International Journal of Pure and Applied Mathematics
Volume 94 No. 3 2014, 379-402
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v94i3.7
ijpam.eu

ON THE RATE OF CONVERGENCE AND DATA DEPENDENCE OF JUNGCK MULTISTEP ITERATIVE SCHEMES

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Abstract: In this paper we prove data dependence of Jungck-Multistep iterative schemes using quasi contractive operators, that is, by using approximate quasi -contractive operators we estimate the point of coincidence of the given operators. Also following Biazar and Amriteimoori [5] we modify Jungck-Multistep iterative schemes and with the help of numerical example compare their rate of convergence. Our results generalize some of the result in literature of fixed point theory.

AMS Subject Classification: 47H10
Key Words: Jungck multistep iteration, data dependence, quasi contractive

1. Introduction

Data dependence of fixed points and rate of convergence of fixed point iterations is one of among the issues involved in fixed point theory and has been studied by many authors like Rus and Muresan [34], Rus et al. [32, 33], Berinde [5], Espinola and Petrusel [15], Markin [23], Chifu and Petrusel [8], Olantiwo [26, 27], Soltuz [35, 37], Soltuz and Grosan [36], Chugh and Kumar [10, 11, 12, 13, 14], Gursoy et al. [16], Hussain et al. [17, 18], Berinde [6], Biazar [7], and several references therein.

Received: January 11, 2014

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Let X be a Banach space, Y an arbitrary set and $S, T : Y \to X$ such that $T(Y) \subseteq S(Y)$. For $x_0 \in Y$, the iterative scheme:

$$Sx_{n+1} = Tx_n, \quad n = 0, 1, \tag{1}$$

was introduced by Jungck [21] in 1976 and it becomes the Picard iterative scheme when $S = I_d$ (identity mapping) and $Y = X$.

Singh et al. [38] defined the Jungck-Mann iterative scheme as

$$Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Tx_n, \tag{2}$$

where $\{\alpha_n\}$ is a sequences of positive numbers in $[0, 1]$.

In [28], Olatinwo defined the Jungck-Ishikawa and Jungck-Noor [24] iterative schemes as

$$\begin{cases} Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Ty_n \\ Sy_n = (1 - \beta_n)Sx_n + \beta_n Tx_n \end{cases} \tag{3}$$

and

$$\begin{cases} S_1x_{n+1} = (1 - \alpha_n)S_1x_n + \alpha_n T_1y_n \\ S_1y_n = (1 - \beta_n)S_1x_n + \beta_n T_1z_n \\ S_1z_n = (1 - \gamma_n)S_1x_n + \gamma_n T_1x_n \end{cases} \tag{4}$$

respectively, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in $[0, 1]$.

Recently, Chugh and Kumar defined the Jungck-SP [11] iterative schemes as

$$\begin{cases} S_1x_{n+1} = (1 - \alpha_n)S_1y_n + \alpha_n T_1y_n \\ S_1y_n = (1 - \beta_n)S_1z_n + \beta_n T_1z_n \\ S_1z_n = (1 - \gamma_n)S_1x_n + \gamma_n T_1x_n \end{cases} \tag{5}$$

Olaleru and Akewe [29] defined the Jungck multistep Noor iteration as

$$\begin{cases} S_1x_{n+1} = (1 - \alpha_n)S_1x_n + \alpha_n T_1y_n^1 \\ S_1y_n^i = (1 - \beta_n^i)S_1x_n + \beta_n^i T_1y_n^{i+1}, \quad i = 1, 2, \ldots, k-2 \\ S_1y_n^{p-1} = (1 - \beta_n^{k-1})S_1x_n + \beta_n^{k-1} T_1x_n \end{cases} \tag{6}$$

Recently Hudson Akewe [2] defined the Jungck multi step SP iteration as

$$\begin{cases} S_2x_{n+1} = (1 - \alpha_n)S_2y_n^1 + \alpha_n T_2y_n^1 \\ S_2y_n^i = (1 - \beta_n^i)S_2y_n^{i+1} + \beta_n^i T_2y_n^{i+1}, \quad i = 1, 2, \ldots, k-2 \\ S_2y_n^{p-1} = (1 - \beta_n^{k-1})S_2x_n + \beta_n^{k-1} T_2x_n \end{cases} \tag{7}$$

where $\{\alpha_n\}$ and $\{\beta_n^i\}$ are sequences of positive numbers in $[0,1]$.

Remark 1.1. If $X = Y$ and $S = I_d$ (identity mapping), then the Jungck multi step SP (1.7), Jungck multi step Noor (1.6), Jungck-SP (1.5), Jungck-Noor (1.4), Jungck-Ishikawa (1.3) and the Jungck-Mann (1.2) iterative schemes, respectively, become the multi step SP, multi step Noor, SP, Noor, Ishikawa and the Mann iterative schemes.

Bizare and Amriteimoori [7] improved the picard iteration under following conditions:

(i) Initial approximation is chosen in the interval $[a, b]$, where function is defined.

(ii) Function has continuous derivative on (a, b).

(iii) $|T'(x)| < 1$ for all $x \in [a, b]$

(iv) $a \le T(x) \le b$ for all $x \in [a, b]$.

Definition 1.1 ([7]). Let $\{x_n\}$ converges to α. If there exists an integer constant q and a real +ve constant C such that

$$\lim_{n\to\infty} \lim \left| \frac{x_{n+1} - \alpha}{(x_n - \alpha)^q} \right| = C,$$

q is called order and C is called constant of convergence.

Theorem 1.2 ([7], [3]). *Let $f \in C^q[a, b]$, if $f^k(x) = 0$ for $k = 1, 2, \ldots, q-1$ and $f^q(x) \neq 0$ then sequence x_n is of order q.*

To improve the order of convergence of fixed iterative schemes, such that $f'(\alpha), f''(\alpha), \ldots, f^k(\alpha) = 0$. We determine λ_i $(i = 1, 2, \ldots, k)$ from the following equation $x + \lambda_1 x + \lambda_2 x^2 + \ldots + \lambda_k x^k = f(x) + \lambda_1 x + \lambda_2 x^2 + \ldots + \lambda_k x^k$ which becomes $x = \frac{f(x)+\lambda_1 x+\lambda_2 x^2+\ldots+\lambda_k x^k}{1+\lambda_1+\lambda_2 x+\ldots+\lambda_k x^{k-1}} = f_\lambda(x)$ this is fixed point equation form. Now the assumption that $f'_\lambda(\alpha) = f''_\lambda(\alpha) = \ldots = f^{k-1}_\lambda(\alpha) = 0$ yields to a system of linear equations which after solving [7] converted into upper triangular matrix which have nonzero diagonal entries. It means determinant is nonzero. So we determine λ_i $(i = 1, 2, \ldots, k)$ uniquely.

Now the new Picard iteration becomes

$$x_{n+1} = f_\lambda(x_n), \quad n = 1, 2, \ldots \tag{8}$$

where

$$f_\lambda(x) = \frac{f(x) + \lambda_1 x + \lambda_2 x^2 + \ldots + \lambda_k x^k}{1 + \lambda_1 + \lambda_2 x + \ldots + \lambda_k x^{k-1}}$$

Bhagwati Parsad and Ritu Shani [27] modify Ishikawa and Jungck-Ishikawa iteration as:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n f_\lambda(y_n)$$
$$y_n = (1-\beta_n)x_n + \beta_n f_\lambda(x_n)$$

and

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n f_\lambda(y_n)$$
$$Sy_n = (1-\beta_n)Sx_n + \beta_n f_\lambda(x_n) \tag{9}$$

Then new modified Jungck-SP, Jungck-Noor, Jungck-Multistep iterative schemes are New Jungck modified multistep Noor iteration scheme

$$\begin{cases} Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n f_\lambda y_n^1 \\ Sy_n^i = (1-\beta_n^i)Sx_n + \beta_n^i f_\lambda y_n^{i+1}, \quad i = 1,2,\dots,k-2 \\ Sy_n^{p-1} = (1-\beta_n^{k-1})Sx_n + \beta_n^{k-1} f_\lambda x_n \end{cases} \tag{10}$$

New modified Jungck modified multistep SP iteration scheme

$$\begin{cases} Sx_{n+1} = (1-\alpha_n)Sy_n^1 + \alpha_n f_\lambda(y_n^1) \\ Sy_n^i = (1-\beta_n^i)Sy_n^{i+1} + \beta_n^i f_\lambda(y_n^{i+1}), i = 1,2,\dots,k-2 \\ Sy_n^{p-1} = (1-\beta_n^{k-1})Sx_n + \beta_n^{k-1} f_\lambda(x_n) \end{cases} \tag{11}$$

Where $\{\alpha_n\}$ and $\{\beta_n^i\}_{i=1}^k$ are real sequences in $[0,1]$.

