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EQUIVALENCE OF TWO WIDELY RESEARCHED PROBLEMS IN HYPERGRAPH THEORY

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Abstract: The problems of enumerating (i) all the minimal transversals and (ii) all the minimal dominating sets, in a given hypergraph, have received a lot of attention because of their applications in Computer Science. This article explores the possibilities of these two problems being solution-wise equivalent - that is, each solution to one of them being a solution to the other - in the domain of Sperner hypergraphs, culminating in identifying the only class of such hypergraphs in which the equivalence holds.

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1. Introduction

The *cardinality* (or, *size*) [8] of a finite set V is denoted by $\mid V \mid$. The *power set* of V is the set of all subsets (including the empty set ϕ) of V, and is denoted by 2^V. The set of all nonempty subsets of V is denoted by 2^{V*}; that is, $2^{V*} = 2^V - \{\phi\}$.

A *simple hypergraph* [2] is an ordered couple $H = (V, E)$ where: (i) V is a nonempty finite set and (ii) E is a set of nonempty subsets of V such that

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$\bigcup_{X \in E} X = V$. Each member of V is a *vertex* in H; and each member of E is a *hyperedge* (or, an *edge*) in H. A hyperedge X with $| X |= 1$ is a *loop*. H is *loop-free* if $| X |> 1$ for every hyperedge X. Two distinct vertices x and y are *adjacent* if there is a hyperedge that contains both x and y. H is *Sperner* if no hyperedge is a subset of another. Sperner hypergraphs are necessarily simple, though not conversely [3].

A set $T \in 2^{V*}$ is a *transversal* [2] in H if $T \cap X \neq \phi$ for each hyperedge X(or, T meets each of the hyperedges in H). A transversal T is *minimal* if no proper subset of T is a transversal.

A set $D \in 2^{V*}$ is a *dominating set* [1] in H if either $D = V$ or each $y \in V - D$ is adjacent to some $x \in D$. A dominating set D is *minimal* if no proper subset of D is a dominating set.

The problem of identifying the minimal transversals in a given hypergraph H will be denoted by $HYP - TRANS - H$, and the problem of identifying the minimal dominating sets in H will be denoted by $HYP - DOM - H$.

If $Y \in 2^{V*}$, then Y is a solution to $HYP - TRANS - H$ (written $Y \in HYP - TRANS - H$) if Y is a minimal transversal in H. And Y is a solution to $HYP - DOM - H$ (written $Y \in HYP - DOM - H$) if Y is a minimal dominating set in H.

The hypergraphs considered in this article are all assumed loop-free and of the Sperner type. A given hypergraph H will be assumed to be the ordered couple $H = (V, E)$ unless alternate notations are explicit. This research is of theoretical interest, motivations for it coming from: (i) minimal transversals and related problems, dealt with in [4]; (ii) minimal dominating sets and related topics, covered in [6], and (iii) transversals and dominating sets, treated in [5].

The following polynomial-time equivalence is considered in [6]: If P and Q are enumeration problems for hypergraph properties $\alpha(X)$ and $\beta(X)$, respectively (where X is a nonempty subset of vertices in the given hypergraph), then P is *at least as hard* as Q if an output-polynomial time algorithm for Q implies an output-polynomial time algorithm for P; and P is *equivalent* to Q if each of the two enumeration problems is at least as hard as the other.

The equivalence considered in this article (Section 3), while being prompted by the one in [6], focuses on behaviour of solutions instead of their enumeration. So, the problems $HYP - TRANS - H$ and $HYP - DOM - H$ have been viewed not as enumeration problems but, rather, as problems admitting common solutions under specific conditions.

2. Trim Hypergraphs

Let $H = (V, E)$. A hyperedge X is *redundant* in H if $E - \{X\}$ covers X; that is, $X \subseteq \bigcup_{Y \in E - \{X\}} Y$. If H has no redundant hyperedges then H is a *trim* hypergraph. Evidently a trim hypergraph is Sperner, though not conversely. Trim hypergraphs are dealt with in some detail in [7].

