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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR DAMPED LINEAR HYPERBOLIC EQUATIONS WITH DIRICHLET BOUNDARY CONDITIONS

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Abstract: We consider damped linear hyperbolic equations with Dirichlet boundary conditions. We prove the existence, uniqueness, and regularity of the solution. We apply semi-discretization in time technique.

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1. Introduction

In this paper, we study the initial boundary value problem for the following damped linear hyperbolic equations with Dirichlet boundary conditions.

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \eta \frac{\partial u}{\partial t} - \Delta u = f(x,t),\ x \in \Omega,\ t \in]0,T[, \\ u(t,x) = 0,\ \text{on } \Sigma, \\ u(0,x) = u_0(x),\ \frac{\partial u(0,x)}{\partial t} = u_1(x),\ x \in \Omega. \end{cases} \tag{1}$$

where η is the constant damping coefficient, $\eta \in \mathbb{R}$ and T finite, Ω is a bounded open domain in $\mathbb{R}^n$, $n \geq 1$, with a smooth boundary $\partial\Omega$. We denote by Q the

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cylinder of $\mathbb{R}_x^n \times \mathbb{R}_t$, $Q = \Omega\times]0,T[$ and by Σ the lateral boundary of Q, $\Sigma = \partial\Omega\times]0,T[$, $\Delta u = \Sigma_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$ is the Laplacian, and linear function $f : \Omega\times]0,T[\longrightarrow \mathbb{R}$ for $i = 0, 1$, the functions $u_i : \Omega \longrightarrow \mathbb{R}$, are given. we find a function $u = u(x,t)$, is a real-valued satisfies (1).

This Problem has its origin in a physical problem, we study a model that describes the transverse vibrations of a membrane Ω fixed at its ends and in the presence of damping η. Let $u(x,t)$ be the vertical position of $x \in \Omega$ at time $t \in [0,T]$, is retarded by a damping force proportional to the velocity of the membrane, then u satisfies (1).

This problem has been already investigated by many authors.

For example in ([2]), ([7]), ([17]).

We define some function spaces required to establish the existence and uniqueness of solution to (1). We use the function spaces for any $1 \leq p < \infty$, $L^p(\Omega)$ is the space of real measurable functions $u : \Omega \longrightarrow \mathbb{R}$ for the Lebesgue measure dx, it is a Banach space for the following norm

$$\|u\|_{L^p(\Omega)} = (\int_\Omega |u(x)|^p dx)^{\frac{1}{p}} < +\infty$$

for $p = 2$, $L^2(\Omega)$ is a Hilbert space for the scalar product

$$(u, v) = \int_\Omega u(x)v(x)dx,$$

the corresponding norm being denoted $\|u\|$,

$$\|u\| = (\int_\Omega u^2(x)dx)^{\frac{1}{2}}.$$

if X is a Banach space, $1 \leq p < \infty$, $L^p(0,T;X)$ is the space of measurable functions u of $]0,T[$ into X for the Lebesgue measure dt, which is Banach space for the following norm

$$\|u\|_{L^p(0,T;X)} = (\int_0^T \|u(t)\|_X^p dt)^{\frac{1}{p}} < +\infty$$

if $X = L^p(\Omega)$, then $L^p(]0,T[; L^p(\Omega)) = L^p(Q)$.

$L^\infty(0,T;X)$ is the space of measurable functions from $]0,T[$ into X which are essentially bounded, the space is Banach for the following norm

$$\|u\|_{L^\infty(0,T;X)} = \sup_{t\in]0,T[} ess\|u(t)\|_X$$

we denote by $\mathscr{C}([0,T];X)$ the space of continuous functions from $[0,T]$ into X, the space is Banach for the following norm

$$\|u\|_{\mathscr{C}([0,T];X)} = \sup_{t\in[0,T]} \|u(t)\|_X,$$

and by $\mathscr{C}^k([0,T];X)$, $k \in \mathbb{N}$ the space of k times continuously differentiable functions from $[0,T]$ into X, it is a Banach spaces for the following norm

$$\|u\|_{\mathscr{C}^k([0,T];X)} = \Sigma_{j=0}^k \|\frac{d^j u}{dt^j}\|_{\mathscr{C}([0,T];X)}$$

we denote by $\mathscr{C}^\infty(\Omega)$ the space of infinitely times continuously differentiable functions on Ω. The space $\mathscr{C}^\infty(\Omega)$ of real functions on Ω, with a compact support in Ω, is denoted by $\mathscr{D}(\Omega)$, as in the theory of distributions of L.Schwartz in ([8]), $\mathscr{D}'(\Omega)$ is the space of distributions on Ω.

We introduce the Sobolev spaces, for $m \in \mathbb{N}, 1 \leq p \leq \infty$, $W^{m,p}(\Omega)$ is the space of functions u in $L^p(\Omega)$ whose distribution derivatives of order $\leq m$ are in $L^p(\Omega)$. This is a Banach space for the norm

$$\|u\|_{W^{m,p}(\Omega)} = \Sigma_{|\alpha|\leq m} \|D^\alpha u\|_{L^p(\Omega)}$$

where

$$D^\alpha u = \frac{\partial^{\alpha_1+\cdots+\alpha_n} u}{\partial x_1^{\alpha_1}\cdots \partial x_n^{\alpha_n}},\ \alpha = \{\alpha_1, \cdots, \alpha_n\} \in \mathbb{N}^n,$$

$$|\alpha| = \alpha_1 + \cdots + \alpha_n \text{ and } D_i u = \frac{\partial u}{\partial x_i}, \quad i = 1, 2, \cdots, n.$$

When $p = 2$ we write $W^{m,2}(\Omega) = H^m(\Omega)$ and this is a Hilbert space for the scalar product

$$(u,v)_{H^m(\Omega)} = \Sigma_{|\alpha|\leq m}(D^\alpha u, D^\alpha v).$$

we use the space $H_0^1(\Omega)$

$$H_0^1(\Omega) = \text{the closure of } \mathscr{D}(\Omega) \text{ in } H^1(\Omega) \tag{2}$$

for study the problem (1) we introduce the space

$$V = H_0^1(\Omega), \tag{3}$$

V is a Hilbert space for the scalar product

$$(u,v)_V = \Sigma_{i=1}^n (\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i})$$

and that the corresponding norm

$$\|u\|_V = \{(u,u)_V\}^{\frac{1}{2}}$$

with dual $V' = H^{-1}(\Omega)$, the corresponding norm on V' is

$$\|l\|_* = \sup_{u\in V} \frac{|\langle l,u\rangle|}{\|u\|_V}$$

for any $l \in V'$ and $u \neq 0$, where $\langle .,.\rangle$ is the scalar product between V and V'. To given linear continuous operator $A \in \mathscr{L}(V,V')$, we can associate a bilinear continuous form a on V by setting

$$a(u,v) = \langle Au, v\rangle,\ \forall u,v \in V, \tag{4}$$

such that $\mathscr{L}(V,V')$ is the space of linear continuous operators from V into V'. conversely, to given a bilinear continuous form a on V, we can associate with a a linear continuous operator A from V into V', and from the properties of a that A is linear continuous, and by the continuity of a if

$$|a(u,v)| \leq C\|u\|_V\|v\|_V,\ C > 0,\ \forall u,v \in V, \tag{5}$$

then

$$\|A\|_* \leq C \tag{6}$$

If $a(u,v) = (u,v)_V$ is the scalar product of V, that a is coercive,

$$a(u,u) \geq \alpha\|u\|_V^2,\ 0 \leq \alpha \leq 1,\ \forall u \in V. \tag{7}$$

The Riesz representation theorem to show that each a linear continuous form on H can be represented with the aid of scalar product. Let H' the dual space of H, to given $\phi \in H'$ there exists a unique $f \in H$ such that the application $\phi \longrightarrow f$ is an isomorphism and isometric that allows us to identify H to the dual space H'.

In general, but not always, it is also convenient to identify H to its dual H'. We write position typical where is not place of performance this identification. Let $H = L^2(\Omega)$, and $V = H_0^1(\Omega)$ is a dense in $L^2(\Omega)$ since that V it is a Banach space reflexive, we assume that the canonical injection of V in H being continuous, then identify, $H' \equiv H$ and $H \subset V'$ from the following assertion:

For given $f \in H$, the application $v \in V \longrightarrow (f,v)$ of V into $\mathbb{R}$ is linear continuous on H and by priority on V, denote by $Tf \in V'$ such that

$$\langle Tf, v\rangle = (f,v),\ \forall f \in H,\ \forall v \in V$$

where $T : H \longrightarrow V'$ satisfies the following proprieties

(i) $\|Tf\|_* \leq C\|f\|$, for any $f \in H$,

(ii) T is injective

(iii) $T(H)$ is a dense in V'

from T, we have $H \subset V'$ then we obtain

$$V \subset H \equiv H' \subset V', \tag{8}$$

where each space is dense in the following, the injections being continuous.

We recall some basic results for using in the proof of our main results.

2. Preliminaries

Lemma 1. *If* $u(x,t) \in L^1(\Omega \times [0,T])$ *satisfies*

$$\int_\Omega u(x,t)dx \leq C + B\int_0^t (\int_\Omega u(x,s)dx)ds,\ C \in \mathbb{R},\ B > 0$$

then

$$\int_\Omega u(x,t)dx \leq Ce^{Bt},\ x \in \Omega,\ t \in [0,T].$$

Proof. we have the inequality

$$\int_\Omega u(x,t)dx \leq C + B\int_0^t (\int_\Omega u(x,s)dx)ds \leq$$
$$C + B\int_0^t [C + B\int_0^s (\int_\Omega u(x,\sigma)dx)d\sigma]ds =$$
$$C(1+Bt) + B^2\int_0^t (t-s)(\int_\Omega u(x,s)dx)ds$$

then, we have

$$\int_\Omega u(x,t)dx \leq C(1+Bt) + B^2\int_0^t (t-s)(\int_\Omega u(x,s)dx)ds \leq$$
$$C(1+Bt) + B^2\int_0^t (t-s)[C + B(\int_0^s (\int_\Omega u(x,\sigma)dx)d\sigma]ds =$$
$$C(1+Bt+B^2\frac{t^2}{2}) + B^3\int_0^t \frac{(t-s)^2}{2}(\int_\Omega u(x,s)dx)ds$$

by backward we obtain :

$$\int_{\Omega} u(x,t)dx \leq C(1 + Bt + B^2\frac{t^2}{2} + \cdots + B^n\frac{t^n}{n!}) + B^{n+1}\int_0^t \frac{(t-s)^n}{n!}(\int_{\Omega} u(x,s)dx)ds$$

pass to the limit as $n \longrightarrow +\infty$, we obtain

$$\int_{\Omega} u(x,t)dx \leq Ce^{Bt}.$$

□

Lemma 2. *If $u \in L^p(0,T;X)$ and $\frac{\partial u}{\partial t} \in L^p(0,T;X), 1 \leq p \leq \infty$ then u is a continuous from $[0,T]$ into X, almost everywhere on $[0,T]$.*

Proof. We find u solution of the problem (1) in the space $L^{\infty}(0,T;X)$ then we need the derivative $\frac{\partial u}{\partial t}$ in the space $L^{\infty}(0,T;X)$ we prove $\frac{\partial u}{\partial t} \in L^p(0,T;X)$ if $u \in L^p(0,T;X)$ where $1 \leq p \leq \infty$.

In ([9]), $\mathscr{D}'(0,T;X)$ is the space of distributions from $]0,T[$ into X, defined by

$$\mathscr{D}'(0,T;X) = \mathscr{L}(\mathscr{D}(]0,T[);X).$$

if $u \in \mathscr{D}'(0,T;X)$, the distributional derivative is defined by

$$\frac{\partial u}{\partial t}(\varphi) = -u(\frac{d\varphi}{dt}),\ \varphi \in \mathscr{D}(]0,T[) \tag{9}$$

if $u \in L^p(0,T;X)$, the corresponding distribution is also defined by u from $]0,T[$ into X, such that

$$u(\varphi) = \int_0^T u(t)\varphi(t)dt,\ \varphi \in \mathscr{D}(]0,T[),$$

the integral $u(\varphi) \in X$; we can also defined $\frac{\partial u}{\partial t} \in \mathscr{D}'(0,T;X)$ by (9). □

Let V is a Banach space separable and reflexive and K is a closed convex set in V.