The iterative scheme (1.1) was used by Jungck [21] to prove some of the common fixed point's results using the following Jungck-contraction

$$d(Tx,Ty) \le \alpha d(Sx,Sy), \quad 0 \le \alpha < 1\,. \tag{12}$$

Olatinwo [28] used the following more general contractive definitions than (1.12) to prove the stability and strong convergence results for the Jungck-Ishikawa iteration process:

(a) There exists a real number $a \in [0,1)$ and a monotone increasing function $\phi : R^+ \to R^+$ such that $\phi(0) = 0$ and $\forall\ x,y \in Y$, we have

$$\|Tx - Ty\| \le \phi(\|Sx - Tx\|) + a\|Sx - Sy\|\,. \tag{13}$$

(b) There exists real numbers $M \geq 0$, $a \in [0, 1)$ and a monotone increasing function $\phi : R^+ \to R^+$ such that $\phi(0) = 0$ and $\forall\, x, y \in Y$, we have

$$\|Tx - Ty\| \leq \frac{\phi(\|Sx - Tx\|) + a\|Sx - Sy\|}{1 + M\|Sx - Tx\|}\,. \tag{14}$$

Olatinwo [25] used the convergence of Jungck-Noor iterative scheme (1.4) to approximate the coincidence points of some pairs of generalized contractive-operators satisfying

$$\|Tx - Ty\| \leq \{2\delta\|Sx - Tx\| + \delta\|Sx - Sy\|\} \tag{15}$$

$\forall\, x, y \in Y$, where δ is a real number $\in [0, 1)$.

In this paper we prove data dependence results for more generalized Jungck multistep SP and Noor iterative schemes using quasi contractive operators satisfying (1.13) and show the comparison between rate of convergence of Jungck multistep SP, Noor and modified Jungck multistep SP, Noor with the help of a numerical example.

To prove data dependence result we will use following definition and results.

Definition 1.3 ([4]). Let T_1, T_2 be two operators. We say T_2 is approximate operator of T_1 if for all $x \in X$and for a fixed $\epsilon > 0$, we have $\|T_1x - T_2x\| \leq \epsilon$.

Lemma 1.4 ([37]). *Let $\{\alpha_n\}_{n=0}^{\infty}$ be a nonnegative sequence for which there exists $n_0 \in I$, such that for all $n \geq n_0$it satisfies the following inequality:*

$$\alpha_n \leq (1 - \lambda_n)\alpha_n + \lambda_n\sigma_n,$$

where $\lambda_n \in (0, 1)$, $\forall\ n \in N$, $\sum\limits_{n=1}^{\infty} \lambda_n = \infty$ and $\sigma_n \geq 0\ \forall\ n \in N$. Then, $0 \leq \lim\limits_{n\to\infty} \sup \alpha_n \leq \lim\limits_{n\to\infty} \sup \sigma_n$.

Theorem 1.5 ([2]). *Let $(X, \|\ \|)$ be an arbitrary Banach space and $S, T : Y \to X$ are nonself operators on an arbitrary set Y such that $T(Y) \subseteq S(Y)$, where $S(Y)$ is complete subspace of X and S is injective operator. Let z be coincidence point of S and T, i.e., $Sz = Tz = p$ (say). Suppose S and T satisfy condition (1.13). For $y_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be Jungck multistep SP iterative scheme defined by (1.7), where $\{\alpha_n\}$ and $\{\beta_n^i\}_{i=1}^{k}$ are sequences of positive number in $[0, 1]$ satisfying $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$. Then the Jungck multistep SP iterative scheme $\{Sx_n\}$ converges strongly to p.*

Theorem 1.6 ([29]). *Let $(X, \|\ \|)$ be an arbitrary Banach space and $S, T : Y \to X$are nonself operators on arbitrary set Y such that $T(Y) \subseteq S(Y)$, where $S(Y)$ is complete subspace of X and S is injective operator. Let z be coincidence point of S and T, i.e., $Sz = Tz = p$ (say). Suppose S and T satisfy condition (1.13). For $y_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be Jungck multistep Noor iterative scheme defined by (1.6), where $\{\alpha_n\}$ and $\{\beta_n^i\}_{i=1}^{k}$ are sequences of positive number in $[0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the Jungck multistep Noor iterative scheme $\{Sx_n\}$ converges strongly to p.*

2. Main Results

Theorem 2.1. *Let $T_1, S_1 : Y \to E$ be mappings satisfying (1.13). Let T_2, S_2 be approximate operators of T_1, S_1, respectively, as in Definition 1.4. and $\{S_1x_n\}_{n=0}^{\infty}$, $\{S_2u_n\}_{n=0}^{\infty}$ be two Jungck multi step Noor iterative schemes defined by (1.6) associated to T_1, S_1 and T_2, S_2, respectively, where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n^i\}_{n=0}^{\infty}$ are real sequences in $[0,1)$ satisfying* $\begin{cases} \text{(i)} \ \frac{1}{2} \leq \alpha_n(1-\delta), \ \forall\, n \\ \text{(ii)} \ \sum\limits_{n=0}^{\infty} \alpha_n = \infty \end{cases}$.
Let $p = T_1q_1 = S_1p_1$ and $q = S_2q_2 = T_2p_2$, then we have the following estimate:

$$\|p - q\| \leq \frac{2k\epsilon}{1-\delta}\,.$$

Proof. For a given $x_0 \in E$ and $u_0 \in E$ we consider the following iterative schemes for T_1 and T_2

$$\begin{cases} S_1x_{n+1} = (1-\alpha_n)S_1x_n + \alpha_n T_1 y_n^1 \\ S_1y_n^i = (1-\beta_n^i)S_1x_n + \beta_n^i T_1 y_n^{i+1}, \quad i = 1, 2, \ldots, k-2 \\ S_1y_n^{p-1} = (1-\beta_n^{p-1})S_1x_n + \beta_n^{p-1} T_1 x_n, \end{cases} \tag{16}$$

and

$$\begin{cases} S_2u_{n+1} = (1-\alpha_n)S_2u_n + \alpha_n T_2 v_n^1 \\ S_2v_n^i = (1-\beta_n^i)S_2u_n + \beta_n^i T_2 v_n^{i+1}, \quad i = 1, 2, \ldots, k-2 \\ S_2v_n^{p-1} = (1-\beta_n^{p-1})S_2u_n + \beta_n^{p-1} T_2 u_n, \end{cases} \tag{17}$$

then using (1.13), (2.1) and (2.2), yield the following estimates:

$$\|S_1x_{n+1} - S_2u_{n+1}\| = \|(1-\alpha_n)(S_1x_n - S_2u_n) + \alpha_n(T_1y_n^1 - T_2v_n^1)\|$$

$$
\begin{aligned}
&\leq (1-\alpha_n)\|S_1x_n - S_2u_n\| + \alpha_n\|T_1y_n^1 - T_2v_n^1\| \\
&= (1-\alpha_n)\|S_1x_n - S_2u_n\| \\
&\quad + \alpha_n\|T_1y_n^1 - T_1v_n^1 + T_1v_n^1 - T_2v_n^1\| \\
&\leq (1-\alpha_n)\|S_1x_n - S_2u_n\| + \alpha_n\|T_1y_n^1 - T_1v_n^1\| \\
&\quad + \alpha_n\|T_1v_n^1 - T_2v_n^1\| \\
&\leq (1-\alpha_n)\|S_1x_n - S_2u_n\| + \alpha_n\delta\|S_1y_n^1 - S_2v_n^1\| \\
&\quad + \alpha_n\varphi(\|S_1y_n^1 - T_1y_n^1\|) + 2\alpha_n\epsilon \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
\|S_1y_n^1 - S_2v_n^1\| &= \|(1-\beta_n^1)(S_1x_n - S_2u_n) + \beta_n^1(T_1y_n^2 - T_2v_n^2)\| \\
&\leq (1-\beta_n^1)\|S_1x_n - S_2u_n\| + \beta_n^1\|T_1y_n^2 - T_2v_n^2\| \\
&\leq (1-\beta_n^1)\|S_1x_n - S_2u_n\| + \beta_n^1\delta\|T_1y_n^2 - T_2v_n^2\| \\
&\quad + \beta_n^1\varphi(\|S_1y_n^2 - T_1y_n^2\|) + 2\beta_n^1\epsilon \qquad (19)
\end{aligned}
$$

and

$$
\begin{aligned}
\|S_1y_n^2 - S_2v_n^2\| &= \|(1-\beta_n^2)(S_1x_n - S_2u_n) + \beta_n^2(T_1y_n^3 - T_2v_n^3)\| \\
&\leq (1-\beta_n^2)\|S_1x_n - S_2u_n\| + \beta_n^2\|T_1y_n^3 - T_2v_n^3\| \\
&\leq (1-\beta_n^2)\|S_1x_n - S_2u_n\| + \beta_n^2\delta\|S_1y_n^3 - S_2v_n^3\| \\
&\quad + \beta_n^2\varphi(\|S_1y_n^3 - T_1y_n^3\|) + 2\beta_n^2\epsilon \qquad (20)
\end{aligned}
$$