Proposition 2.1. *Every transversal is a dominating set, in any hypergraph.*

Proposition 2.2. *In a trim hypergraph H, a set $X \in 2^{V*}$ is a dominating set if and only if X is a transversal.*

Proposition 2.3. *A Sperner hypergraph H is trim if and only if every dominating set is a transversal.*

Proofs of 2.1, 2.2 and 2.3 are given in [3].

Proposition 2.4. *If V is the minimal transversal in $H = (V, E)$, then H is not loop-free.*

Proof. If $\mid V \mid = 1$ then the conclusion is obvious. In the case $\mid V \mid > 1$, let x, $y \in V$ be distinct. Then $V - \{y\}$ is not a transversal, and so $\{y\}$ is a hyperedge in H. □

3. The Equivalence of HYP-TRANS-H and HYP-DOM-H in Trim Hypergraphs

Let $H = (V, E)$ be a given hypergraph. Let $\alpha(X)$ and $\beta(X)$ be two different hypergraph properties. For instance, $\alpha(X)$ could be 'X is a minimal dominating set'and $\beta(X)$ could be 'X is a transversal.' Let $\alpha(H)$ denote the problem of identifying the sets X (in 2^{V*}) such that $\alpha(X)$ is true; and let $\beta(H)$ be the corresponding problem for $\beta(X)$. Then $\alpha(H)$ is *solution-wise included* in $\beta(H)$, written $\alpha(H) \prec \beta(H)$, if $\beta(X)$ is satisfied whenever $\alpha(X)$ is. $\alpha(H)$ and $\beta(H)$ are *solution-wise equivalent*, written $\alpha(H)(\equiv)\beta(H)$, if $\alpha(H) \prec \beta(H)$ as well as $\beta(H) \prec \alpha(H)$.

In 3.1 through 3.3, H is assumed Sperner and loop-free.

3.1: Proposition. *If H is trim, then $HYP - TRANS - H(\equiv)HYP - DOM - H$.*

Proof. Let $Y \in HYP - TRANS - H$. Then Y is a proper subset of V (by 2.4) and Y is a dominating set (by 2.1). Were some proper subset X of Y a dominating set in H, then X would be a transversal (by 2.3), contradicting $Y \in HYP - TRANS - H$. So Y is a minimal dominating set in H, whence $HYP - TRANS - H \prec HYP - DOM - H$.

On the other hand, let $Y \in HYP - DOM - H$. Then Y is a transversal in H (by 2.3). Were some proper subset X of Y a transversal in H, then X would be a dominating set (by 2.1), going against $Y \in HYP - DOM - H$. So Y is a minimal transversal in H, giving $HYP - DOM - H \prec HYP - TRANS - H$ as well. □

Proposition 3.2. *If $HYP - TRANS - H (\equiv) HYP - DOM - H$ then H is trim.*

Proof. Let D be a given dominating set in H. Let D_1 be any minimal dominating set contained in D. Then, by hypothesis, D_1 is a minimal transversal in H and so D is a transversal in H. Then H is trim (by 2.3). □

Propositions 3.1 and 3.2 imply the following proposition.

Proposition 3.3. *$HYP - TRANS - H (\equiv) HYP - DOM - H$ if and only if H is trim.*

4. Resume

The only class of loop-free Sperner hypergraphs in which the HYP-TRANS and HYP-DOM problems are solution-wise equivalent is the class of trim hypergraphs.

Consequently, enumerating the minimal transversals in a trim hypergraph H is equivalent to enumerating the minimal dominating sets in H. A fortiori, an output-polynomial time algorithm for HYP-TRANS-H is one for HYP-DOM-H, and vice-versa. A future direction of research could be investigation of output-polynomial time algorithm to enumerate minimal transversals / minimal dominating sets in trim hypergraphs.

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