Theorem 3. *We assume that K is a closed convex set unbounded in V. Let A is a pseudo-monotone operator from K into V', and coercive in the following sense:*

There exists $v_0 \in K$ *such that*

$$\frac{\langle A(v), v - v_0\rangle}{\|v\|_V} \longrightarrow +\infty \text{ as } \|v\|_V \longrightarrow +\infty,\ v \in K \tag{10}$$

then, for $f \in V'$*, there exists* $u \in K$ *such that*

$$\langle A(u), v - u\rangle \geq \langle f, v - u\rangle,\ \forall v \in K.$$

Proof. We note that A is pseudo - monotone from V into V' if satisfies the following conditions:

First: A is bounded,

Second: as $j \longrightarrow +\infty$, u_j tending to u weakly in V and $\limsup\langle A(u_j), u_j - u\rangle \leq 0$, then,
$\liminf\langle A(u_j), u_j - v\rangle \geq \langle A(u), u - v\rangle$, as $j \longrightarrow +\infty$.

We give the following Theorem

Theorem 4. *We assume that* K *is a convex closed bounded noempty. Let* A *is a operator pseudo-monotone from* K *into* V'*. Then for* $f \in V'$*, there exists* u *in* K *such that*

$$\langle A(u), v - u\rangle \geq \langle f, v - u\rangle,\ \forall v \in K.$$

The proof of the Theorem is in ([3], P. 245).

Let

$$B_R = \{v \mid v \in V,\ \|v\|_V \leq R\},\ K_R = K \bigcap B_R,$$

since K_R is a closed convex and bounded, then from Theorem 4, there exists $u_R \in K_R$ such that

$$\langle A(u_R), v - u_R\rangle \geq \langle f, v - u_R\rangle\ \forall v \in K_R, \tag{11}$$

choosing $R \geq R_0$ such that $\|v_0\|_V \leq R_0$. Then by taking $v = v_0$ in (11) we deduce from (10), that

$$\|u_R\|_V \leq C$$

and u_R is solution of (11), we have $\|u_R\| \leq C$ and if choosing $R > C$, then u_R is solution of

$$\langle A(u), v - u\rangle \geq \langle f, v - u\rangle,\ \forall v \in K.$$

indeed, we have $\|u_R\|_V \leq C$, then $A(u_R)$ remain in a bounded set of V' and there exists a subsequence $R \longrightarrow \infty$ such that

$$u_R \longrightarrow u \text{ weakly in } V,\ A(u_R) \longrightarrow \chi \text{ weakly in } V',$$

since K is weakly closed, $u \in K$. We have

$$\langle A(u_R), u_R - u\rangle \leq \langle f, u_R - u\rangle$$

on $R \geq \|u\|_V = C$, then

$$\limsup \langle A(u_R), u_R - u\rangle \leq 0$$

and from the pseudo-monotone,

$$\liminf \langle A(u_R), u_R - v\rangle \geq \langle A(u), u - v\rangle \tag{12}$$

and since

$$\langle A(u_R), u_R - v\rangle \leq \langle f, u_R - v\rangle \longrightarrow \langle f, u - v\rangle \ \forall v \in K,$$

we deduce from (12) that

$$\langle A(u), u - v\rangle \leq \langle f, u - v\rangle \ \forall v \in K,$$

then we have

$$\langle A(u), v - u\rangle \geq \langle f, v - u\rangle$$

□

3. Main Results

3.1. Existence and Uniqueness of Solutions

Theorem 5. *Assume that Ω be a bounded open. We give f, u_0, u_1 with*

$$f \in L^2(Q), \tag{13}$$

$$u_0 \in H_0^1(\Omega), \tag{14}$$

$$u_1 \in L^2(\Omega). \tag{15}$$

There exists a unique solution u satisfies

$$u \in L^\infty(0, T; H_0^1(\Omega)), \tag{16}$$

$$\frac{\partial u}{\partial t} \in L^\infty(0, T; L^2(\Omega)), \tag{17}$$

$$\frac{\partial^2 u}{\partial t^2} + \eta \frac{\partial u}{\partial t} - \Delta u = f \ in \quad Q, \tag{18}$$

$$u(0) = u_0, \tag{19}$$

$$\frac{\partial u}{\partial t}(0) = u_1. \tag{20}$$

Proof. Proof of the existence. The proof of this Theorem will be made in three steps.

Step 1: The existence is proved by the Faedo - Galerkin method, in ([3]), we take $u' = \frac{\partial u}{\partial t}$, $u'' = \frac{\partial^2 u}{\partial t^2}$, since the space $V = H_0^1(\Omega)$ is a separable space. We introduce a sequence of functions $w_1, \cdots, w_m, \cdots$, such that, $w_i \in H_0^1(\Omega)$, for any $i = 1, 2, \cdots, m, \cdots$ and for any m, $w_1, \cdots, w_m$ are linearly independent elements of $H_0^1(\Omega)$, the finite linear combinations of w_i are dense in the space $H_0^1(\Omega)$.

We find an approximate solution $u_m = u_m(t)$ of $(18)-(19)-(20)$ as follows

$$u_m(t) = \Sigma_{i=1}^m g_{im}(t) w_i, \tag{21}$$

where g_{im}, are obtained by the conditions

$$\begin{cases} (u''_m(t), w_j) + \eta(u'_m(t), w_j) + a(u_m(t), w_j) = (f(t), w_j), \\ \hfill \text{for } j = 1, \cdots, m, \end{cases} \tag{22}$$

where

$$a(u, v) = \Sigma_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx \tag{23}$$

the system (22) of linear ordinary differential equations is given with the initial conditions, as $m \longrightarrow +\infty$

$$u_m(0) = u_{0m}, \ u_{0m} = \Sigma_{i=1}^m \alpha_{im} w_i \longrightarrow u_0 \text{ in } H_0^1(\Omega) \tag{24}$$

$$u'_m(0) = u_{1m}, \ u_{1m} = \Sigma_{i=1}^m \beta_{im} w_i \longrightarrow u_1 \text{ in } L^2(\Omega) \tag{25}$$

From the linearly independent of $w_1, \cdots, w_m$, we have
$det(w_i, w_j) \neq 0, \ i = 1, \cdots, m$ and $j = 1, \cdots, m$ then from the general results on the systems of differential equations, these results guarantees the existence of a solution of $(22) - (24) - (25)$ in the intervalle $[0, t]$. The following a priori estimates show that $t = T$.

Step 2: We multiply (22) by g'_{jm}, add these relations for
$j = 1, \cdots, m$, which gives

$$(u''_m(t), u'_m(t)) + \eta(u'_m(t), u'_m(t)) + a(u_m(t), u'_m(t)) \quad =$$

$$(f(t), u'_m(t)) \tag{26}$$

then

$$\frac{1}{2}\frac{d}{dt}\int_\Omega [(u'_m(t))^2 + a(u_m(t), u_m(t))]dx + \eta \int_\Omega (u'_m(t))^2 dx = (f(t), u'_m(t)) \tag{27}$$

then after an integration on t and using Cauchy-Schwarz inequality, we obtain

$$\frac{1}{2}[\|u'_m(t)\|^2 + \|u_m(t)\|_V^2] + \eta \int_0^t \|u'_m(\sigma)\|^2 d\sigma \leq \frac{1}{2}[\|u_{1m}\|^2 + \|u_{0m}\|_V^2] + \int_0^t \|f(\sigma)\|\|u'_m(\sigma)\| d\sigma \tag{28}$$

then

$$\frac{1}{2}[\|u'_m(t)\|^2 + \|u_m(t)\|_V^2] + \eta \int_0^t \|u'_m(\sigma)\|^2 d\sigma \leq C + \frac{1}{2}\int_0^t \|f(\sigma)\|^2 d\sigma + \frac{1}{2}\int_0^t \|u'_m(\sigma)\|^2 d\sigma \tag{29}$$

where $C > 0$ is independent of m from (13) we have

$$\int_0^T \|f(\sigma)\|^2 d\sigma \leq C$$

we conclude that

$$\frac{1}{2}[\|u'_m(t)\|^2 + \|u_m(t)\|_V^2] + \eta \int_0^t \|u'_m(\sigma)\|^2 d\sigma \leq C + \frac{1}{2}\int_0^t \|u'_m(\sigma)\|^2 d\sigma \tag{30}$$

we use the Lemma 1, we obtain

$$\frac{1}{2}[\|u'_m(t)\|^2 + \|u_m(t)\|_V^2] + \eta \int_0^t \|u'_m(\sigma)\|^2 d\sigma \leq C \tag{31}$$

then if $\eta \geq 0$ we obtain

$$\|u'_m(t)\| \leq C \text{ and } \|u_m\|_V \leq C \tag{32}$$

if $\eta < 0$ we conclude from (30) that

$$\frac{1}{2}[\|u'_m(t)\|^2 + \|u_m\|_V^2] \leq C + (\frac{1}{2} - \eta) \int_0^t \|u'_m(\sigma)\|^2 d\sigma$$

we use the Lemma 1, we obtain

$$\frac{1}{2}[\|u'_m(t)\|^2 + \|u_m(t)\|_V^2] \leq C$$

then we obtain

$$\|u'_m(t)\| \leq C \text{ and } \|u_m\|_V \leq C \tag{33}$$

then we conclude that $t = T$, from $(32) - (33)$ we obtain the result, letting $m \longrightarrow +\infty$,

$$\begin{aligned} u_m \text{ remain in a bounded set of } L^\infty(0, T; H_0^1(\Omega)) \\ \text{and } u'_m \text{ remain in a bounded set of } L^\infty(0, T; L^2(\Omega)) \end{aligned} \tag{34}$$

Step 3: Pass to the limit

From Dunford-pettis theorem in ([16]) to show that the space $L^\infty(0, T; H_0^1(\Omega)$ be a given with dual $L^1(0, T; H^{-1}(\Omega))$ and the space $L^\infty(0, T; L^2(\Omega))$ be a given with dual $L^1(0, T; L^2(\Omega))$ by a consequence there exists a subsequence u_μ of u_m such that

$$u_\mu \longrightarrow u \text{ weakly star in } L^\infty(0, T; H_0^1(\Omega)) \tag{35}$$

$$\text{and } u'_\mu \longrightarrow u' \text{ weakly star in } L^\infty(0, T; L^2(\Omega)) \tag{36}$$

from (34) to show that u_m is a bounded in $L^2(0, T; H_0^1(\Omega))$ and u'_m is a bounded in $L^2(0, T; L^2(\Omega))$. Then to show that u_m remain in a bounded set of $H^1(Q)$. Then from Rellich-Kondrachoff theorem in ([4]) to show that

$$\text{the injection of } H^1(Q) \text{ in } L^2(Q) \text{ is compact}$$

we assume that subsequence u_μ of u_m satisfies $(35) - (36)$

$$u_\mu \longrightarrow u \text{ strongly in } L^2(\Omega) \text{ and almost everywhere}$$

pass to the limit in (22) and using for $m = \mu$, let j is a fixed and $\mu > j$, then from (22)

$$(u''_\mu, w_j) + \eta(u'_\mu, w_j) + a(u_\mu, w_j) = (f, w_j) \tag{37}$$

from $(35) - (36)$ we have

$$a(u_\mu, w_j) \longrightarrow a(u, w_j) \text{ weakly star in } L^\infty(0, T; H_0^1(\Omega)),$$

$$(u'_\mu, w_j) \longrightarrow (u', w_j) \text{ weakly star in } L^\infty(0,T;L^2(\Omega)),$$

and

$$(u''_\mu, w_j) = \frac{d}{dt}(u'_\mu, w_j) \longrightarrow (u'', w_j) \text{ in } \mathscr{D}'(0,T)$$

we conclude from (37) that

$$\frac{d^2}{dt^2}(u, w_j) + \eta\frac{d}{dt}(u, w_j) + a(u, w_j) = (f, w_j) \tag{38}$$

this for j is a fixed arbitrary. We multiply (38) by g_{jm}, add these relation for $j = 1, \cdots, m$ we conclude that

$$\frac{d^2}{dt^2}(u, v) + \eta\frac{d}{dt}(u, v) + a(u, v) = (f, v), \ \forall v \in V \tag{39}$$

then u satisfies (18) and (17) – (16). For show that (19) is satisfying, from (35) – (36) and the Lemma 2 we have, $u_\mu(0) \longrightarrow u(0)$ weakly in $L^2(\Omega)$, and from (24), $u_\mu(0) = u_{0\mu} \longrightarrow u_0$ in $H_0^1(\Omega)$, then we have (19). For show that (20) is satisfying we prove the Lemma

Lemma 6. *Let Q be a bounded open set in $\mathbb{R}^n_x \times \mathbb{R}_t$, u_μ and u are functions in $L^q(Q)$, $1 < q < \infty$, such that*

$$\|u_\mu\|_{L^q(Q)} \le C, \ u_\mu \longrightarrow u \text{ almost everywhere in } Q$$

then $u_\mu \longrightarrow u$ weakly in $L^q(Q)$.