Combining (2.3), (2.4) and (2.5), we have

$$
\begin{aligned}
\|S_1x_{n+1} - S_2u_{n+1}\| &\leq (1-\alpha_n)\|S_1x_n - S_2u_n\| \\
&\quad + \alpha_n\varphi(\|S_1y_n^1 - T_1y_n^1\|) + 2\alpha_n\epsilon \\
&\quad + \alpha_n\delta\{(1-\beta_n^1)\|S_1x_n - S_2u_n\| \\
&\quad + \beta_n^1\delta\|S_1y_n^2 - S_2v_n^2\| + \beta_n^1\varphi(\|S_1y_n^2 - T_1y_n^2\|) + \beta_n^1\epsilon\} \\
&\leq [(1-\alpha_n(1-\delta)) - \delta\alpha_n\beta_n^1]\|S_1x_n - S_2u_n\| \\
&\quad + \delta^2\alpha_n\beta_n^1\|S_1y_n^2 - S_2v_n^2\| + \alpha_n\varphi(\|S_1y_n^1 - T_1y_n^1\|) \\
&\quad + \delta\alpha_n\beta_n^1\varphi(\|S_1y_n^2 - T_1y_n^2\|) + 2\alpha_n\epsilon + 2\delta\alpha_n\beta_n^1\epsilon \\
&\leq [(1-\alpha_n(1-\delta)) - \delta\alpha_n\beta_n^1(1-\delta(1-\beta_n^2))]\|S_1x_n \\
&\quad - S_2u_n\| + \delta^3\alpha_n\beta_n^1\beta_n^2\|S_1y_n^3 - S_2v_n^3\| \\
&\quad + \alpha_n\varphi(\|S_1y_n^1 - T_1y_n^1\|) + \delta\alpha_n\beta_n^1\varphi(\|S_1y_n^2 - T_1y_n^2\|) \\
&\quad + \delta^3\alpha_n\beta_n^1\beta_n^2\varphi(\|S_1y_n^3 - T_1y_n^3\|) + 2\alpha_n\epsilon \\
&\quad + 2\delta\alpha_n\beta_n^1\epsilon + 2\delta\alpha_n\beta_n^1\beta_n^2 \qquad (21)
\end{aligned}
$$

Thus inductively, we get

$$\begin{aligned}\|S_1x_{n+1}-S_2u_{n+1}\| \leq{} & [(1-\alpha_n(1-\delta))-\delta\alpha_n\beta_n^1(1-\delta(1-\beta_n^2)\\ & -\delta^2\beta_n^2(1-\beta_n^3)-\ldots-\delta^{p-3}\beta_n^2\beta_n^3\ldots\\ & \beta_n^{k-3}(1-\beta_n^{k-2}))]\|S_1x_n-S_2u_n\|\\ & +\delta^{k-2}\alpha_n\beta_n^1\beta_n^2\ldots\beta_n^{k-2}\|S_1y_n^{k-1}-S_2v_n^{k-1}\|\\ & +\alpha_n\varphi(\|S_1y_n^1-T_1y_n^1\|)+\delta\alpha_n\beta_n^1\varphi(\|S_1y_n^2-T_1y_n^2\|)\\ & +\delta^3\alpha_n\beta_n^1\beta_n^2\varphi(\|S_1y_n^3-T_1y_n^3\|)+\ldots\\ & +\delta^{k-2}\alpha_n\beta_n^1\beta_n^2\ldots\beta_n^{k-2}\varphi(\|S_1y_n^{k-1}-T_1y_n^{k-1}\|)\\ & +2[\alpha_n\epsilon+\delta\alpha_n\beta_n^1\epsilon+\delta\alpha_n\beta_n^1\beta_n^2\epsilon+\ldots\\ & +\delta^{k-2}\alpha_n\beta_n^1\beta_n^2\ldots\beta_n^{k-2}\epsilon]\end{aligned} \tag{22}$$

Using (2.1), (2.2) and (1.13)

$$\begin{aligned}\|S_1y_n^{k-1}-S_2v_n^{k-1}\| &\leq \|(1-\beta_n^{k-1})(S_1x_n-S_2u_n)+\beta_n^{k-1}(T_1x_n-T_2u_n)\|\\ &\leq (1-\beta_n^{k-1})\|S_1x_n-S_2u_n\|+\beta_n^{k-1}\|T_1x_n-T_2u_n\|\\ &\leq (1-\beta_n^{k-1}(1-\delta))\|S_1x_n-S_2u_n\|\\ &\quad +\beta_n^{k-1}\varphi(\|S_1x_n-T_1x_n\|)+2\beta_n^{k-1}\epsilon\end{aligned} \tag{23}$$

Now by combining (2.7) and (2.8)

$$\begin{aligned}\|S_1x_{n+1}-S_2u_{n+1}\| \leq{} & [(1-\alpha_n(1-\delta))-\delta\alpha_n\beta_n^1(1-\delta(1-\beta_n^2)-\ldots\\ & -\delta^{k-3}\beta_n^2\ldots\beta_n^{k-3}(1-\beta_n^{k-1}(1-\delta))]\|S_1x_n-S_2u_n\|\\ & +\alpha_n\varphi(\|S_1y_n^1-T_1y_n^1\|)+\delta\alpha_n\beta_n^1\varphi(\|S_1y_n^2-T_1y_n^2\|)\\ & +\delta^3\alpha_n\beta_n^1\beta_n^2\varphi(\|S_1y_n^3-T_1y_n^3\|)+\ldots\\ & +\delta^{k-2}\alpha_n\beta_n^1\ldots\beta_n^{k-2}\varphi(\|S_1x_n-T_1x_n\|)\\ & +2[\alpha_n\epsilon+\delta\alpha_n\beta_n^1\epsilon+\ldots+\delta^{k-2}\alpha_n\beta_n^1\ldots\beta_n^{k-2}\epsilon],\end{aligned}$$

which further implies,

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| \leq{} & [(1-\alpha_n(1-\delta))-\delta\alpha_n\beta_n^1+L]\|S_1x_n-S_2u_n\|\\ & +\alpha_n\varphi(\|S_1y_n^1-T_1y_n^1\|)+\delta\alpha_n\beta_n^1\varphi(\|S_1y_n^2-T_1y_n^2\|)\\ & +\delta^3\alpha_n\beta_n^1\beta_n^2\varphi(\|S_1y_n^3-T_1y_n^3\|)+\ldots\\ & +\delta^{k-2}\alpha_n\beta_n^1\ldots\beta_n^{k-2}\varphi(\|S_1x_n-T_1x_n\|)\\ & +2[\alpha_n\epsilon+\delta\alpha_n\beta_n^1\epsilon+\ldots+\delta^{p-2}\alpha_n\beta_n^1\ldots\beta_n^{k-2}\epsilon],\end{aligned} \tag{24}$$

where

$$\begin{aligned}
L &= \delta^2\alpha_n\beta_n^1(1-\beta_n^2)\ldots+\delta^{p-3}\alpha_n\beta_n^1\beta_n^2\ldots\beta_n^{p-3}(1-\beta_n^{k-2}(1-\delta))\\
&\le \delta^2\alpha_n\beta_n^1\beta_n^2\ldots+\delta^{k-3}\alpha_n\beta_n^1\beta_n^2\ldots\beta_n^{k-3}\\
&\le \delta^2\alpha_n\beta_n^1\beta_n^2+\delta^3\alpha_n\beta_n^1\beta_n^2\beta_n^3\ldots+\delta^{k-3}\alpha_n\beta_n^1\beta_n^2\ldots\beta_n^{k-3}\\
&\le \delta^2\alpha_n\beta_n^1\beta_n^2+\delta^3\alpha_n\beta_n^1\beta_n^2\ldots+\delta^{k-3}\alpha_n\beta_n^1\beta_n^2\\
&= \delta^2\alpha_n\beta_n^1\beta_n^2[1+\delta+\delta^2+\ldots\delta^{k-5}]\\
&= \delta^2\alpha_n\beta_n^1\beta_n^2\frac{[1-\delta^{k-4}]}{[1-\delta]}<\delta\alpha_n\beta_n^1 \qquad (25)
\end{aligned}$$

