b,R})$ is a semialgebraic subset of $\mathbb{P}^r(R)$ ([2], Proposition 2.2.7). Since $S(X, r, \leq b)$ is the image of $\bigcup_{i=1}^b \text{Im}(\rho_i)$, we get that each set $S(X, r, \leq b)$ is semialgebraic. Since $S(X, R, 1) = X(R)$, by induction on $b$ we get that each set $S(X, R, b)$ is semialgebraic. Hence we get that in the decomposition of $\mathbb{P}^r(R)$ into pairwise disjoint sets $S(X, R, b)$, $b \geq 1$, only finitely many sets $S(X, R, b)$ have dimension $r$; we say that these integers $b$ are the \textit{typical $X$-ranks} over $R$. We prove the following result.

**Theorem 1.** Let $X \subset \mathbb{P}^r$ be a non-degenerate geometrically irreducible variety defined over $R$. Let $L \supset R$ be another real closed field. Assume $X_{\text{reg}}(R) \neq \emptyset$. Then the generic $X$-rank is the minimum typical $X$-ranks and the typical $X$-ranks over $R$ and over $L$ are the same.

With our definition (not the unique possible definition) it is easy to check that any typical $X$-rank is at least the generic $X$-rank and that the generic $X$-rank is a typical rank if and only if $X_{\text{reg}}(R) \neq \emptyset$ (see also [1], Theorem 2). The integer $b$ is a typical $X$-rank if and only if $S(X, b, R)$ is Zariski dense in $\mathbb{P}^r(R)$.

Since any finite products $\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n_s}$ of projective space $s$ is defined over $\mathbb{Q}$ and its Segre embedding $u_{n_1,\ldots,n_s} : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s} \rightarrow \mathbb{P}^N$ is defined over $\mathbb{Q}$, we get the following corollary.

**Corollary 1.** For all integers $s \geq 2$, $n_i > 0$, $1 \leq i \leq s$, the typical ranks of tensors of format $(n_1, \ldots, n_s)$ are the same for all real closed fields.

See [4] and [5] for recent results on the typical ranks of three-ways tensors. In the following observation we recall a case in which the generic rank is also the maximal rank and so the generic rank is the only typical rank.

**Remark 1.** Fix integers $s \geq 2$ and $n_i > 0$, $1 \leq i \leq s$. Assume $n_s \geq \prod_{i=1}^{n_1}(n_i + 1) - 1$. Then every tensor of format $(n_1, \ldots, n_s)$ has rank at most $\prod_{i=1}^{n_1}(n_i + 1)$ and $\prod_{i=1}^{n_1}(n_i + 1)$ is the generic rank ([3], formula (††) at p. 267 and part (1) of Theorem 3.1). Hence $\prod_{i=1}^{n_1}(n_i + 1)$ is the only typical rank for
formats of type \( (n_1, \ldots, n_s) \).

There are many other examples for varieties \( X \subset \mathbb{P}^r \) (e.g. most space curves and almost all curves in \( \mathbb{P}^4 \)).

Proof of Theorem 1. The condition “\( X_{\text{reg}}(R) \neq \emptyset \)” is equivalent to require that \( X(R) \) is Zariski dense in \( X(\overline{R}) \) and it implies that \( V_b(X)(R) \) is Zariski dense in \( V_b(X)(\overline{R}) \). Since \( X(R) \) spans \( \mathbb{P}^r(\overline{R}) \), we have \( S(X, R, b) = \emptyset \) for all \( b > r \). We get that for each \( b > 0 \) the set \( \sigma_b(X)(R) \) is Zariski dense in \( \sigma_b(\overline{R}) \). Since \( \sigma_b(\overline{R}) = \mathbb{P}^r(\overline{R}) \) if and only if \( b \) is at least the generic \( X \)-rank, while \( \dim(\sigma_b(\overline{R})) < r \) if \( b \) is smaller, we get that the generic \( X \)-rank is the minimal typical \( X \)-rank. Using the sets \( S(X, R, b) \) and \( S(X, L, b) \) and that the dimensions of semialgebraic sets don’t change extending the real closed base field (use [2], Proposition 5.2.1), we get that typical \( X \)-ranks are the same over \( R \) and over \( L \).

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References


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