Proof. Suppose that on a measurable set E, we note that
$1 \le p \le +\infty$, $1 \le q \le +\infty$ are conjugated, $\frac{1}{p} + \frac{1}{q} = 1$, let v_μ be a sequence of $L^p(E)$, tending to v, weakly in $L^p(E)$, as $\mu \longrightarrow +\infty$, if $\lim \int_E v_\mu \xi dx = \int_E v\xi dx$, for any $\xi \in L^p(E)$.
Let N is an increasing sequence tending to $+\infty$, we introduce

$$E_N = \{(x,t) \mid (x,t) \in Q, \ |u_\mu(x,t) - u(x,t)| \le 1, \text{ for } \mu \ge N\}$$

E_N are measurable set increases with N and measure$(E_N) \longrightarrow$ measure(Q), as $N \longrightarrow +\infty$. Let Φ_N the set of functions φ in $L^{q'}(Q)$, such that $L^{q'}(Q)$ denote the conjugate space of $L^q(Q)$,
$\frac{1}{q} + \frac{1}{q'} = 1$, with a support in E_N and let $\Phi = \bigcup_{N \longrightarrow +\infty} \Phi_N$, Φ is dense in $L^q(Q)$. If we take $\varphi \in \Phi$, then from Lebesgue dominated convergence theorem we obtain

$$\int_0^T \int_\Omega \varphi(x,t)(u_\mu(x,t) - u(x,t))dxdt \longrightarrow 0, \text{ as } \mu \longrightarrow +\infty \tag{40}$$

indeed, we have u_μ and u are functions in $L^q(Q)$, $1 < q < \infty$, such that $\|u_\mu\|_{L^q(Q)} \leq C$, $u_\mu \longrightarrow u$ almost everywhere in Q, since $|\varphi(x,t)(u_\mu(x,t) - u(x,t))| \leq |\varphi(x,t)|$, and $\varphi \in \Phi_{N_0}$, we take $\mu \geq N_0$, then $\varphi(x,t)(u_\mu(x,t) - u(x,t)) \longrightarrow 0$ almost everywhere, as Φ is dense in $L^{q'}(Q)$, then $\int_0^T \int_\Omega \varphi(x,t)u_\mu(x,t)dxdt \longrightarrow$ $\int_0^T \int_\Omega \varphi(x,t)u(x,t)dxdt$, as $\mu \longrightarrow +\infty$, then $u_\mu \longrightarrow u$ weakly in $L^q(Q)$. □

From (40),

$$(u''_\mu, w_j) \longrightarrow (u'', w_j), \text{ weakly star in } L^\infty(0,T;L^2(\Omega))$$

then from Lemma 2 with $X = \mathbb{R}$

$$(u'_\mu(0), w_j) \longrightarrow (u', w_j)\mid_{t=0} = (u'(0), w_j)$$

and from (25),

$$(u'_\mu(0), w_j) \longrightarrow (u_1, w_j),$$

then

$$(u'(0), w_j) = (u_1, w_j),$$

for any j, then we have (20). □

Remark 7. From (2) and (3), $u = 0$ on Σ, then the condition $u = 0$ on Σ is satisfies in (16).
In ([4]) from (16) – (17) and Lemma 2, we obtain that u is continuous from $[0,T]$ into $L^2(\Omega)$ then (19) has a sense. To verify that (20) has a sense, using the equation (18) can be written as

$$\frac{\partial^2 u}{\partial t^2} = f + \Delta u - \eta \frac{\partial u}{\partial t} \tag{41}$$

since $\Delta \in \mathscr{L}(H_0^1(\Omega), H^{-1}(\Omega))$, we have $\Delta u \in L^\infty(0,T;H^{-1}(\Omega))$ from (13) – (17) we have $f \in L^2(0,T;L^2(\Omega))$, $\frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega)$
from (41) we obtain that

$$\frac{\partial^2 u}{\partial t^2} \in L^2(0,T;L^2(\Omega)) + L^\infty(0,T;H^{-1}(\Omega) + L^2(\Omega))$$

then

$$\frac{\partial^2 u}{\partial t^2} \in L^2(0,T;H^{-1}(\Omega) + L^2(\Omega))$$

since $\frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega))$, and from Lemma 2, that $\frac{\partial u}{\partial t}$ is continuous from $[0,T]$ into $H^{-1}(\Omega) + L^2(\Omega)$, such that (20) has a sense.

Proof of the Uniqueness

Theorem 8. *The equations defined by* (18) – (19) – (20) *in Theorem* 5, *has a unique solution* u.

Proof. Assume that u, v are two solutions of (18) – (19) – (20), then $w = u - v$ satisfies

$$\frac{\partial^2 w}{\partial t^2} + \eta \frac{\partial w}{\partial t} - \Delta w = 0, \tag{42}$$

$$w(0) = 0, \quad w'(0) = 0, \tag{43}$$

$$w \in L^\infty(0, T; H_0^1(\Omega)), \tag{44}$$

$$w' \in L^\infty(0, T; L^2(\Omega)). \tag{45}$$

by taking $w' = \frac{\partial w}{\partial t}$, $w'' = \frac{\partial^2 w}{\partial t^2}$
we multiply (42) by w', we obtain that

$$(w'', w') + \eta(w', w') - (\Delta w, w') = 0, \tag{46}$$

by using integration by parts and (43), we obtain

$$\frac{d}{dt}[\|w'(t)\|^2 + \|w(t)\|_V^2] + 2\eta\|w'(t)\|^2 = 0 \tag{47}$$

integration from 0 to t and from (43) we obtain

$$\|w'(t)\|^2 + \|w(t)\|_V^2 + 2\eta \int_0^t \|w'(\sigma)\|^2 d\sigma = 0 \tag{48}$$

if $\eta \geq 0$, we obtain that

$$\|w'(t)\|^2 + \|w(t)\|_V^2 \leq 0 \tag{49}$$

and $w(t) = 0$, $\forall t \in [0, T]$
if $\eta < 0$, we write

$$\|w'(t)\|^2 + \|w(t)\|_V^2 \leq -2\eta \int_0^t (\|w'(\sigma)\|^2 + \|w(\sigma)\|_V^2) d\sigma$$

using the Lemma 1, we obtain (49), then $w(t) = 0$, $\forall t \in [0, T]$.
To justify the previous conclusion we use Method in linear hyperbolic equations. Let $s \in]0, T]$, we introduce

$$\psi(t) = \begin{cases} -\int_t^s w(\sigma) d\sigma, & t \leq s; \\ 0, & t > s. \end{cases}$$

and $\psi(t) = w_1(t) - w_1(s)$ if $t \leq s$ such that $w_1(t) = \int_0^t w(\sigma)d\sigma$. We multiply (42) by $\psi(t)$ we obtain

$$\int_0^s (w'', \psi)dt + \eta \int_0^s (w', \psi)dt - \int_0^s (\Delta w, \psi)dt = 0$$

by using integration by parts and (43), we obtain

$$-\int_0^s (w', \psi')dt + \eta \int_0^s (w', \psi)dt + \int_0^s a(w, \psi)dt = 0$$

then, since $\psi' = w$ and $\psi(0) = -w_1(s)$

$$-\frac{1}{2}\|w(s)\|^2 - \eta \int_0^s \|\psi'\|^2 dt - \frac{1}{2}\|w_1(s)\|_V^2 = 0$$

then

$$\frac{1}{2}\|w(s)\|^2 + \eta \int_0^s \|\psi'\|^2 dt + \frac{1}{2}\|w_1(s)\|_V^2 = 0$$

if $\eta \geq 0$, we obtain

$$\|w(s)\|^2 + \|w_1(s)\|_V^2 \leq 0, \tag{50}$$

and $w(t) = 0$ for $t \in [0, T]$
if $\eta < 0$, we write

$$\|w(s)\|^2 + \|w_1(s)\|_V^2 \leq -2\eta \int_0^s (\|\psi'(t)\|^2 + \|w_1(t)\|_V^2)dt$$

using the Lemma 1, we obtain (50), then $w(t) = 0$ for $t \in [0, T]$.

Remark 9. In the case of the uniqueness the sequence u_m of approximate solutions converges to u.

□

3.2. A Regularity Result

Theorem 10. *The hypotheses are those of theorem* 5 *with another*

$$\frac{\partial f}{\partial t} \in L^2(Q), \tag{51}$$

$$u_0 \in H_0^1(\Omega) \bigcap H^2(\Omega), \tag{52}$$

$$u_1 \in H_0^1(\Omega), \tag{53}$$

then there exists a unique solution u *of* (18) – (19) – (20) *satisfies*

$$u \in L^{\infty}(0,T;H_0^1(\Omega)\bigcap H^2(\Omega)), \tag{54}$$

$$u' \in L^{\infty}(0,T;H_0^1(\Omega)), \tag{55}$$

$$u'' \in L^{\infty}(0,T;L^2(\Omega)). \tag{56}$$

Proof. **Proof of the Existence**
The Proof of this Theorem will be made in two steps, the existence is proved by the Faedo-Galerkin method, in ([3]).
From the approximate solution u_m of (22) – (24) – (25), we take $w_j,\ j = 1,2,\cdots,m$ is a basis in the space $H_0^1(\Omega)\bigcap H^2(\Omega)$,
from (24) – (25) we assume that

$$u_{0m} \longrightarrow u_0 \text{ in } H_0^1(\Omega)\bigcap H^2(\Omega), \tag{57}$$

$$u_{1m} \longrightarrow u_1 \text{ in } H_0^1(\Omega). \tag{58}$$

We prove in step 1 an additional a priori estimate to show that the existence of a solution with (55) – (56), and we prove (54) in step 2 by using the equation (18).
Step 1: We deduce from (22) that

$$(u_m''(0) + \eta u_m'(0), w_j) = (f(0) + \Delta u_{0m}, w_j),\ j = 1,\cdots,m \tag{59}$$

from (51) and Lemma 2, $f(0) \in L^2(\Omega)$, and from (57)

$$\|\Delta u_{0m}\| = (\int_{\Omega}(\Delta u_{0m})^2 dx)^{\frac{1}{2}} < C,$$

we multiply (59) by $g_{jm}''(0)$ and add these relations for $j = 1,\cdots,m$ which gives

$$\int_{\Omega}(u_m''(0))^2 dx = (f(0) + \Delta u_{0m}, u_m''(0)) - \eta(u_m'(0), u_m''(0))$$

by using the Cauchy-Schwarz inequality, we obtain

$$\int_{\Omega}(u_m''(0))^2 dx \le [(\int_{\Omega}(f(0))^2 dx)^{\frac{1}{2}} + (\int_{\Omega}(\Delta u_{0m})^2 dx)^{\frac{1}{2}} + \eta(\int_{\Omega}(u_m'(0))^2 dx)^{\frac{1}{2}}](\int_{\Omega}(u_m''(0))^2 dx)^{\frac{1}{2}}$$

then we conclude by using (57) − (58) and $f(0) \in L^2(\Omega)$ that

$$(\int_\Omega (u''_m(0))^2 dx)^{\frac{1}{2}} \leq C. \tag{60}$$

by differentiating (22) with respect to t, we obtain

$$\begin{cases} (u'''_m(t), w_j) + \eta(u''_m(t), w_j) + a(u'_m(t), w_j) = (f'(t), w_j) \\ \hfill \text{for } j = 1, \cdots, m, \end{cases} \tag{61}$$

we multiply (61) by $g''_{jm}(t)$, add these relations for $j = 1, \cdots, m$,
we obtain

$$(u'''_m(t), u''_m(t)) + \eta(u''_m(t), u''_m(t)) + a(u'_m(t), u''_m(t)) = (f'(t), u''_m(t))$$

then

$$\frac{1}{2}\frac{d}{dt}\int_\Omega [(u''_m(t))^2 + \|u'_m(t)\|_V^2]dx + \eta\int_\Omega (u''_m(t))^2 dx = (f'(t), u''_m(t))$$