Since $\delta\frac{[1-\delta^{k-3}]}{[1-\delta]}<1$, this imply

$$\{(1-\alpha_n(1-\delta))-\delta\alpha_n\beta_n^1+L<(1-\alpha_n(1-\delta))\,. \qquad (26)$$

Hence using (2.10) and (2.11) in (2.9), we get

$$\begin{aligned}
\|S_1x_{n+1}-S_2u_{n+1}\| &\le [1-(1-\alpha_n(1-\delta))]\|S_1x_n-S_2u_n\|\\
&\quad+\alpha_n\varphi(\|S_1y_n^1-T_1y_n^1\|)+\alpha_n\varphi(\|S_1y_n^2-T_1y_n^2\|)\\
&\quad+\alpha_n\varphi(\|S_1y_n^3-T_1y_n^3\|)+\ldots\\
&\quad+\alpha_n\varphi(\|S_1x_n-T_1x_n\|)+2(k-1)\epsilon\\
&= [1-(1-\alpha_n(1-\delta))]\|S_1x_n-S_2u_n\|\\
&\quad+\alpha_n(1-\delta)\left\{\frac{\begin{pmatrix}\varphi(\|S_1y_n^1-T_1y_n^1\|)\\ +\varphi(\|S_1y_n^2-T_1y_n^2\|)+\ldots\\ +\alpha_n\varphi(\|S_1x_n-T_1x_n\|)\\ +2(k-1)\epsilon\end{pmatrix}}{(1-\delta)}\right\}\\
&\le [1-(1-\alpha_n(1-\delta))]\|S_1x_n-S_2u_n\|\\
&\quad+\alpha_n(1-\delta)\left\{\frac{\begin{pmatrix}\varphi(\|S_1y_n^1-T_1y_n^1\|)\\ +\varphi(\|S_1y_n^2-T_1y_n^2\|)+\ldots\\ +\alpha_n\varphi(\|S_1x_n-T_1x_n\|)\\ +2k\epsilon\end{pmatrix}}{(1-\delta)}\right\} \qquad (27)
\end{aligned}$$

Let us denote

$$\begin{aligned}
a_n &= \|S_1x_n-S_2u_n\|,\\
r_n &= \alpha_n(1-\delta),
\end{aligned}$$

and

$$\sigma_n : \frac{\begin{pmatrix} \{\varphi(\|S_1y_n^1 - T_1y_n^1\|) + \varphi(\|S_1y_n^2 - T_1y_n^2\|) + \dots \\ +\varphi(\|S_1x_n - T_1x_n\|) + 2(k-1)\epsilon\} \end{pmatrix}}{(1-\delta)} .$$

Now from Theorem 1.6 we have $\lim_{n\to\infty} \|S_1x_n - p\| = 0$, $\lim_{n\to\infty} \|S_1x_n - T_1x_n\| = \lim_{n\to\infty} \|S_1y_n - T_1y_n\| = \lim_{n\to\infty} \|S_1z_n - T_1z_n\| = 0$.

Because $S_1x_{n}{}_{n=0}^{\infty}$, $S_1y_{n}{}_{n=0}^{\infty}$, $S_1z_{n}{}_{n=0}^{\infty}$ converges to common fixed point

$$\begin{aligned} \lim_{n\to\infty} \varphi(\|S_1x_n - T_1x_n\|) &= \lim_{n\to\infty} \varphi(\|S_1z_n - T_1z_n\|) \\ &= \lim_{n\to\infty} \varphi(\|S_1y_n - T_1y_n\|) = 0\,. \end{aligned}$$

Since φ is a continuous, hence using Lemma 1.5, (2.12) yields

$$\|p - q\| \le \frac{2k\epsilon}{1-\delta}.$$

□

Remark 2.1. Since the iteration (1.2), (1.3) and (1.4) are special cases of iterative scheme (1.6), so. Theorem 1.7 generalizes existing result for (1.2), (1.3) and (1.4). By taking $k = 1, 2$ and 3 and using Remark 1.1 in Theorem 2.1, data dependence results for the iterative schemes (1.2), (1.3) and (1.4) and for Mann, Ishikawa, Noor iterative schemes can be obtained easily.

Theorem 2.2. *Let $T_1, S_1 : Y \to E$ be mappings satisfying (1.13). Let T_2, S_2 be approximate operators of T_1, S_1, respectively, as in Definition 1.4 and $\{S_1x_n\}_{n=0}^{\infty}$, $\{S_2u_n\}_{n=0}^{\infty}$ be two Jungck multi step SP iterative schemes defined by (1.7) associated to T_1, S_1 and T_2, S_2, respectively, where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in $[0,1)$ satisfying* $\begin{cases} \text{(i)} \ \ \frac{1}{2} \le \alpha_n(1-\delta), \ \forall\ n \\ \text{(ii)} \ \ \sum_{n=0}^{\infty} \alpha_n = \infty \end{cases}$.
Let $p = T_1q_1 = S_1p_1$ and $q = S_2q_2 = T_2p_2$, then we have the following estimate:

$$\|p - q\| \le \frac{2k\epsilon}{1-\delta}.$$

Proof. For a given $x_0 \in E$ and $u_0 \in E$ we consider the following iterative schemes for T_1 and T_2

$$\begin{cases} S_1x_{n+1} = (1-\alpha_n)S_1y_n^1 + \alpha_nT_1y_n^1 \\ S_1y_n^i = (1-\beta_n^i)S_1y_n^{i+1} + \beta_n^iT_1y_n^{i+1}, \quad i = 1, 2, \dots, k-2 \\ S_1y_n^{p-1} = (1-\beta_n^{p-1})S_1x_n + \beta_n^{p-1}T_1x_n, \end{cases} \tag{28}$$

and

$$\begin{cases} S_2u_{n+1} = (1-\alpha_n)S_2v_n^1 + \alpha_n T_2v_n^1 \\ S_2v_n^i = (1-\beta_n^i)S_2v_n^{i+1} + \beta_n^i T_2v_n^{i+1}, \quad i = 1,2,\dots,k-2 \\ S_2v_n^{p-1} = (1-\beta_n^{p-1})S_2u_n + \beta_n^{p-1}T_2u_n, \end{cases} \tag{29}$$

then using (1.13), (2.13) and (2.14), yield the following estimates:

$$\begin{aligned} \|S_1x_{n+1} - S_2u_{n+1}\| &= \|(1-\alpha_n)(S_1y_n^1 - S_2v_n^1) + \alpha_n(T_1y_n^1 - T_2v_n^1)\| \\ &\le (1-\alpha_n)\|S_1y_n^1 - S_2v_n^1\| + \alpha_n\|T_1y_n^1 - T_2v_n^1\| \\ &= (1-\alpha_n)\|S_1y_n^1 - S_2v_n^1\| \\ &\quad + \alpha_n\|T_1y_n^1 - T_1v_n^1 + T_1v_n^1 - T_2v_n^1\| \\ &\le (1-\alpha_n)\|S_1y_n^1 - S_2v_n^1\| + \alpha_n\|T_1y_n^1 - T_1v_n^1\| \\ &\quad + \alpha_n\|T_1v_n^1 - T_2v_n^1\| \\ &\le (1-\alpha_n(1-\delta)\|S_1y_n^1 - S_2v_n^1\| \\ &\quad + \alpha_n\varphi(\|S_1y_n^1 - T_1y_n^1\|) + 2\alpha_n\epsilon, \end{aligned} \tag{30}$$

$$\begin{aligned} \|S_1y_n^1 - S_2v_n^1\| &= \|(1-\beta_n^1)(S_1y_n^2 - S_2v_n^2) + \beta_n^1(T_1y_n^2 - T_2v_n^2)\| \\ &\le (1-\beta_n^1)\|S_1y_n^2 - S_2v_n^2\| + \beta_n^1\|T_1y_n^2 - T_2v_n^2\| \\ &\le (1-\beta_n^1(1-\delta))\|S_1y_n^2 - S_2v_n^2\| \\ &\quad + \beta_n^1\varphi(\|S_1y_n^2 - T_1y_n^2\|) + 2\beta_n^1\epsilon \end{aligned} \tag{31}$$

and

$$\begin{aligned} \|S_1y_n^2 - S_2v_n^2\| &= \|(1-\beta_n^2)(S_1y_n^3 - S_2v_n^3) + \beta_n^2(T_1y_n^3 - T_2v_n^3)\| \\ &\le (1-\beta_n^2)\|S_1y_n^3 - S_2y_n^3\| + \beta_n^2\|T_1y_n^3 - T_2v_n^3\| \\ &\le (1-\beta_n^2(1-\delta))\|S_1x_n - S_2u_n\| \\ &\quad + \beta_n^2\varphi(\|S_1y_n^3 - T_1y_n^3\|) + 2\beta_n^2\epsilon \end{aligned} \tag{32}$$