integrating from 0 to t and using Cauchy-Schwarz inequality we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega [(u''_m(t))^2 + \|u'_m(t)\|_V^2]dx + \eta\int_0^t (\int_\Omega (u''(\sigma))^2 dx)d\sigma \leq \\ &\frac{1}{2}\int_\Omega [(u''_m(0))^2 + \|u'_m(0)\|_V^2]dx + \int_0^t \|f'(\sigma)\|\|u''_m(\sigma)\|d\sigma \leq \\ &\frac{1}{2}\int_\Omega [(u''_m(0))^2 + \|u'_m(0)\|_V^2]dx + \frac{1}{2}\int_0^t (\int_\Omega (f'(\sigma))^2 dx)d\sigma + \\ &\hfill \frac{1}{2}\int_0^t (\int_\Omega (u''_m(\sigma))^2 dx)d\sigma \end{aligned}$$

by using (51) − (58) − (60), we obtain

$$\begin{aligned} &\frac{1}{2}[\|u''_m(t)\|^2 + \|u'_m\|_V^2] + \eta\int_0^t (\int_\Omega (u''_m(\sigma))^2 dx)d\sigma \leq \\ &\hfill C + \frac{1}{2}\int_0^t (\int_\Omega (u''_m(\sigma))^2 dx)d\sigma \end{aligned} \tag{62}$$

we use the Lemma 1, we obtain

$$\frac{1}{2}\int_\Omega [(u''_m(t))^2 + \|u'_m(t)\|_V^2]dx + \eta\int_0^t (\int_\Omega (u''_m(\sigma))^2 dx)d\sigma \leq C \tag{63}$$

then if $\eta \geq 0$ we obtain

$$\|u'_m\|_V \leq C \text{ and } \|u''_m\| \leq C \tag{64}$$

if $\eta < 0$ from (62) we obtain

$$\frac{1}{2}[\|u''_m(t)\|^2 + \|u'_m(t)\|_V^2] \leq C + (\frac{1}{2} - \eta)\int_0^t (\int_\Omega (u''_m(\sigma))^2 dx) d\sigma \tag{65}$$

we use the Lemma 1 we obtain

$$\frac{1}{2}[\|u''_m(t)\|^2 + \|u'_m\|_V^2] \leq C$$

then we obtain

$$\|u'_m\|_V \leq C \; and \; \|u''_m(t)\| \leq C \tag{66}$$

then from $(64)-(66)$ we obtain the result

$$\begin{gathered} u'_m \text{ remain in a bounded set of } L^\infty(0,T;H_0^1(\Omega)), \\ u''_m \text{ remain in a bounded set of } L^\infty(0,T;L^2(\Omega)). \end{gathered}$$

Then there exists a subsequence u_μ of u_m as in the proof of the existence Theorem 5 such that, u satisfies $(55)-(56)$, then we have $u \in L^\infty(0,T;H_0^1(\Omega))$ from the theorem 5, for proved (54), to verify that

$$u \in L^\infty(0,T;H^2(\Omega)) \tag{67}$$

Step 2: We prove (67).
We conclude from (18) that

$$\Delta u = u'' + \eta u' - f \tag{68}$$

from (13) and (51), $f \in L^\infty(0,T;L^2(\Omega))$ and with (56) we conclude from (68) that

$$\Delta u \in L^\infty(0,T;L^2(\Omega))$$

put

$$\Delta u = h,$$

since $\Delta : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ is an isomorphism with continuous inverse $G = \Delta^{-1}$, and since $u \in L^\infty(0,T;H_0^1(\Omega))$ we have

$$u(t) = Gh(t) \text{ almost everywhere} \tag{69}$$

in ([4], [6]) we have the theorems for regularity of solutions of linear elliptic equations are given

$$G \in \mathscr{L}(L^2(\Omega), H^2(\Omega)) \tag{70}$$

then from (69) – (70), we obtain

$$u \in L^\infty(0, T; H^2(\Omega)).$$

□

Proof of the Uniqueness

Theorem 11. *The solution u in the theorem 10 is a unique.*

Proof. Assume that u, v are two solutions given in the theorem 10, then $w = u - v$ satisfies

$$\frac{\partial^2 w}{\partial t^2} + \eta \frac{\partial w}{\partial t} - \Delta w = 0 \tag{71}$$

$$w(0) = 0, \quad w'(0) = 0, \tag{72}$$

$$w \in L^\infty(0, T; H_0^1(\Omega) \bigcap H^2(\Omega)), \tag{73}$$

$$w' \in L^\infty(0, T; H_0^1(\Omega)) \tag{74}$$

$$w'' \in L^\infty(0, T; L^2(\Omega)) \tag{75}$$

we multiply (71) by w', we obtain

$$(w'', w') + \eta(w', w') - (\Delta w, w') = 0$$

by integration by parts and (72), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega [(w'(t))^2 + \|w(t)\|_V^2] dx + \eta \int_\Omega (w'(t))^2 dx = 0$$

integration from 0 to t and from (72) we obtain

$$\frac{1}{2}\int_\Omega [(w'(t))^2 + \|w(t)\|_V^2] dx + \eta \int_0^t (\int_\Omega (w'(\sigma))^2 dx) d\sigma = 0$$

if $\eta \geq 0$, we obtain that

$$\int_\Omega [(w'(t))^2 + \|w(t)\|_V^2] dx \leq 0, \tag{76}$$

then $w(t) = 0$, for $t \in [0, T]$
if $\eta < 0$, we write

$$\int_\Omega [(w'(t))^2 + \|w(t)\|_V^2]dx \le -2\eta \int_0^t (\int_\Omega [(w'(\sigma))^2 + \|w(\sigma)\|_V^2]dx)d\sigma$$

using the Lemma 1, we obtain (76), then $w(t) = 0$ for $t \in [0, T]$.
For justify the previous conclusion the uniqueness applying the same argument in uniqueness Theorem 8, then the sequence u_m of approximate solutions converges to u. □

3.3. Semi Discretization and Variational inequalities

In ([3], P. 432), we apply semi-discretization in time, to establish the existence and uniqueness of solution, then
we give the following Theorem.
Such that from (8) we have

$$V \subset H \equiv H' \subset V' \tag{77}$$

and from (7), we have

$$-\Delta \in \mathscr{L}(V, V'),\ \langle -\Delta v, v\rangle \ge \alpha\|v\|_V^2,\ 0 \le \alpha \le 1,\ v \in V. \tag{78}$$

Theorem 12. *We assume that* (77) – (78) *are satisfied.*
Let K is a closed convex set in V. We give

$$f \in L^2(0, T; H), \tag{79}$$

$$u_0 \in K, \tag{80}$$

$$u_1 \in H. \tag{81}$$

There exists a unique solution u, such that

$$u \in \mathscr{C}([0, T]; V) \bigcap \mathscr{C}^1([0, T]; H). \tag{82}$$

$$u(t) \in K,\ \textit{almost everywhere},\ u'(t) \in H. \tag{83}$$

And in ***the cases 1.,2.*** *We obtain the inequality*

$$\int_0^T \langle u''(t) + \eta u'(t) - \Delta u(t) - f(t), v(t) - u(t)\rangle dt +$$

$$\|v^1 - u^1\|^2 + \Sigma_{n=1}^{N-1}\|v^{n-1} - u^{n-1} - (v^n - u^n)\|^2 \geq 0 \tag{84}$$

$\forall v \in \mathscr{C}([0,T];V),\ v' \in \mathscr{C}([0,T];H),\ v'' \in L^2(0,T,V'),$
$v(t) \in K$, *almost everywhere,* $v'(t) \in H$.
And in ***the cases 2., 3.*** *We obtain the inequality*

$$\int_0^T \langle u''(t) + \eta u'(t) - \Delta u(t) - f(t), v(t) - u(t)\rangle dt + \Sigma_{n=1}^{N-1}\|v^{n+1} - u^{n+1} - (v^n - u^n)\|^2 + \Sigma_{n=1}^{N-1}\|v^{n-1} - u^{n-1} - (v^n - u^n)\|^2 \geq 0 \tag{85}$$

$\forall v \in \mathscr{C}([0,T];V),\ v' \in \mathscr{C}([0,T];H),\ v'' \in L^2(0,T;V'),$
$v(t) \in K,$ *almost everywhere,* $v'(t) \in H$.
And in ***the cases 2.,4.,5.*** *We obtain the inequality*

$$\int_0^T \langle u''(t) + \eta u'(t) - \Delta u(t) - f(t), v(t) - u(t)\rangle dt + (\frac{1+\eta k}{2})\|v^1 - u^1\|^2 + (\frac{1+\eta k}{2})\Sigma_{n=1}^{N-1}\|v^{n+1} - u^{n+1} - (v^n - u^n)\|^2 + \Sigma_{n=1}^{N-1}\|v^{n-1} - u^{n-1} - (v^n - u^n)\|^2 \geq 0 \tag{86}$$

$\forall v \in \mathscr{C}([0,T];V),\ v' \in \mathscr{C}([0,T];H),\ v'' \in L^2(0,T;V'),$
$v(t) \in K,$ *almost everywhere,* $v'(t) \in H$.

Proof. **Semi-discretization in time** We introduce

$$k = \triangle t = \frac{T}{N} \tag{87}$$

N is an integer fixed in $\mathbb{N}$ and u^n an approximation to u at time nk. We introduce

$$f^n = \frac{1}{k}\int_{nk}^{(n+1)k} f(\sigma)d\sigma,\ n \geq 1; \tag{88}$$

we take

$$u^0 = u_0 \tag{89}$$

and we define u_n by

$$\begin{cases} (\frac{u^{n+1}-2u^n+u^{n-1}}{k^2} + \eta\frac{u^{n+1}-u^n}{k}, v - u^n) + \\ \langle -\Delta u^n - f^n, v - u^n\rangle \geq 0 \\ \forall v \in K,\ \forall v' \in H,\ u^n \in K,\ n = 1, \cdots, N-1. \end{cases} \tag{90}$$

The system (90) is an elliptic variational inequality, has a unique solution. Indeed, (90) is equivalent to, for any $v \in K,\ v' \in H$

$$\langle -\Delta u^n - (\frac{2}{k^2} + \frac{\eta}{k})u^n, v - u^n\rangle \geq (f^n - (\frac{1}{k^2} + \frac{\eta}{k})u^{n+1} - \frac{1}{k^2}u^{n-1}, v - u^n),$$

then by applying the Theorem 3, to the operator

$$-\Delta - (\frac{2}{k^2} + \frac{\eta}{k})I.$$

Where I is the identity operator, then we prove that operator
$-\Delta - (\frac{2}{k^2} + \frac{\eta}{k})I$ is coercive, we have
$\langle -\Delta v, v_0\rangle = a(v, v_0) = \Sigma_{i=1}^N \int_\Omega \frac{\partial v}{\partial x_i}\frac{\partial v_0}{\partial x_i}dx$, $v_0 \in K$, and from (4) – (7) we obtain that

$$\langle -(\Delta + (\frac{2}{k^2} + \frac{\eta}{k}))v, v - v_0\rangle =$$
$$\langle -\Delta v, v\rangle - \langle -\Delta v, v_0\rangle - ((\frac{2}{k^2} + \frac{\eta}{k})v, v - v_0) \geq$$
$$\alpha\|v\|_V^2 - a(v, v_0) - ((\frac{2}{k^2} + \frac{\eta}{k})v, v - v_0)$$

then for any $v \in K$

$$\frac{\langle -(\Delta + (\frac{2}{k^2} + \frac{\eta}{k}))v, v - v_0\rangle}{\|v\|_V} \longrightarrow +\infty$$
$$\text{as } \|v\|_V \longrightarrow +\infty$$

and we prove that operator $-\Delta - (\frac{2}{k^2} + \frac{\eta}{k})I$ is pseudo monotone, first we prove that operator is bounded then for any $u, v \in K$, from (5) we have

$$\langle -\Delta u - (\frac{2}{k^2} + \frac{\eta}{k})u, v\rangle = \langle -\Delta u, v\rangle - (\frac{2}{k^2} + \frac{\eta}{k})(u, v) \leq$$
$$C\|u\|_V\|v\|_V - (\frac{2}{k^2} + \frac{\eta}{k})(u, v) \leq C\|u\|_V\|v\|_V$$

then

$$\langle -\Delta u - (\frac{2}{k^2} + \frac{\eta}{k})u, v\rangle \leq C\|u\|_V\|v\|_V$$

then

$$\| - \Delta - (\frac{2}{k^2} + \frac{\eta}{k})\|_* \leq C$$

second if $u_j \longrightarrow u$ weakly in K, as $j \longrightarrow +\infty$, then
$\lim(-\Delta - (\frac{2}{k^2} + \frac{\eta}{k}))(u_j) = (-\Delta - (\frac{2}{k^2} + \frac{\eta}{k}))(u)$,
for any $-\Delta - (\frac{2}{k^2} + \frac{\eta}{k}) \in V'$ since

$$\limsup\langle(-\Delta - (\frac{2}{k^2} + \frac{\eta}{k}))(u_j), u_j - v\rangle =$$
$$\liminf\langle(-\Delta - (\frac{2}{k^2} + \frac{\eta}{k}))(u_j), u_j - v\rangle =$$
$$\langle(-\Delta - (\frac{2}{k^2} + \frac{\eta}{k}))(u), u - v\rangle$$

$\forall v \in K$, then

$$\liminf\langle(-\Delta - (\frac{2}{k^2} + \frac{\eta}{k}))(u_j), u_j - v\rangle \geq$$
$$\langle(-\Delta - (\frac{2}{k^2} + \frac{\eta}{k})(u), u - v\rangle$$

then by applying the Theorem 3, we deduce the system (90) has a unique solution. We say that the system (90) is a semi-discrete approximation of the inequalitys (84), (85), (86).