Combining (2.15), (2.16) and (2.17), we have

$$\begin{aligned} \|S_1x_{n+1} - S_2u_{n+1}\| &\le (1-\alpha_n(1-\delta))(1-\beta_n^1(1-\delta)) \\ &\quad \times (1-\beta_n^2(1-\delta))\|S_1y_n^3 - S_2v_n^3\| \\ &\quad + (1-\alpha_n(1-\delta))(1-\beta_n^1(1-\delta)) \\ &\quad \times \beta_n^2\varphi(\|S_1y_n^3 - T_1y_n^3\|) \\ &\quad + 2(1-\alpha_n(1-\delta))(1-\beta_n^1(1-\delta))\beta_n^2\epsilon \end{aligned}$$

$$
\begin{aligned}
&+ (1-\alpha_n(1-\delta))\beta_n^1 \varphi(\|S_1 y_n^2 - T_1 y_n^2\|) \\
&+ 2(1-\alpha_n(1-\delta))\beta_n^1 \epsilon + \alpha_n \varphi(\|S_1 y_n^1 - T_1 y_n^1\|) \\
&+ 2\alpha_n \epsilon
\end{aligned} \tag{33}
$$

Thus inductively, we get

$$
\begin{aligned}
\|S_1 x_{n+1} - S_2 u_{n+1}\| \leq\; & [(1-\alpha_n(1-\delta))(1-\beta_n^1(1-\delta)) \ldots \\
& (1-\beta_n^{k-3}(1-\delta))] \|S_1 y_n^{k-2} - S_2 v_n^{k-2}\| \\
& + \alpha_n \varphi(\|S_1 y_n^1 - T_1 y_n^1\|) + (1-\alpha_n(1-\delta)) \\
& \beta_n^1 \varphi(\|S_1 y_n^2 - T_1 y_n^2\|) \ldots \\
& + (1-\alpha_n(1-\delta))(1-\beta_n^1(1-\delta)) \ldots \\
& (1-\beta_n^{k-4}(1-\delta))\beta_n^{k-3} \varphi(\|S_1 y_n^{k-2} - T_1 y_n^{k-2}\|) \\
& + 2[\alpha_n \epsilon + (1-\alpha_n(1-\delta))\beta_n^1 \epsilon + \ldots \\
& + (1-\alpha_n(1-\delta))(1-\beta_n^1(1-\delta)) \ldots \\
& (1-\beta_n^{k-4}(1-\delta))\beta_n^{k-3} \epsilon]
\end{aligned} \tag{34}
$$

Using (1.13), (2.13) and (2.14)

$$
\begin{aligned}
\|S_1 y_n^{k-2} - S_2 v_n^{k-2}\| &\leq \|(1-\beta_n^{k-2})(S_1 y_n^{k-1} - S_2 v_n^{k-1}) \\
&\quad + \beta_n^{k-2}(T_1 y_n^{k-1} - T_2 v_n^{k-1})\| \\
&\leq (1-\beta_n^{k-2})\|S_1 y_n^{k-1} - S_2 v_n^{k-1}\| \\
&\quad + \beta_n^{k-2}\|T_1 y_n^{k-1} - T_2 v_n^{k-1}\| \\
&\leq (1-\beta_n^{k-2}(1-\delta))\|S_1 y_n^{k-1} - S_2 v_n^{k-1}\| \\
&\quad + \beta_n^{k-2} \varphi(\|S_1 y_n^{k-1} - T_1 y_n^{k-1}\|) + 2\beta_n^{k-2}\epsilon
\end{aligned} \tag{35}
$$

Now by combining (2.19) and (2.20)

$$
\begin{aligned}
\|S_1 x_{n+1} - S_2 u_{n+1}\| \leq\; & [(1-\alpha_n(1-\delta))(1-\beta_n^1(1-\delta)) \ldots \\
& (1-\beta_n^{k-3}(1-\delta))(1-\beta_n^{k-2}(1-\delta))] \|S_1 y_n^{k-1} - S_2 v_n^{k-1}\| \\
& + \alpha_n \varphi(\|S_1 y_n^1 - T_1 y_n^1\|) \\
& + (1-\alpha_n(1-\delta))\beta_n^1 \varphi(\|S_1 y_n^2 - T_1 y_n^2\|) \ldots \\
& + (1-\alpha_n(1-\delta))(1-\beta_n^1(1-\delta)) \ldots \\
& (1-\beta_n^{k-3}(1-\delta))\beta_n^{k-2} \varphi(\|S_1 y_n^{k-1} - T_1 y_n^{k-1}\|) \\
& \\
& + 2[\alpha_n \epsilon + (1-\alpha_n(1-\delta))\beta_n^1 \epsilon + \ldots
\end{aligned}
$$

$$+ (1-\alpha_n(1-\delta))(1-\beta_n^1(1-\delta))\dots$$
$$(1-\beta_n^{k-3}(1-\delta))\beta_n^{k-2}\epsilon].$$

Using (1.13), (2.13) and (2.14)

$$\begin{aligned}\|S_1y_n^{k-1}-S_2v_n^{k-1}\| &\le \|(1-\beta_n^{k-1})(S_1x_n-S_2u_n)+\beta_n^{k-1}(T_1x_n-T_2u_n)\| \\ &\le (1-\beta_n^{k-1})\|S_1x_n-S_2u_n\|+\beta_n^{k-1}\|T_1x_n-T_2u_n\| \\ &\le (1-\beta_n^{k-1}(1-\delta))\|S_1x_n-S_2u_n\| \\ &\quad + \beta_n^{k-1}\varphi(\|S_1x_n-T_1x_n\|)+2\beta_n^{k-1}\epsilon \end{aligned} \tag{36}$$

which further implies

$$\begin{aligned}\|S_1x_{n+1}-S_2u_{n+1}\| \le\ & [(1-\alpha_n(1-\delta))(1-\beta_n^1(1-\delta))\dots \\ & (1-\beta_n^{k-3}(1-\delta))(1-\beta_n^{k-1}(1-\delta))]\|S_1x_n-S_2u_n\| \\ & +\alpha_n\varphi(\|S_1y_n^1-T_1y_n^1\|) \\ & +(1-\alpha_n(1-\delta))\beta_n^1\varphi(\|S_1y_n^2-T_1y_n^2\|)\dots \\ & +(1-\alpha_n(1-\delta))(1-\beta_n^1(1-\delta))\dots \\ & (1-\beta_n^{k-2}(1-\delta))\beta_n^{k-1}\varphi(\|S_1x_n-T_1x_n\|) \\ & +2[\alpha_n\epsilon+(1-\alpha_n(1-\delta))\beta_n^1\epsilon+\dots \\ & +(1-\alpha_n(1-\delta))(1-\beta_n^1(1-\delta))\dots \\ & (1-\beta_n^{k-2}(1-\delta))\beta_n^{k-1}\epsilon], \end{aligned} \tag{37}$$

where

$$\begin{aligned}&[(1-\alpha_n(1-\delta))][(1-\beta_n^1(1-\delta))]\dots[(1-\beta_n^{k-3}(1-\delta))][(1-\beta_n^{k-1}(1-\delta))] \\ &\quad \le [(1-\alpha_n(1-\delta))]\end{aligned} \tag{38}$$