Proof of the Existence

We introduce

$$u_k(t) = u^n \text{ in } [nk, (n+1)k[,\ n = 0, \cdots, N-1 \tag{91}$$

then we prove the Lemma

Lemma 13. *We take $k \longrightarrow 0$, we have*

$$\begin{aligned} &u_k \textit{ remain in a bounded set of } L^\infty(0,T;V),\\ &u'_k \textit{ remain in a bounded set of } L^\infty(0,T;H). \end{aligned} \tag{92}$$

Proof. We refer to ([13]), ([3], P. 222), ([5]) for the proof of Lemma 13.

we want to solve the initial-value problem such that $\eta \in \mathbb{R}$

$$\frac{\partial^2 u}{\partial t^2} + \eta\frac{\partial u}{\partial t} - \Delta u = f, \text{ on }]0,T[, \tag{93}$$

$$u(0) = u_0,\ \frac{\partial u}{\partial t}(0) = u_1. \tag{94}$$

Where $f \in L^2(0,T;H),\ u_0 \in K,\ u_1 \in H$.

Since the space V is a separable space and consider a sequence of linearly independent elements of V, $w_1, \cdots, w_k, \cdots$, which is total in V. Using the Faedo- Galerkin method we define for each k an approximate solution u_k of (93) – (94)

$$u_k(t) = \Sigma_{i=1}^{k} g_{ik}(t) w_i, \tag{95}$$

then

$$\frac{\partial^2}{\partial t^2}(u_k, w_j) + \eta \frac{\partial}{\partial t}(u_k, w_j) + a(u_k, w_j) = (f, w_j), \qquad j = 1, \cdots, k, \tag{96}$$

$$u_k(0) = u_{0k}, \tag{97}$$

$$u_k'(0) = u_{1k}. \tag{98}$$

Where u_{0k} is the projection in V of u_0 on the space spanned by $w_1, \cdots, w_k$, and u_{1k} is the projection in V of u_1 on the space spanned by $w_1, \cdots, w_k$.
Equations $(95)-(96)-(97)-(98)$ are equivalent to a linear initial-value problem for an ordinary k-dimensional differential equation. They possess a unique solution defined for all time and in particular on $[0, T]$, the function u_k is in $\mathscr{C}([0, T]; V)$ and $u_k' \in \mathscr{C}([0, T]; V)$,
u_k'' is in $L^2(0, T; V)$.
A priori estimates are obtained by multiplying (96) by g_{jk}' and summing these relations for $j = 1, \cdots, k$. We obtain

$$(u_k'', u_k') + \eta(u_k', u_k') + a(u_k, u_k') = (f, u_k'). \tag{99}$$

$$\frac{d}{dt}[\|u_k'\|^2 + a(u_k, u_k)] + 2\eta\|u_k'\|^2 = 2(f, u_k') \leq \|f\|^2 + \|u_k'\|^2. \tag{100}$$

We use the Lemma 1, we obtain

$$u_k \text{ remain in a bounded set of } L^\infty(0, T; V),$$
$$u_k' \text{ remain in a bounded set of } L^\infty(0, T; H).$$

Then we have (92). □

Thus there exists a subsequence, remain denoted u_k, and u, such that

$$u \in L^\infty(0, T; V),\ u' \in L^\infty(0, T; H), \tag{101}$$

as $k \longrightarrow +\infty$,

$$u_k \longrightarrow u \text{ in } L^\infty(0, T; V) \text{ weak-star},$$

$$u'_k \longrightarrow u' \text{ in } L^\infty(0,T;H) \text{ weak-star }.$$

Then passing to the limit in $(95)-(96)-(97)-(98)$ we see that u is a solution of $(93)-(94)$ which satisfies (101).
To conclude the proof of existence it remains to show the continuity properties $u \in \mathscr{C}([0,T];V),\ u' \in \mathscr{C}([0,T];H)$.
We give the following Lemma

Lemma 14. *Let X and Y be two Banach spaces such that*

$$X \subset Y \tag{102}$$

with a continuous injection.
If a function φ belongings to $L^\infty(0,T;X)$ and is weakly continuous with values in Y, then φ is weakly continuous with values in X.

The proof of the Lemma 14 is in ([12]), and in ([14], Lemma 1.4, Chapter III).
It follows from the Lemma 14 and (101) that u is weakly continuous from $[0,T]$ in V. Similarly, we infer from (93) that

$$u'' = f - \eta u' + \Delta u$$

and $u'' \in L^2(0,T;V')$, since $f \in L^2(0,T;H),\ u' \in L^\infty(0,T;H)$, $u \in L^\infty(0,T;V)$ which implies $-\Delta u \in L^\infty(0,T;V')$.
We give the following Lemma

Lemma 15. *Let X be a given Banach space with dual X' and let u and g be two functions belonging to $L^1(0,T;X)$. Then the following three conditions are equivalent*

(i) *u is almost everywhere equal to a primitive function g, there exists $\xi \in X$ such that*

$$u(t) = \xi + \int_0^t g(s)ds, \text{ for almost everywhere } t \in [0,T]. \tag{103}$$

(ii) *For every test function $\varphi \in \mathscr{D}(]0,T[)$,*

$$\int_0^T u(t)\varphi'(t)dt = -\int_0^T g(t)\varphi(t)dt \tag{104}$$

(iii) *For each $\xi \in X'$,*

$$\frac{d}{dt}\langle u,\xi\rangle = \langle g,\xi\rangle \tag{105}$$

in the scalar distribution sense on $]0, T[$.
If (i)-(ii)-(iii) *are satisfied we say that* g *is the* X*-valued distribution derivative of* u*, and* u *is almost everywhere equal to a continuous function from* $[0, T]$ *into* X.

The proof of the Lemma 15 is in ([14], Lemma 1.1, Chapter III).
From Lemma 15, then shows that u is continuous from $[0, T]$ in V', Lemma 14 and (101) imply that u' is weakly continuous from $[0, T]$ in H.
We give the following Lemma

Lemma 16. *We assume that* w *is such that*

$$w \in L^2(0, T; V), \ w' \in L^2(0, T; H), \tag{106}$$

and

$$w'' - \Delta w \in L^2(0, T; H). \tag{107}$$

Then, after modification on a set of measure zero, u *is continuous from* $[0, T]$ *into* V*,* u' *is continuous from* $[0, T]$ *into* H *and, in the sense of distributions on* $]0, T[$,

$$(w'' - \Delta w, w') = \frac{1}{2}\frac{d}{dt}\{\|w'\|^2 + a(w, w)\}. \tag{108}$$

The proof of the Lemma 16 is in ([13], P.79).
We deduce from Lemma 16 that u satisfies an equation similar to (100), namely

$$\frac{d}{dt}[\|u'\|^2 + a(u, u)] + 2\eta\|u'\|^2 = 2(f, u')$$

This shows that the function

$$t \longmapsto \|u'(t)\|^2 + a(u(t), u(t))$$

is continuous on $[0, T]$. In conjunction with the above properties of weak continuity, we conclude that $u \in \mathscr{C}([0, T]; V)$ and
$u' \in \mathscr{C}([0, T]; H)$, then we have (82)

$$u \in \mathscr{C}([0, T]; V) \bigcap \mathscr{C}^1([0, T]; H)$$

For show that (83) we have if the set K is a closed and convex of functions $v \in \mathscr{C}([0, T]; V)$ such that $v(t) \in K$ almost everywhere, then we have
$u_k \in K$ for any k and since K is weakly closed in $\mathscr{C}([0, T]; V)$, we have $u \in K$.
And from (82) we have $u'(t) \in H$.
For show that (84), (85), (86), we consider the function v satisfies

$$v \in \mathscr{C}^2([0, T]; V), \ v(t) \in K, \ v'(t) \in H, \ \text{for any } t \in [0, T] \tag{109}$$

we put

$$v^n = v(nk),\ n = 0, \cdots, N-1,$$

$$v_k = \ll \text{step function defined by } v_k(t) = v^n \text{ in }]nk, (n+1)k[\gg$$

$$\tilde{v}_k = \ll \text{piecewise linear function, continuous in } [0, T]$$

such that

$$\tilde{v}_k(nk) = v^{n-1},\ n = 1, 2, \cdots \text{and } \tilde{v}_k(0) = v^0 \gg .$$

We note that

$$\int_0^T \left(\frac{d^2\tilde{v}_k}{dt^2} + \eta\frac{d\tilde{v}_k}{dt}, v_k - u_k\right)dt =$$
$$\Sigma_{n=0}^{N-1} \int_{nk}^{(n+1)k} \left(\frac{d^2\tilde{v}_k}{dt^2} + \eta\frac{d\tilde{v}_k}{dt}, v_k - u_k\right)dt =$$
$$\Sigma_{n=1}^{N-1}((v^{n+1} - 2v^n + v^{n-1}) + \eta(v^{n+1} - v^n), v^n - u^n), \tag{110}$$

and that

$$\int_0^T \langle -\Delta u_k, v_k - u_k\rangle dt = k^2\Sigma_{n=0}^{N-1}\langle -\Delta u^n, v^n - u^n\rangle. \tag{111}$$

We define

$$f_k = f^n \text{ in } [nk, (n+1)k[, \quad n = 0, \cdots, N-1, \tag{112}$$

we have

$$\int_0^T \langle f_k, v_k - u_k\rangle dt = k^2\Sigma_{n=0}^{N-1}\langle f^n, v^n - u^n\rangle. \tag{113}$$

We take $v = v^n$ in the system (90), and we multiply by k^2 we obtain that

$$((u^{n+1} - 2u^n + u^{n-1}) + \eta k(u^{n+1} - u^n), v^n - u^n) +$$
$$k^2\langle -\Delta u^n - f^n, v^n - u^n\rangle \geq 0, \tag{114}$$

then

$$((v^{n+1} - 2v^n + v^{n-1}) + \eta k(v^{n+1} - v^n), v^n - u^n) +$$
$$k^2\langle -\Delta u^n - f^n, v^n - u^n\rangle =$$
$$((u^{n+1} - 2u^n + u^{n-1}) + \eta k(u^{n+1} - u^n), v^n - u^n) +$$
$$k^2\langle -\Delta u^n - f^n, v^n - u^n\rangle +$$
$$\frac{1+\eta k}{2}[\|v^{n+1} - u^{n+1}\|^2 - \|v^n - u^n\|^2 -$$
$$\|v^{n+1} - u^{n+1} - (v^n - u^n)\|^2] +$$

$$\frac{1}{2}[\|v^{n-1}-u^{n-1}\|^2-\|v^n-u^n\|^2- \\ \|v^{n-1}-u^{n-1}-(v^n-u^n)\|^2] \quad (115)$$

From the properties of the norm in $L^2(\Omega)$, we have the inequality

$$\|v^{n+1}-u^{n+1}-(v^n-u^n)\|^2 \geq \|v^{n+1}-u^{n+1}\|^2-\|v^n-u^n\|^2 \quad (116)$$

if we have **the case 1.:** $\|v^{n+1}-u^{n+1}\|^2-\|v^n-u^n\|^2<0$,
and $\frac{1+\eta k}{2}<0$
by using **the case 1.,** on the member

$$\frac{1+\eta k}{2}[\|v^{n+1}-u^{n+1}\|^2-\|v^n-u^n\|^2-\|v^{n+1}-u^{n+1}-(v^n-u^n)\|^2]$$

of the equality (115) and using the inequality (116), we obtain the inequality

$$\frac{1+\eta k}{2}[\|v^{n+1}-u^{n+1}\|^2-\|v^n-u^n\|^2- \\ \|v^{n+1}-u^{n+1}-(v^n-u^n)\|^2] \geq \\ \|v^{n+1}-u^{n+1}\|^2-\|v^n-u^n\|^2 \quad (117)$$

if we have **the case 2.**:
$\|v^{n-1}-u^{n-1}\|^2-\|v^n-u^n\|^2<0$ or $\|v^{n-1}-u^{n-1}\|^2-\|v^n-u^n\|^2>0$
by using **the case 2.,** on the member

$$\frac{1}{2}[\|v^{n-1}-u^{n-1}\|^2-\|v^n-u^n\|^2-\|v^{n-1}-u^{n-1}-(v^n-u^n)\|^2]$$

of the equality (115) and using the inequality (116), we obtain the inequality

$$\frac{1}{2}[\|v^{n-1}-u^{n-1}\|^2-\|v^n-u^n\|^2- \\ \|v^{n-1}-u^{n-1}-(v^n-u^n)\|^2] \geq \\ -\|v^{n-1}-u^{n-1}-(v^n-u^n)\|^2 \quad (118)$$

by using the inequalitys (114) – (117) – (118), on the equality (115) we obtain

$$((v^{n+1}-2v^n+v^{n-1})+\eta k(v^{n+1}-v^n),v^n-u^n)+ \\ k^2\langle -\Delta u^n-f^n,v^n-u^n\rangle \geq \\ \|v^{n+1}-u^{n+1}\|^2-\|v^n-u^n\|^2- \\ \|v^{n-1}-u^{n-1}-(v^n-u^n)\|^2 \quad (119)$$

summing to n, we deduce

$$\Sigma_{n=1}^{N-1}[((v^{n+1}-2v^n+v^{n-1})+\eta k(v^{n+1}-v^n),v^n-u^n)+ k^2\langle -\Delta u^n-f^n,v^n-u^n\rangle]\geq \|v^N-u^N\|^2-\|v^1-u^1\|^2- \Sigma_{n=1}^{N-1}\|v^{n-1}-u^{n-1}-(v^n-u^n)\|^2\geq -\|v^1-u^1\|^2- \Sigma_{n=1}^{N-1}\|v^{n-1}-u^{n-1}-(v^n-u^n)\|^2 \quad (120)$$

with (110) – (111) – (113), we conclude of (120) the inequality

$$\int_0^T(\frac{d^2\tilde{v}_k}{dt^2}+\eta\frac{d\tilde{v}_k}{dt},v_k-u_k)dt+ \int_0^T\langle -\Delta u_k-f_k,v_k-u_k\rangle dt-k^2\langle -\Delta u^0,v^0-u^0\rangle+ k^2\langle f^0,v^0-u^0\rangle+\|v^1-u^1\|^2+ \Sigma_{n=1}^{N-1}\|v^{n-1}-u^{n-1}-(v^n-u^n)\|^2\geq 0 \quad (121)$$

as $k\longrightarrow 0$, $\frac{d^2\tilde{v}_k}{dt^2}\longrightarrow v''$ strongly in $L^2(0,T;V')$, and $\frac{d\tilde{v}_k}{dt}\longrightarrow v'$ strongly in $\mathscr{C}([0,T];H)$, $v_k\longrightarrow v$ strongly in $\mathscr{C}([0,T];V)$, $f_k\longrightarrow f$ in $L^2(0,T;H)$ and since $-\Delta$ pseudo-monotone, we have as $k\longrightarrow 0$

$$\liminf\int_0^T\langle -\Delta u_k,u_k\rangle dt\geq\int_0^T\langle -\Delta u,u\rangle dt,$$

and since $k^2f^0\longrightarrow 0$ in H as $k\longrightarrow 0$, we conclude of (121) that

$$\int_0^T[(v''+\eta v',v-u)+\langle -\Delta u-f,v-u\rangle]dt+\|v^1-u^1\|^2+ \Sigma_{n=1}^{N-1}\|v^{n-1}-u^{n-1}-(v^n-u^n)\|^2\geq 0 \quad (122)$$

for any v satisfies (109).
if v is given as in the inequality (84), there exists v_j satisfies the conditions (109) and such that $v_j\longrightarrow v$ weakly in $\mathscr{C}([0,T];V)$, $v_j'\longrightarrow v'$ weak-star in $\mathscr{C}([0,T];H)$, $v_j''\longrightarrow v''$ in $L^2(0,T;V')$.
We take $v=v_j$ in (122) and pass to the limit, we conclude that (84).
If we have **the case 3.:**
$\|v^{n+1}-u^{n+1}\|^2-\|v^n-u^n\|^2>0$ and $\frac{1+\eta k}{2}<0$
by using **the case 3.**, on the member

$$\frac{1+\eta k}{2}[\|v^{n+1}-u^{n+1}\|^2-\|v^n-u^n\|^2-\|v^{n+1}-u^{n+1}-(v^n-u^n)\|^2]$$

of the equality (115) and using the inequality (116), we obtain the inequality

$$\frac{1+\eta k}{2}[\|v^{n+1}-u^{n+1}\|^2-\|v^n-u^n\|^2- \\ \|v^{n+1}-u^{n+1}-(v^n-u^n)\|^2] \geq \\ -\|v^{n+1}-u^{n+1}-(v^n-u^n)\|^2 \tag{123}$$

by using the inequalitys (114) – (123) – (118) on the equality (115) we obtain

$$((v^{n+1}-2v^n+v^{n-1})+\eta k(v^{n+1}-v^n), v^n-u^n)+ \\ k^2\langle -\Delta u^n - f^n, v^n-u^n\rangle \geq \\ -\|v^{n+1}-u^{n+1}-(v^n-u^n)\|^2-\|v^{n-1}-u^{n-1}-(v^n-u^n)\|^2 \tag{124}$$

summing to n, we deduce

$$\Sigma_{n=1}^{N-1}[((v^{n+1}-2v^n+v^{n-1})+\eta k(v^{n+1}-v^n), v^n-u^n)+ \\ k^2\langle -\Delta u^n - f^n, v^n-u^n\rangle] \geq \\ -\Sigma_{n=1}^{N-1}\|v^{n+1}-u^{n+1}-(v^n-u^n)\|^2- \\ \Sigma_{n=1}^{N-1}\|v^{n-1}-u^{n-1}-(v^n-u^n)\|^2 \tag{125}$$

with (110) – (111) – (113), we conclude of (125) the inequality

$$\int_0^T (\frac{d^2\tilde{v}_k}{dt^2}+\eta\frac{d\tilde{v}_k}{dt}, v_k-u_k)dt+ \\ \int_0^T \langle -\Delta u_k - f_k, v_k-u_k\rangle dt- \\ k^2\langle -\Delta u^0, v^0-u^0\rangle + k^2\langle f^0, v^0-u^0\rangle + \\ \Sigma_{n=1}^{N-1}\|v^{n+1}-u^{n+1}-(v^n-u^n)\|^2+ \\ \Sigma_{n=1}^{N-1}\|v^{n-1}-u^{n-1}-(v^n-u^n)\|^2 \geq 0 \tag{126}$$

as $k \longrightarrow 0$, $\frac{d^2\tilde{v}_k}{dt^2} \longrightarrow v''$ strongly in $L^2(0,T;V')$, and $\frac{d\tilde{v}_k}{dt} \longrightarrow v'$ strongly in $\mathscr{C}([0,T];H)$, $v_k \longrightarrow v$ strongly in $\mathscr{C}([0,T];V)$, $f_k \longrightarrow f$ in $L^2(0,T,H)$ and since $-\Delta$ pseudo-monotone, we have as $k \longrightarrow 0$

$$\liminf \int_0^T \langle -\Delta u_k, u_k\rangle dt \geq \int_0^T \langle -\Delta u, u\rangle dt,$$

and since $k^2 f^0 \longrightarrow 0$ in H as $k \longrightarrow 0$, we conclude of (126) that

$$\int_0^T [(v''+\eta v', v-u)+\langle -\Delta u - f, v-u\rangle]dt+$$

$$\Sigma_{n=1}^{N-1}\|v^{n+1}-u^{n+1}-(v^n-u^n)\|^2+ \\ \Sigma_{n=1}^{N-1}\|v^{n-1}-u^{n-1}-(v^n-u^n)\|^2\geq 0 \tag{127}$$

for any v satisfies (109).
If v is given as in the inequality (85) there exists v_j satisfies the conditions (109) and such that $v_j\longrightarrow v$ weakly in $\mathscr{C}([0,T];V)$, $v'_j\longrightarrow v'$ weak-star in $\mathscr{C}([0,T];H)$, $v''_j\longrightarrow v''$ in $L^2(0,T;V')$.
We take $v=v_j$ in (127) and pass to the limit, we conclude that (85).
If we have **the case 4.:**
$\|v^{n+1}-u^{n+1}\|^2-\|v^n-u^n\|^2<0$ and $\frac{1+\eta k}{2}>0$
and **the case 5.:**
$\|v^{n+1}-u^{n+1}\|^2-\|v^n-u^n\|^2>0$ and $\frac{1+\eta k}{2}>0$
in **the cases 4., 5.,** we use the inequalitys (114) – (118) on the equality (115) we obtain the inequality

$$((v^{n+1}-2v^n+v^{n-1})+\eta k(v^{n+1}-v^n),v^n-u^n)+ \\ k^2\langle-\Delta u^n-f^n,v^n-u^n\rangle\geq \\ \frac{1+\eta k}{2}[\|v^{n+1}-u^{n+1}\|^2-\|v^n-u^n\|^2- \\ \|v^{n+1}-u^{n+1}-(v^n-u^n)\|^2]-\|v^{n-1}-u^{n-1}-(v^n-u^n)\|^2 \tag{128}$$

summing to n, we deduce

$$\Sigma_{n=1}^{N-1}[((v^{n+1}-2v^n+v^{n-1})+\eta k(v^{n+1}-v^n),v^n-u^n)+ \\ k^2\langle-\Delta u^n-f^n,v^n-u^n\rangle]\geq \\ \frac{1+\eta k}{2}[\|v^N-u^N\|^2-\|v^1-u^1\|^2- \\ \Sigma_{n=1}^{N-1}\|v^{n+1}-u^{n+1}-(v^n-u^n)\|^2]- \\ \Sigma_{n=1}^{N-1}\|v^{n-1}-u^{n-1}-(v^n-u^n)\|^2\geq \\ \frac{1+\eta k}{2}[-\|v^1-u^1\|^2- \\ \Sigma_{n=1}^{N-1}\|v^{n+1}-u^{n+1}-(v^n-u^n)\|^2]- \\ \Sigma_{n=1}^{N-1}\|v^{n-1}-u^{n-1}-(v^n-u^n)\|^2 \tag{129}$$

with (110) – (111) – (113), we conclude of (129) the inequality

$$\int_0^T(\frac{d^2\tilde{v}_k}{dt^2}+\eta\frac{d\tilde{v}_k}{dt},v_k-u_k)dt+ \\ \int_0^T\langle-\Delta u_k-f_k,v_k-u_k\rangle dt-k^2\langle-\Delta u^0,v^0-u^0\rangle+$$

$$k^2\langle f^0, v^0-u^0\rangle + (\frac{1+\eta k}{2})\|v^1-u^1\|^2 + (\frac{1+\eta k}{2})\Sigma_{n=1}^{N-1}\|v^{n+1}-u^{n+1}-(v^n-u^n)\|^2 + \Sigma_{n=1}^{N-1}\|v^{n-1}-u^{n-1}-(v^n-u^n)\|^2 \geq 0 \quad (130)$$

as $k \longrightarrow 0$, $\frac{d^2\tilde{v}_k}{dt^2} \longrightarrow v''$ strongly in $L^2(0,T;V')$, and $\frac{d\tilde{v}_k}{dt} \longrightarrow v'$ strongly in $\mathscr{C}([0,T];H), v_k \longrightarrow v$ strongly in $\mathscr{C}([0,T];V),\ f_k \longrightarrow f$ in $L^2(0,T;H)$
and since $-\Delta$ pseudo-monotone, we have as $k \longrightarrow 0$

$$\liminf \int_0^T \langle -\Delta u_k, u_k\rangle dt \geq \int_0^T \langle -\Delta u, u\rangle dt,$$

and since $k^2 f^0 \longrightarrow 0$ in H as $k \longrightarrow 0$, we conclude of (130) that

$$\int_0^T [(v''+\eta v', v-u) + \langle -\Delta u - f, v-u\rangle]dt + (\frac{1+\eta k}{2})\|v^1-u^1\|^2 + (\frac{1+\eta k}{2})\Sigma_{n=0}^{N-1}\|v^{n+1}-u^{n+1}-(v^n-u^n)\|^2 + \Sigma_{n=0}^{N-1}\|v^{n-1}-u^{n-1}-(v^n-u^n)\|^2 \geq 0 \quad (131)$$

for any v satisfies (109).
If v is given as in the inequality (86) there exists v_j satisfies the conditions (109) and such that $v_j \longrightarrow v$ weakly in $\mathscr{C}([0,T];V)$, $v_j' \longrightarrow v'$ weak-star in $\mathscr{C}([0,T];H)$, $v_j'' \longrightarrow v''$ in $L^2(0,T;V')$.
We take $v = v_j$ in (131) and pass to the limit, we conclude that (86). □

Proof of the Uniqueness

Regularity parabolic and variational inequalities hyperbolic

We approach parabolic equations by elliptic equations the following step is to approach hyperbolic equations by parabolic equations, this is the regularity parabolic method, that allows us to proved the Uniqueness. We apply this method to the evolution inequalities of type hyperbolic or related to operators well-posed of a sense of Petrowski.