Since

$$[(1-\alpha_n(1-\delta))] \le \alpha_n(1-\delta)) \le \alpha_n\,. \tag{39}$$

Hence using (2.23) and (2.24) in (2.22), we get

$$\begin{aligned}\|S_1x_{n+1}-S_2u_{n+1}\| \le\ & [1-(1-\alpha_n(1-\delta))]\|S_1x_n-S_2u_n\| \\ & +\alpha_n\varphi(\|S_1y_n^1-T_1y_n^1\|)+\alpha_n\varphi(\|S_1y_n^2-T_1y_n^2\|) \\ & +\alpha_n\varphi(\|S_1y_n^3-T_1y_n^3\|)+\dots \\ & +\alpha_n\varphi(\|S_1x_n-T_1x_n\|)+2k\epsilon \end{aligned}$$

$$= [1-(1-\alpha_n(1-\delta))]\|S_1x_n - S_2u_n\| + \alpha_n(1-\delta)\left\{\frac{\begin{pmatrix}\varphi(\|S_1y_n^1 - T_1y_n^1\|) \\ +\varphi(\|S_1y_n^2 - T_1y_n^2\|) + \dots \\ +\alpha_n\varphi(\|S_1x_n - T_1x_n\|) \\ +2k\epsilon\end{pmatrix}}{(1-\delta)}\right\} \tag{40}$$

Let us denote

$$a_n = \|S_1x_n - S_2u_n\|,$$
$$r_n = \alpha_n(1-\delta),$$

and

$$\sigma_n : \frac{\begin{pmatrix}\{\varphi(\|S_1y_n^1 - T_1y_n^1\|) \\ +\varphi(\|S_1y_n^2 - T_1y_n^2\|) + \dots \\ +\varphi(\|S_1x_n - T_1x_n\|) + 2k\epsilon\}\end{pmatrix}}{(1-\delta)}.$$

Now from Theorem 1.5, we have

$$\begin{aligned}\lim_{n\to\infty}\|Sx_n - p\| = 0 \\ &= \lim_{n\to\infty}\|S_1x_n - T_1x_n\| \\ &= \lim_{n\to\infty}\|S_1y_n^1 - T_1y_n^1\| \\ &= \lim_{n\to\infty}\|S_1y_n^2 - T_1y_n^2\| = \dots \\ &= \lim_{n\to\infty}\|S_1y_n^{k-1} - T_1y_n^{k-1}\| = 0.\end{aligned}$$

Since φ is continuous, we have

$$\begin{aligned}\lim_{n\to\infty}\varphi(\|x_n - T_1x_n\|) &= \lim_{n\to\infty}\varphi(\|y_n^1 - T_1y_n^1\|) \\ &= \lim_{n\to\infty}\varphi(\|y_n^2 - T_1y_n^2\|) = \dots \\ &= \lim_{n\to\infty}\varphi(\|y_n^{k-1} - T_1y_n^{k-1}\|) = 0\end{aligned}$$

Hence using Lemma 1, (2.25) yields

$$\|p-q\| \le \frac{2k\epsilon}{1-\delta}.$$

Remark 2.2. Since the iteration (1.5) is special case of iterative scheme (1.7), so. Theorem 1.6 generalizes existing result for (1.5). By taking $k = 3$ and using Remark 1.1 in Theorem 2.2, data dependence results for the iterative schemes (1.5) and for SP iterative scheme can be obtained easily.

Numerical Example.

$$p_1(x) = e^{(1-x)^2} - 1 - x,$$
$$p_2(x) = \frac{3}{8} + \frac{35}{8}x^4 - \frac{15}{4}x^2$$

To find the fixed point we write $p_1(x)$ and $p_2(x)$ as

$$e^{(1-x)^2} - 1 - x = 0 \text{ and } \frac{3}{8} + \frac{35}{8}x^4 - \frac{15}{4}x^2 = 0, \quad X = [0,1]$$

and

$$\begin{cases} S_1(x) = 30x^2 \\ T_1(x) = 35x^4 + 3 \end{cases}$$

and

$$\begin{cases} S_2(x) = x + 1 \\ T_2(x) = e^{(1-x)^2} \end{cases},$$

for Jungck multi step SP and Noor iteration

$$\begin{cases} S_1(x) = x \\ T_1(x) = e^{(1-x)^2} - 1 \end{cases}$$

and

$$\begin{cases} S_2(x) = x \\ T_2(x) = \frac{\sqrt{90+1050x^4}}{30} \end{cases}$$

for Jungck modified multi step SP and Noor iteration as both $p_1(x)$ and $p_2(x)$ has unique root in the interval $(0, 1)$ so we convert this in the fixed point form and take $\alpha = 0.5$ and $\alpha = 0.35$ respectively. Now we solve it by

$$T_\lambda(x) = \frac{T(x) + \lambda_1 x + \lambda_2 x^2 + \ldots + \lambda_k x^k}{1 + \lambda_1 + \lambda_2 x + \ldots + \lambda_k x^{k-1}}.$$

For respective value of α, $\lambda_1, \lambda_2, \lambda_3 \dots \lambda_k$ can be determined uniquely from system of linear equations as in [3] for $\alpha = .5$ and $T_2(x) = e^{(1-x)^2} - 1$ we have

$$\begin{pmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 1 & 2\alpha & 3\alpha^2 & 4\alpha^3 \\ 0 & 0 & 2 & 6\alpha & 12\alpha^2 \\ 0 & 0 & 0 & 6 & 24\alpha \\ 0 & 0 & 0 & 0 & 24 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} T'_\lambda(\alpha) \\ T_\lambda^{(2)}(\alpha) \\ T_\lambda^{(3)}(\alpha) \\ T_\lambda^{(4)}(\alpha) \\ T_\lambda^{(5)}(\alpha) \end{pmatrix} = \begin{pmatrix} 1.28403 \\ -3.85208 \\ 8.98818 \\ -32.1006 \\ 104.006 \end{pmatrix}.$$

After solving the system of linear equations we have $\lambda_1 = 6.0858$, $\lambda_2 = -17757$, $\lambda_3 = 22.2698$, $\lambda_4 = -14.0173$, $\lambda_5 = 4.3336$ for second polynomial equation where $\alpha = .35$ and

$$T_1(x) = \frac{\sqrt{90 + 1050x^4}}{30}.$$

System of linear equations become

$$\begin{pmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 1 & 2\alpha & 3\alpha^2 & 4\alpha^3 \\ 0 & 0 & 2 & 6\alpha & 12\alpha^2 \\ 0 & 0 & 0 & 6 & 24\alpha \\ 0 & 0 & 0 & 0 & 24 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} -T'_\lambda(\alpha) \\ -T_\lambda^{(2)}(\alpha) \\ -T_\lambda^{(3)}(\alpha) \\ -T_\lambda^{(4)}(\alpha) \\ -T_\lambda^{(5)}(\alpha) \end{pmatrix} = \begin{pmatrix} -0.291842 \\ -2.25304 \\ -8.53984 \\ 32.6662 \\ 422.235 \end{pmatrix}.$$

Hence we get $\lambda_1 = 0.0129005$, $\lambda_2 = -0.330016$, $\lambda_3 = 3.01516$, $\lambda_4 = 17.5910$, $\lambda_5 = -19.186$.

3. Experiment

Now using the value of $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k$ and iterative schemes we have following table and graphs.

We assume $\alpha_n = a$, $\beta_n^i = b$, $x_0 = 0.6$.

Observations on the Basis of Graphs and Tables

1. Simple multi step SP iteration for $p_1(x)$ converge faster than modified multi step SP iteration and speed of convergence increases as value of a and b increases between $[0, 1]$ the result holds same for $p_2(x)$ but in this case simple multi step SP iteration does not converges for the value $a = 0.9$ and $b = 0.9$ but modified multi step SP iteration converges for all value of a and b.