Hypotheses

Let V, H are Hilbert spaces with

$$V \subset H,\ V \text{ is dense in } H,\ V \longrightarrow H, \text{ continuous.} \quad (132)$$

From (8), we have $V \subset H \equiv H' \subset V'$.
Put, for simplicity

$$L^2(0,T;V) = L^2(V),\ L^2(0,T;H) = L^2(H),\\ L^2(0,T;V') = L^2(V'). \tag{133}$$

And put

$$\mathscr{V} = L^2(0,T;V \times V) = L^2(V) \times L^2(V),\\ \mathscr{H} = L^2(0,T;V \times H) = L^2(V) \times L^2(H). \tag{134}$$

Identifying $\mathscr{H}$ to its dual $\mathscr{H}'$, we have

$$\mathscr{V} \subset \mathscr{H} \equiv \mathscr{H}' \subset \mathscr{V}',\\ \text{such that } \mathscr{V}' = L^2(V) \times L^2(V'). \tag{135}$$

Operator $A = -\Delta$
We give $A = -\Delta$ with

$$-\Delta \in \mathscr{L}(V, V'),\ (-\Delta)^* = -\Delta, \tag{136}$$

and there exists c, α such that

$$\langle -\Delta v, v\rangle + c\|v\|^2 \geq \alpha\|v\|_V^2,\ c > 0,\ \alpha > 0,\ \forall v \in V \tag{137}$$

The scalar product on V is

$$(u,v)_V = ((-\Delta + c)u, v),\ u, v \in V. \tag{138}$$

Operator $-\Delta$ on $L^2(V)$
We define the operator $-\Delta$ from $L^2(V) \longrightarrow L^2(V')$ is given by

$$(-\Delta v)(t) = -\Delta(v(t)) \text{ almost everywhere.} \tag{139}$$

Operator $\mathscr{A}$.
We give $k > 0$, we define

$$\mathscr{A} = \begin{pmatrix} kI & -I \\ -\Delta & kI \end{pmatrix},\ \mathscr{A} \in \mathscr{L}(\mathscr{V}, \mathscr{V}'). \tag{140}$$

If $v = \{v_1, v_2\} \in \mathscr{V}$, then

$$\mathscr{A} v = \{kv_1 - v_2,\ -\Delta v_1 + kv_2\} \in \mathscr{V}'.$$

The scalar product on V is given by (138), the scalar product in $\mathscr{H}$ is given by

$$(u,v)=\int_0^T[(u_1,v_1)_V+(u_2,v_2)]dt$$

then

$$\begin{aligned}\langle\mathscr{A}v,v\rangle=&\int_0^T[(kv_1-v_2,v_1)_V+\langle-\Delta v_1+kv_2,v_2\rangle]dt=\\ &\int_0^T[k((-\Delta+c)v_1,v_1)-((-\Delta+c)v_2,v_1)+\\ &\langle-\Delta v_1,v_2\rangle+k(v_2,v_2)]dt\geq\\ &\int_0^T[k\alpha\|v_1\|_V^2+k\|v_2\|^2-c\|v_1\|\|v_2\|]dt\end{aligned}$$

but

$$\|v\|\leq d\|v\|_V,\ \forall v\in V,\ d>0, \tag{141}$$

then we conclude that if

$$k>\frac{cd}{2\sqrt{\alpha}} \tag{142}$$

then

$$\begin{gathered}\text{there exists } \alpha_0>0 \text{ such that}\\ \langle\mathscr{A}v,v\rangle\geq\alpha_0\|v\|_{\mathscr{V}}^2,\ \forall v\in\mathscr{V}.\end{gathered} \tag{143}$$

The semigroups $G(s)$ and $g(s)$
We give the semigroup $G(s)$ in $L^2(V),\ L^2(H)$ and $L^2(V')$.
$G(s)$ is a semigroup of contractions in $L^2(H)$.
We denote by $-\Lambda$ is the infinitesimal generator of $G(s)$, and
$D(\Lambda;L^2(H))$ is the domain of Λ in $L^2(H)$. We can associate to $G(s)$ the semigroup $g(s)$ in $\mathscr{V}$, $\mathscr{H}$, $\mathscr{V}'$, is given by

$$g(s)=\begin{pmatrix}G(s)&0\\0&G(s)\end{pmatrix} \tag{144}$$

and we denote by $-L$ is the infinitesimal generator of $g(s)$, is given by

$$L=\begin{pmatrix}\Lambda&0\\0&\Lambda\end{pmatrix} \tag{145}$$

with the domain

$$D(L;\mathscr{H})=D(\Lambda;L^2(V))\times D(\Lambda;L^2(H)). \tag{146}$$

The convexs $\mathscr{K}_i$
We give the convexs $\mathscr{K}_i,\ i=1,2$ with

$$\text{the set } \mathscr{K}_i, \text{ is a convex closed of } L^2(V),\ 0 \in \mathscr{K}_i,\ i=1,2 \tag{147}$$

and we assume

$$\begin{aligned}\text{there exists } \sigma > 0 \text{ and } w_0 \in L^2(V) \text{ such that}\\ \sigma\mathscr{K}_2 + w_0 \subset \mathscr{K}_1.\end{aligned} \tag{148}$$

Remark 17. In the applications, using the set $\mathscr{K}_1 = L^2(V)$, such that (148) is satisfying for any $\mathscr{K}_2$.

Compatibility
Perform the following hypotheses

$$G(s)(-\Delta)(v) = -\Delta G(s)v,\ \forall s \geq 0,\ \forall v \in L^2(V), \tag{149}$$

$$G(s)\mathscr{K}_i \subset \mathscr{K}_i,\ \forall s \geq 0,\ i=1,2, \tag{150}$$

$$\begin{aligned}\text{there exists } \rho > 0 \text{ such that, } \forall s \geq 0,\ \forall v \in \mathscr{K}_i,\ i=1,2,\\ G(s)v + G^*(s)v - G^*(s)G(s)v + (\rho-1)v \in \rho\mathscr{K}_i.\end{aligned} \tag{151}$$

Such that $G^*(s)$ is the adjoint semigroup of $G(s)$.
We give the following Theorem

Theorem 18. *We assume that* $-\Delta$ *is given with* $(136)-(137)$ *and* $\mathscr{A}$ *is given by* (140) *with* (142). *We assume that* $(147)-(148)-(149)-(150)-(151)$ *are satisfied. With another*

$$\int_0^T ((-\Delta + c)\Lambda v_1, v_1)dt \geq 0,\ \forall v_1 \in D(\Lambda; L^2(V)). \tag{152}$$

Let $f \in D(\Lambda; L^2(H))$ *and put* $F = \{0, f\} \in \mathscr{H}$.
Then there exists a unique function u *satisfies*

$$u \in \mathscr{K}_1 \times \mathscr{K}_2,\ u \in D(L;\mathscr{K}), \text{ and} \tag{153}$$

$$(Lu, v-u) + \langle \mathscr{A}u, v-u\rangle \geq (F, v-u),\ \forall v \in \mathscr{K}. \tag{154}$$

The proof of the Theorem is in ([3], P. 349).
From the proof of the Theorem 18 we give the following remarks

Remark 19. The variational inequality (154) is not of type parabolic because $\mathscr{A}$ is restrict to $\mathscr{V}$ is not coercive on $\mathscr{V}$, but is only coercive on $\mathscr{H}$, this is the situation typical of hyperbolic operator or well-posed of a sense of Petrowski.

Remark 20. Before we give applications expounding (154), since $\mathscr{K} = \mathscr{K}_1 \times \mathscr{K}_2$, then from (154) we have the inequalitys

$$\begin{aligned} u_1 \in D(\Lambda; L^2(V)),\ u_1 \in \mathscr{K}_1, \\ \int_0^T [(\Lambda u_1, v_1 - u_1)_V + (ku_1 - u_2, v_1 - u_1)_V]dt \geq 0, \\ \forall v_1 \in \mathscr{K}_1, \end{aligned} \tag{155}$$

$$\begin{aligned} u_2 \in D(\Lambda; L^2(H)) \bigcap L^2(V),\ u_2 \in \mathscr{K}_2, \\ \int_0^T [(\Lambda u_2, v_2 - u_2) + \langle -\Delta u_1 + ku_2, v_2 - u_2\rangle]dt \geq \\ \int_0^T (f, v_2 - u_2)dt,\ \forall v_2 \in \mathscr{K}_2. \end{aligned} \tag{156}$$

In particular, $\mathscr{K}_1 = L^2(V)$ then from (155) we deduce the equation

$$\Lambda u_1 + ku_1 - u_2 = 0. \tag{157}$$

Remark 21. Third a priori estimate in the proof of the Theorem 18 is not satisfying if

$$\mathscr{K}_2 \text{ is bounded in } L^2(V), \tag{158}$$

then (148) is not utility.

Remark 22. Since $\mathscr{A}$ is given by (140), but with $k = 0$ we have

$$\mathscr{A} = \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix} \tag{159}$$

and perform the following hypothese : $\mathscr{K}_1 = L^2(V)$, $\mathscr{K}_2$ is bounded in $L^2(V)$ and if $u_j \in D(\Lambda; L^2(V))$ and $(\Lambda + \epsilon)v_j$ is remain in bounded of $L^2(V)$, then v_j is remain in a bounded set of $L^2(V)$, for any j, as $j \longrightarrow 0$.
In these conditions, the Theorem 18 is also satisfying.
Indeed, by taking

$$\mathscr{B} = \begin{pmatrix} I & 0 \\ 0 & -\Delta + c \end{pmatrix} \tag{160}$$

such that

$$\langle(\mathscr{A}+\epsilon\mathscr{B})v,v\rangle=\epsilon\langle\mathscr{B}v,v\rangle\geq\epsilon\|v\|^2_{\mathscr{V}},$$

then there exists u_ϵ in $\mathscr{K}_1\times\mathscr{K}_2$ and $D(L;\mathscr{V})$ such that there exists

$$(Lu_\epsilon,v-u_\epsilon)+\langle(\mathscr{A}+\epsilon\mathscr{B})u_\epsilon,v-u_\epsilon\rangle\geq(F,v-u_\epsilon),\ \forall v\in\mathscr{K},$$

then

$$((-\Delta+c)(\Lambda u_{\epsilon1}-u_{\epsilon2}+\epsilon u_{\epsilon1}),v_1-u_{\epsilon1})\geq 0,\forall v_1\in L^2(V)$$

and then

$$(\Lambda+\epsilon)u_{\epsilon1}=u_{\epsilon2}. \tag{161}$$

But $u_{\epsilon2}\in\mathscr{K}_2$ bounded in $L^2(V)$ and then $u_{\epsilon1}$ is remain in a bounded set of $L^2(V)$ and bounded of $D(\Lambda,L^2(V))$.
Then we have the results of first and third a priori estimate in the proof of the Theorem 18, second a priori estimate is unchanged.