Table 1: Jungck multi step SP and Noor for $p_1(x)$ and $p_2(x)$

	Jungck multi step SP for $p_1(x)$				Jungck multi step Noor for $p_1(x)$				Jungck multi step SP for $p_2(x)$				Jungck multi step Noor for $p_2(x)$		
	$a = 0.1,\, b = 0.1$				$a = 0.1,\, b = 0.1$				$a = 0.1,\, b = 0.1$				$a = 0.1,\, b = 0.1$		
n	x_{n+1}	S_{xn}	T_{xn}	n	x_{n+1}	S_{xn}	T_{xn}	n	x_{n+1}	S_{xn}	T_{xn}	n	x_{n+1}	S_{xn}	T_{xn}
1	0.554712	9.23117	7.536	1	0.590034	10.4442	7.536	1	0.459744	1.45974	1.17351	1	0.459744	1.45974	1.17351
2	0.511911	7.86159	6.3139	2	0.58012	10.0962	7.24206	2	0.422943	1.42294	1.33894	2	0.422943	1.42294	1.33894
3	0.473661	6.73065	5.40351	3	0.570286	9.75678	6.96407	3	0.414662	1.41466	1.39514	3	0.414662	1.41466	1.39514
41	0.339981	3.46762	3.46761	96	0.340298	3.47408	3.4695	8	0.412392	1.41239	1.41238	8	0.412392	1.41239	1.41238
42	**0.339981**	**3.46761**	**3.46761**	201	0.339981	3.46762	3.46761	**9**	**0.412391**	**1.41239**	**1.41239**	**52**	**0.412391**	**1.41239**	**1.41239**
43	**0.339981**	**3.46761**	**3.46761**	**202**	**0.339981**	**3.46761**	**3.46761**	**10**	**0.412391**	**1.41239**	**1.41239**	**53**	**0.412391**	**1.41239**	**1.41239**

Table 2: Jungck modified multi step SP and Noor for $p_1(x)$ and $p_2(x)$

	Jungck modified multi step SP for $p_2(x)$				Jungck modified multi step Noor for $p_2(x)$				Jungck modified multi step SP for $p_1(x)$				Jungck modified multi step Noor for $p_1(x)$		
	$a = 0.1,\, b = 0.1$				$a = 0.1,\, b = 0.1$				$a = 0.1,\, b = 0.1$				$a = 0.1,\, b = 0.1$		
n	x_{n+1}	S_{xn}	T_{xn}	n	x_{n+1}	S_{xn}	T_{xn}	n	x_{n+1}	S_{xn}	T_{xn}	n	x_{n+1}	S_{xn}	T_{xn}
1	0.578706	0.6	0.38003	1	0.515037	0.6	0.38003	1	0.52004	0.6	-2.12644	1	0.331642	0.6	-2.12644
2	0.560083	0.578706	0.387059	2	0.470524	0.515037	0.402834	2	0.492146	0.52004	0.154728	2	0.33503	0.331642	0.339882
3	0.543743	0.560083	0.392477	3	0.445915	0.470524	0.409341	3	0.470885	0.492146	0.241343	3	0.337048	0.33503	0.339946
125	0.412391	0.412392	0.412391	25	0.412391	0.412392	0.412391	121	0.339981	0.339982	0.339981	20	0.339981	0.33998	0.339981
126	**0.412391**	**0.412391**	**0.412391**	**26**	**0.412391**	**0.412391**	**0.412391**	**122**	**0.339981**	**0.339981**	**0.339981**	**21**	**0.339981**	**0.339981**	**0.339981**
127	**0.412391**	**0.412391**	**0.412391**	**27**	**0.412391**	**0.412391**	**0.412391**	**123**	**0.339981**	**0.339981**	**0.339981**	**22**	**0.339981**	**0.339981**	**0.339981**

2. Modified multi step Noor iteration converge faster than simple multi step Noor iteration for $p_1(x)$ and speed of convergence increases as value of a and b increases between $[0, 1]$ the result holds same for $p_2(x)$ but in this case simple multi step Noor iteration do not converges for the value $a = 0.9$ and $b = 0.9$ but modified multi step Noor iteration converges for all value of a and b.

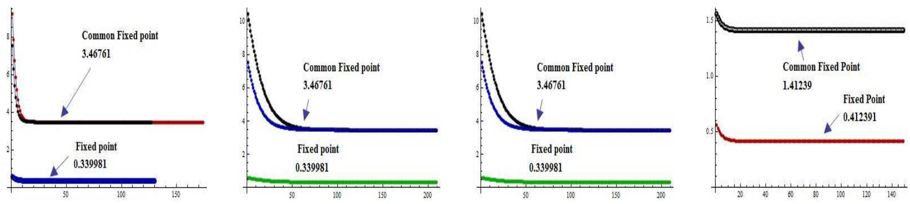

Figure 1: Graphical observations of Jungck multi step SP and Noor for $p_1(x)$ and $p_2(x)$. Where initial approximation is $x_0 = 0.6$, $a = 0.1$ and $b = 0.1$. Here MSP_0, MSP_0, MSP_1, MN_1 show the graph for Table 1.1. The merging point with value 3.46761 and 1.41239 is common fixed point for $p_1(x)$ and $p_2(x)$

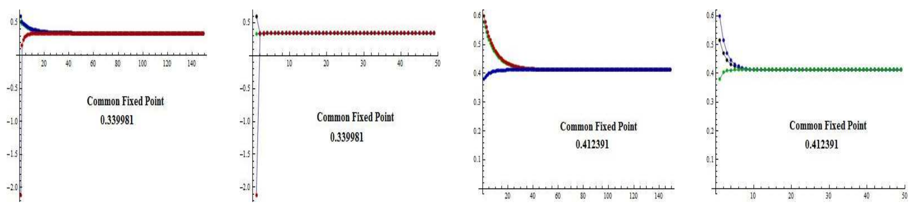

Figure 2: Graphical observations of Jungck modified multi step SP and Noor for $p_1(x)$ and $p_2(x)$. Where initial approximation is $x_0 = 0.6$, $a = 0.1$ and $b = 0.1$. Here MMSP_0, MMN_0, MMSP_1, MMN_1 show the graph for Table 1.2. The merging point with value 0.339981 and 0.41239 is common fixed point for $p_1(x)$ and $p_2(x)$

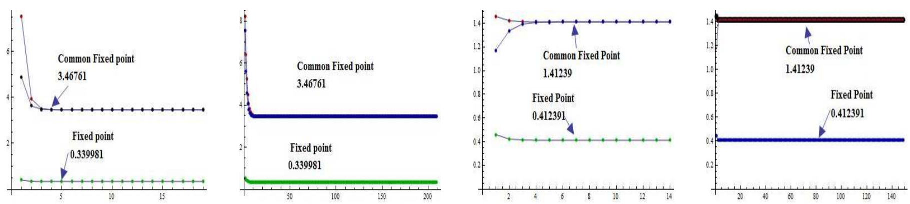

Figure 3: Graphical observations of Jungck multi step SP and Noor for $p_1(x)$ and $p_2(x)$. Where initial approximation is $x_0 = 0.6$, $a = 0.5$ and $b = 0.5$. Here MSP_0, MN_0, MSP_1, MN_1 show the graph for Table 1.3. The merging point with value 3.46761 and 1.41239 is common fixed point for $p_1(x)$ and $p_2(x)$

4. Conclusion

On the analysis of table and graph of multi step and modified multi step SP, Noor iterations for $p_1(x)$ but $p_2(x)$ we conclude that modified multi step Noor

Table 3: Jungck multi step SP and Noor for $p_1(x)$ and $p_2(x)$

	Jungck multi step SP for $p_1(x)$				Jungck multi step Noor for $p_1(x)$				Jungck multi step SP for $p_2(x)$				Jungck multi step Noor for $p_2(x)$		
	$a = 0.5$, $b = 0.5$				$a = 0.5$, $b = 0.5$				$a = 0.5$, $b = 0.5$				$a = 0.5$, $b = 0.5$		
n	x_{n+1}	Sx_n	Tx_n	n	x_{n+1}	Sx_n	Tx_n	n	x_{n+1}	Sx_n	Tx_n	n	x_{n+1}	Sx_n	Tx_n
1	0.404097	4.89883	7.536	1	0.523506	8.22176	7.536	1	0.412067	1.41207	1.17351	1	0.444448	1.44445	1.17351
2	0.347967	3.63243	3.93327	2	0.462308	6.41185	5.62879	2	0.412392	1.41239	1.41293	2	0.413343	1.41334	1.36157
3	0.340824	3.48482	3.51312	3	0.417941	5.24025	4.59879	**3**	**0.412391**	**1.41239**	**1.41239**	3	0.41238	1.41238	1.41081
7	0.339981	3.46762	3.46762	28	0.339981	3.46762	3.46761	**4**	**0.412391**	**1.41239**	**1.41239**	**4**	**0.412391**	**1.41239**	**1.41241**
8	**0.339981**	**3.46761**	**3.46761**	**29**	**0.339981**	**3.46761**	**3.46761**	**5**	**0.412391**	**1.41239**	**1.41239**	**5**	**0.412391**	**1.41239**	**1.41239**
9	**0.339981**	**3.46761**	**3.46761**	**30**	**0.33991**	**3.46761**	**3.46761**	**6**	**0.412391**	**1.41239**	**1.41239**	**6**	**0.412391**	**1.41239**	**1.41239**

Table 4: Jungck modified multi step SP and Noor for $p_1(x)$and $p_2(x)$

	Jungck modified multi step SP for $p_2(x)$				Jungck modified multi step Noor for $p_2(x)$				Jungck modified multi step SP for $p_1(x)$				Jungck modified multi step Noor for $p_1(x)$		
	$a = 0.5$, $b = 0.5$				$a = 0.5$, $b = 0.5$				$a = 0.5$, $b = 0.5$				$a = 0.5$, $b = 0.5$		
n	x_{n+1}	Sx_n	Tx_n	n	x_{n+1}	Sx_n	Tx_n	n	x_{n+1}	Sx_n	Tx_n	n	x_{n+1}	Sx_n	Tx_n
1	0.502517	0.6	0.38003	1	0.416654	0.6	0.38003	1	0.451595	0.6	-2.12644	1	-0.564101	0.6	-2.12644
2	0.456574	0.502517	0.405035	2	0.412523	0.416654	0.412375	2	0.392942	0.451595	0.302308	2	0.0533425	-0.564101	-0.424452
3	0.434268	0.456574	0.410632	3	0.412395	0.412523	0.412391	3	0.365884	0.392942	0.334296	3	0.327481	0.0533425	0.315952
4	0.423276	0.434268	0.411961	4	0.412391	0.412395	0.412391	4	0.352801	0.365884	0.338826	4	0.339577	0.327481	0.339765
19	0.412392	0.412392	0.412391	**5**	**0.412391**	**0.412391**	**0.412391**	19	0.339981	0.339982	0.339981	5	0.339968	0.339577	0.339981
20	0.412391	0.412392	0.412391	**6**	**0.412391**	**0.412391**	**0.412391**	**20**	**0.339981**	**0.339981**	**0.339981**	6	0.339981	0.339968	0.339981
21	**0.412391**	**0.412391**	**0.412391**					**21**	**0.339981**	**0.339981**	**0.339981**	**7**	**0.339981**	**0.339981**	**0.339981**
22	**0.412391**	**0.412391**	**0.412391**												

iteration converge faster than modified multi step SP and multi step SP and Noor iterations for value of a, b between $[0, 1]$.

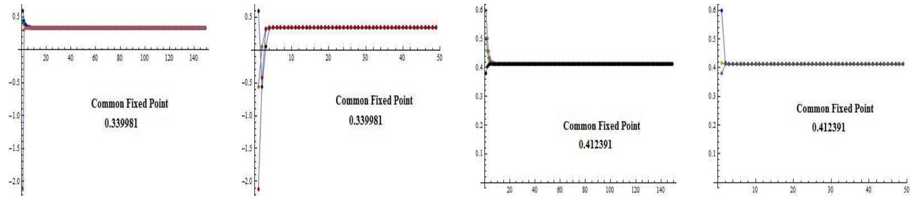

Figure 4: Graphical observations of Jungck multi step SP and Noor for $p_1(x)$ and $p_2(x)$. Where initial approximation is $x_0 = 0.6$, $a = 0.1$ and $b = 0.1$. Here MMSP$_0$, MMN$_0$, MMSP$_1$, MMN$_1$ show the graph for Table 1.4. The merging point with value 0.339981 and 0.41239 is common fixed point for $p_1(x)$ and $p_2(x)$

Table 5: Jungck multi step SP and Noor for $p_1(x)$

	Jungck multi step SP for $p_1(x)$				Jungck multi step Noor for $p_1(x)$		
	$a = 0.9$, $b = 0.9$				$a = 0.9$, $b = 0.9$		
n	x_{n+1}	Sx_n	Tx_n	n	x_{n+1}	Sx_n	Tx_n
1	0.34589	3.58919	7.536	1	0.389028	4.54029	7.536
2	0.34001	3.4682	3.50098	2	0.346955	3.61134	3.80167
3	0.339981	3.46762	3.46777	3	0.340916	3.48672	3.50718
4	**0.339981**	**3.46761**	**3.46761**	7	0.339981	3.46762	3.46763
5	**0.339981**	**3.46761**	**3.46761**	**8**	**0.339981**	**3.46761**	**3.46761**
6	**0.339981**	**3.46761**	**3.46761**	**9**	**0.339981**	**3.46761**	**3.46761**

Table 6: Jungck modified multi step SP and Noor for $p_1(x)$ and $p_2(x)$

	Jungck modified multi step SP for $p_2(x)$				Jungck modified multi step Noor for $p_2(x)$				Jungck modified multi step SP for $p_1(x)$				Jungck modified multi step Noor for $p_1(x)$		
	$a = 0.9$, $b = 0.9$				$a = 0.9$, $b = 0.9$				$a = 0.9$, $b = 0.9$				$a = 0.9$, $b = 0.9$		
n	x_{n+1}	Sx_n	Tx_n	n	x_{n+1}	Sx_n	Tx_n	n	x_{n+1}	Sx_n	Tx_n	n	x_{n+1}	Sx_n	Tx_n
1	0.430876	0.6	0.38003	1	0.41239	0.6	0.38003	1	0.163863	0.6	-2.12644	1	-1.79368	0.6	-2.12644
2	0.414237	0.430876	0.412084	2	0.412391	0.41239	0.412391	2	0.321977	0.163863	0.322627	2	-1.71277	-1.79368	-1.77631
5	0.412393	0.41241	0.412391	**3**	**0.412391**	**0.412391**	**0.412391**	5	0.339963	0.339801	0.339981	8	-0.899783	-1.1189	-1.07683
6	0.412391	0.412393	0.412391	**4**	**0.412391**	**0.412391**	**0.412391**	6	0.339979	0.339963	0.339981	9	-0.514814	-0.899783	-0.837249
7	**0.412391**	**0.412391**	**0.412391**	**5**	**0.412391**	**0.412391**	**0.412391**	7	0.339981	0.339979	0.339981	10	0.336801	-0.514814	-0.353497
8	**0.412391**	**0.412391**	**0.412391**					**8**	**0.339981**	**0.339981**	**0.339981**	11	0.339981	0.336801	0.339966
												12	**0.339981**	**0.339981**	**0.339981**
												13	**0.339981**	**0.339981**	**0.339981**

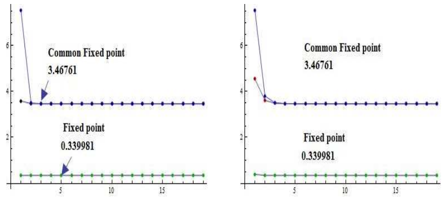

Figure 5: Graphical observations of Jungck multi step SP and Noor for $p_1(x)$. Where initial approximation is $x_0 = 0.6$, $a = 0.9$ and $b = 0.9$. Here MSP_0, MN_0 show the graph for Table 1.5. The merging point with value 3.46761 is common fixed point for $p_1(x)$. While Jungck multi step SP and Noor do not converge for $p_2(x)$

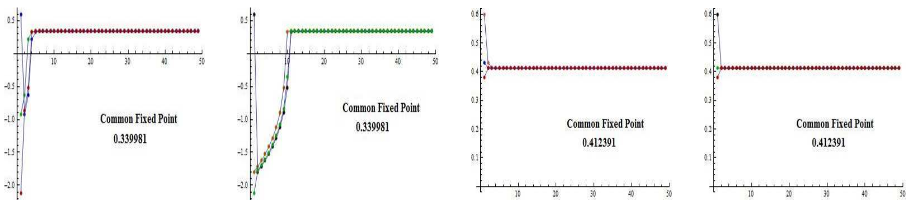

Figure 6: Graphical observations of Jungck multi step SP and Noor for $p_1(x)$ and $p_2(x)$. Where initial approximation is $x_0 = 0.6$, $a = 0.9$ and $b = 0.9$. Here MMSP_0, MMN_0, MMSP_1, MMN_1 show the graph for Table 1.6. The merging point with value 0.339981 and 0.41239 is common fixed point for $p_1(x)$ and $p_2(x)$

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