Remark 23. From the proof of Theorem 18 can be replaced The hypotheses (150) – (151) by the following hypotheses respectively,

$$\text{there exists }\beta\in\mathbb{R},\text{ such that }\forall s\geq 0,\ e^{\beta s}G(s)\mathscr{K}_i\subset\mathscr{K}_i,\quad i=1,2, \tag{162}$$

$$\text{there exists }\rho>0,\text{ such that, }\forall s\geq 0,\ \forall v\in\mathscr{K}_i,\ i=1,2,$$
$$e^{\beta s}G(s)v+e^{-\beta s}G^*(s)v=G^*(s)G(s)v+(\rho-1)v\in\rho\mathscr{K}_i. \tag{163}$$

First application
From the Theorem 18 we prove as application the following Theorem.

Theorem 24. *We give the function $f=f(x,t)$, with*

$$f,\ \frac{\partial f}{\partial t}\in L^2(Q),\ f(x,0)=0. \tag{164}$$

There exists a unique function u satisfies

$$u,\ \frac{\partial u}{\partial x_i},\ \frac{\partial u}{\partial t},\ \frac{\partial^2 u}{\partial x_i\partial t},\ \frac{\partial^2 u}{\partial t^2}\ \in L^2(Q),\ i=1,\cdots,n, \tag{165}$$

$$\frac{\partial^2 u}{\partial t^2}-\Delta u=f,\ \textit{in } Q, \tag{166}$$

$$u(x,0)=0,\ \frac{\partial u}{\partial t}(x,0)=0,\ on\ \Omega, \tag{167}$$

$$\begin{aligned}\frac{\partial u}{\partial t}\geq 0\ on\ \Sigma,\ \frac{\partial u}{\partial n}\geq 0\ on\ \Sigma,\\ \frac{\partial u}{\partial t}\times\frac{\partial u}{\partial n}=0,\ on\ \Sigma.\end{aligned} \tag{168}$$

Such that $\frac{\partial}{\partial n}$ = *normal derivative to* $\partial\Omega$, *directed on the exterior of* Ω. *From* (165), $\Delta u=\frac{\partial^2 u}{\partial t^2}-f\in L^2(Q)$, *such that* $\frac{\partial u}{\partial n}$ *has a sense from ([4]).*

Proof. We apply the Theorem 18 in the following conditions

$$V=H^1(\Omega),$$

$$(-\Delta u,v)=\Sigma_{i=1}^n\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx,$$

$$\mathscr{K}_1=L^2(V)=L^2(0,T;V),$$

$$\mathscr{K}_2=\{v\mid v\in L^2(V),\ v\geq 0,\ \text{almost everywhere on }\Sigma\},$$

$$G(s)\varphi(t)=\begin{cases}\varphi(t-s)\text{ if }t\geq s,\\ 0,\ \text{if }t<s.\end{cases}$$

Here the conditions of the Theorem 18 are satisfied, choosing $k>0$. The operator Λ is $\Lambda=\frac{\partial}{\partial t}$, with domain the null functions for $t=0$, here by using (157), then we have the existence and uniqueness of the couple u_1,u_2 with

$$u_1\in D(\Lambda;L^2(V)),\ \text{such that,}$$

$$u_1\in L^2(V),\ u_1'\in L^2(V),u_1(0)=0, \tag{169}$$

$$u_2\in D(\Lambda;L^2(H)),\ u_2\in\mathscr{K}_2, \tag{170}$$

$$u_1'+ku_1-u_2=0, \tag{171}$$

$$\int_0^T(u_2'+(-\Delta)u_1+ku_2-f,v_2-u_2)dt\geq 0,\quad \forall v_2\in\mathscr{K}_2. \tag{172}$$

As $\mathscr{K}_2$ is cone of summit the origin, (172) is equivalent to

$$\int_0^T(u_2'+(-\Delta)u_1+ku_2-f,v_2)dt\geq 0$$

$$\forall v_2 \in \mathscr{K}_2, \text{ with } \int_0^T (u_2' + (-\Delta)u_1 + ku_2 - f, v_2)dt = 0, \quad \text{if } v_2 = u_2. \tag{173}$$

Using the definition of $\mathscr{K}_2$, we conclude that

$$u_2' - \Delta u_1 + ku_2 = f, \text{ in } Q. \tag{174}$$

If we multiplying by v_2 and integration by parts, we deduce of (174) that

$$\int_\Sigma \frac{\partial u_1}{\partial n} v_2 \, d\Sigma = \int_0^T (u_2' + (-\Delta)u_1 + ku_2 - f, v_2)dt \tag{175}$$

and then

$$\begin{aligned} \int_\Sigma \frac{\partial u_1}{\partial n} v_2 \, d\Sigma \geq 0, \ \forall v_2 \in \mathscr{K}_2, \\ \text{with, } \int_\Sigma \frac{\partial u_1}{\partial n} v_2 \, d\Sigma = 0, \text{ if } v_2 = u_2, \end{aligned} \tag{176}$$

then

$$\frac{\partial u_1}{\partial n} \geq 0, \ u_2 \frac{\partial u_1}{\partial n} = 0. \tag{177}$$

Then put

$$w_i = e^{kt} u_i, \ i = 1, 2.$$

We conclude of (171) and (176) that

$$\begin{aligned} w_1' - w_2 = 0, \\ w_2' - \Delta w_1 = e^{kt} f = f^*, \\ w_2 \geq 0, \text{ on } \Sigma, \ \frac{\partial w_1}{\partial n} = 0, \text{ on } \Sigma, \\ w_2 \frac{\partial w_1}{\partial n} = 0, \text{ on } \Sigma. \end{aligned} \tag{178}$$

Then $u = w_1$ satisfying the conditions of Theorem 24 , with f is replaced by f^*.

As we have equivalent of the research of u, of $\{w_1, w_2\}$ and $\{u_1, u_2\}$ we have further the uniqueness. □

Second application

We consider the problem with periodic solutions in t of inequality of type (166)−(168). We take that the change $w_i = e^{kt}u_i,\ i = 1, 2$, exterminate the periodicity in t then we will verify this.
We give the following Theorem as application on the Theorem 18 and the Theorem 24

Theorem 25. *We give the function $f = f(x,t)$, with*

$$f,\ \frac{\partial f}{\partial t} \in L^2(Q),\ f(x,0) = f(x,T),\ x \in \Omega. \tag{179}$$

Let $k = \eta > 0$.
There exists a unique function u, satisfies (165) *and*

$$\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} - \Delta u = f,\ \textit{in } Q, \tag{180}$$

$$u(x,0) = u(x,T),$$

$$\frac{\partial u}{\partial t}(x,0) = \frac{\partial u}{\partial t}(x,T),\ x \in \Omega, \tag{181}$$

and (168).

Proof. Applying the Theorem 18 with the same hypotheses of the Theorem 24, but the semigroup $G(s)$ is given by

$$G(s)\varphi(t) = \begin{cases} \varphi(t - s + T), & \text{if } t \le s, \\ \varphi(t - s), & \text{if } t \ge s. \end{cases}$$

We obtain the existence and uniqueness of the couple $\{u_1, u_2\}$ with

$$u_1 \in D(\Lambda; L^2(V)),\ \text{such that } u_1 \in L^2(V),\ u_1' \in L^2(V),$$
$$u_1(x,0) = u_1(x,T),\ u_2 \in D(\Lambda; L^2(H)),\ u_2 \in \mathscr{K}_2, \tag{182}$$

and (171) − (172). Then interpret (172) as in the Theorem 24 and that $u = u_1$ satisfying the conditions of the Theorem 24, as we have equivalent of the research of u, of $\{u_1, u_2\}$ we have further the uniqueness.
Then we obtain the Uniqueness in the Theorem 12 when $\eta > 0$. □

To illustrate our results we consider the following example.

3.4. An Example

We consider the initial - boundary value problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \eta \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t), \\ x \in]0,1[, \ t \in]0,T[, \\ u(0,t) = u(1,t) = 0, \ t \in]0,T[, \\ u(x,0) = 2\sin \pi x = u_0(x), \quad x \in]0,1[, \\ \frac{\partial u}{\partial t}(x,0) = -\sin 2\pi x = u_1(x), \ x \in]0,1[. \end{cases} \tag{183}$$

Where $\eta > 0$, and $f(x,t) = -2\eta\pi(\sin \pi x \sin \pi t + \frac{1}{2\pi} \sin 2\pi x \cos 2\pi t)$
the unique solution of (183) is given by

$$u(x,t) = 2 \sin \pi x \cos \pi t - \frac{1}{2\pi} \sin 2\pi x \sin 2\pi t \tag{184}$$

For find the unique numerical solution v, the way we derive the finite difference scheme for (183) then to replace the derivatives involved in (183) by finite differences. But for (183) we have to approximate both the space and the time derivatives.
Let $n \geq 1$ be a given integer, the grid spacing in the x - direction is $\triangle x = \frac{1}{n+1}$.
The grid points are $x_j = j \triangle x$ for $j = 0, 1, \cdots, n+1$.
The discrete time levels are given by $t_m = m \triangle t$ for integers $m \geq 0$, where $\triangle t > 0$ is the time step.
The grid function v, with $v_j^m = v(x_j, t_m)$, approximates u.
Then we define the difference scheme

$$\frac{v_j^{m+1} - 2v_j^m + v_j^{m-1}}{(\triangle t)^2} + \eta \frac{v_j^{m+1} - v_j^m}{\triangle t} - \frac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{(\triangle x)^2} = f(x_j, t_m) \tag{185}$$

for $j = 1, 2, \cdots, n$ and for $m \geq 1$. We require the discrete solution to satisfy the boundary conditions in (183),

$$v_0^m = v_{n+1}^m = 0 \quad \text{for, } m \geq 0.$$

If $\{v_j^m\}_{j=1}^n$ and $\{v_j^{m-1}\}_{j=1}^n$ are known, then the solutions $\{v_{j=1}^{m+1}\}_{j=1}^n$ can be computed from (185). We need to know v at the first two time levels. We have

$$v_j^0 = u_0(x_j) \quad j = 1, 2, \cdots, n. \tag{186}$$

to obtain approximation v_j^1 for $u(x, \triangle t)$ we use a Taylor expansion with respect to time to obtain

$$u(x, \triangle t) = u(x, 0) + (\triangle t)\frac{\partial u}{\partial t}(x, 0) + \frac{(\triangle t)^2}{2}\frac{\partial^2 u}{\partial t^2}(x, 0) + O((\triangle t)^3)$$

$$u(x, \triangle t) = u_0(x) + (\triangle t)u_1(x) + \frac{(\triangle t)^2}{2}u_0''(x) + O((\triangle t)^3).$$

Since

$$\frac{\partial^2 u}{\partial t^2}(x, 0) = \frac{\partial^2 u}{\partial x^2}(x, 0) = u_0''(x).$$

Then, we have the following approximation v_j^1 for $u(x_j, \triangle t)$

$$v_j^1 = v_j^0 + (\triangle t)u_1(x_j) + \frac{(\triangle t)^2}{2(\triangle x)^2}(v_{j-1}^0 - 2v_j^0 - v_{j+1}^0). \tag{187}$$

To write (185) in a more compact form, we let $v^m \in \mathbb{R}^n$ be the vector $v^m = (v_1^m, v_2^m, \cdots, v_n^m)^\tau$, $\tau =$ transferred, and $A \in \mathbb{R}^{n,n}$ the tridiagonal matrix

$$A = \frac{1}{(\triangle x)^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \tag{188}$$

then (185) can be written for $m \geq 1$,

$$v^{m+1} = (\frac{2 + \eta \triangle t}{1 + \eta \triangle t}I - \frac{(\triangle t)^2}{1 + \eta \triangle t}A)v^m - \frac{1}{1 + \eta \triangle t}v^{m-1} + f^m \tag{189}$$

where $I \in \mathbb{R}^{n,n}$, is the identity matrix, and $f^m = (f_1^m, f_2^m, \cdots, f_n^m)^\tau$ with components given by

$$f_j^m = f(x_j, t_m) = -\frac{2\eta\pi(\triangle t)^2}{1 + \eta \triangle t}(\sin \pi x_j \sin \pi t_m + \frac{1}{2\pi}\sin 2\pi x_j \cos 2\pi t_m)$$

for $j = 1, 2, \cdots, n$ and $m \geq 1$, where the initial approximation v^0 and v^1 are determined by (186) and (187).

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