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ON A GENERALIZATION OF COFINITELY LIFTING MODULES

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Abstract: In this paper, we study on cofinitely *Rad*-lifting modules as a proper generalization of modules with the property (P^*) and cofinitely lifting modules, and we obtain the properties of these modules. In particular, we prove that if M with the property (SSP) is a cofinitely *Rad*-lifting module, then $\frac{M}{N}$ is a cofinitely *Rad*-lifting module for every direct summand N of M. We show that π-projective cofinitely (*Rad*-) $\oplus$-supplemented modules are cofinitely (*Rad*-) lifting. We obtain a new characterization of semiperfect rings by using this result. This characterization generalizes the result of Wang and Wu.

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1. Introduction

Throughout this paper, all rings are associative with identity element and all modules are unital left R-modules. Let R be a ring and let M be an R-module. The notation $N \leq M$ means that N is a submodule of M. A submodule K of M is called *cofinite* (in M) if the factor module $\frac{M}{K}$ is finitely generated. A module M is called *extending* if every submodule is essential in a direct summand of M

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[6]. Here a submodule $L \leq M$ is said to be *essential* in M, denoted as $L \trianglelefteq M$, if $L \cap N \neq 0$ for every nonzero submodule $N \leq M$. Dually, a submodule S of M is called *small (in M)*, denoted as $S << M$, if $M \neq S + L$ for every proper submodule L of M. Let $K \leq N \leq M$. If $\frac{N}{K} \ll \frac{M}{K}$, then K is called a *coessential submodule* of N in M. A submodule N of M is called *coclosed* in M if N has no proper coessential submodule. K is called a *coclosure* of N in M if K is a coessential submodule of N and K is coclosed in M [11]. By $Rad(M)$ we denote the intersection of all maximal submodules of M. For any ring R, an R-module M is called *(cofinitely) supplemented* if every (cofinite) submodule N of M has a *supplement*, that is a submodule K minimal with respect to $M = N + K$. Equivalently, $M = N + K$ and $N \cap K << K$ [18].

A module M is called *lifting* if for every submodule N of M there exists a direct summand K of M such that $K \leq N$ and $\frac{N}{K} << \frac{M}{K}$ (i.e. K is a *coessential submodule* of N in M) as a dual notion of extending modules. Mohamed and Müller has generalized the concept of lifting modules to $\oplus$-supplemented modules. M is called *$\oplus$-supplemented* if every submodule N of M has a supplement that is a direct summand of M [12]. Then, Çalışıcı and Pancar have defined a module M *$\oplus$-cofinitely supplemented* if every cofinite submodule of M has a supplement that is a direct summand of M [7].

Wang and Wu call a module M *cofinitely lifting* if every cofinite submodule N of M there exists a direct summand K of M such that $K \leq N$ and $\frac{N}{K} << \frac{M}{K}$ [17]. It is shown in [17, Proposition 2.4] that every cofinitely lifting module is $\oplus$-cofinitely supplemented.

Let M be an R-module and let N and K be any submodules of M. If $M = N+K$ and $N \cap K \subseteq Rad(K)$, then K is called a *Rad-supplement* of N in M [19]. Since $Rad(K)$ is the sum of all small submodules of K, every supplement submodule is a *Rad*-supplement in M. Adapting the concept of (cofinitely) supplemented modules, one calls a module M *(cofinitely) Rad-supplemented* if every (cofinite) submodule has a *Rad*-supplement in M as in [4] and [5]. On the other hand, M is called *(cofinitely) Rad-$\oplus$-supplemented* if every (cofinite) submodule of M has a *Rad*-supplement that is a direct summand of M. By $cgs^{\oplus}$, we denote cofinitely Rad-$\oplus$-supplemented modules in [13].

Recall from Al-Khazzi and Smith [3] that a module M is said to have *the property (P^*)* if for every submodule N of M there exists a direct summand K of M such that $K \leq N$ and $\frac{N}{K} \subseteq Rad(\frac{M}{K})$. The authors have obtained in the same paper the various properties of modules with the property (P^*). Radical modules have the property (P^*). It is clear that every lifting module has the property (P^*) and every module with the property (P^*) is Rad-$\oplus$-supplemented.

As motivated by the above definitions, it is natural to introduce a generalization of modules with the property (P^*). We say that a module M is *cofinitely Rad-lifting* if for every cofinite submodule N of M there exists a direct summand K of M such that $K \leq N$ and $\frac{N}{K} \subseteq Rad(\frac{M}{K})$. A module with the property (P^*) is cofinitely Rad-lifting. Also, a finitely generated cofinitely Rad-lifting is lifting (see Corollary 2.6). It is clear that every cofinitely lifting module is cofinitely Rad-lifting. However, a cofinitely Rad-lifting module is not necessarily cofinitely lifting (see Example 2.8).

In this paper, we provide the properties of cofinitely Rad-lifting modules in Section 2. Some examples are given to separate cofinitely lifting modules, cofinitely Rad-lifting modules and modules with the property (P^*). We show that a cofinitely Rad-lifting module which has a small radical is cofinitely lifting. We give some conditions for factor modules (in particular, direct summands) of a cofinitely Rad-lifting to be cofinitely Rad-lifting. We prove that a π-projective cofinitely (Rad-) $\oplus$-supplemented module is cofinitely (Rad-) lifting. We obtain a new characterization of semiperfect rings by using this result. This characterization generalizes the result of Wang and Wu [17].

2. Cofinitely *Rad*-Lifting Modules

It is well known that a ring R is left perfect if and only if every free left R-module $R^{(I)}$ is lifting. In this section, we shall prove an analogous characterization for semiperfect rings and give various properties of cofinitely Rad-lifting modules.

For the next result see [4, 2.8].

Lemma 2.1. *A module M is cofinitely Rad-lifting if and only if for any cofinite submodule N of M there exist submodules K, K' of M such that $M = K \oplus K'$, $K \leq N$ and $N \cap K' \subseteq Rad(K')$.*

The following Corollary is an immediate consequence of Lemma 2.1.

Corollary 2.2. *Every cofinitely Rad-lifting module is a $cgs^{\oplus}$-module.*

Proof. Let U be any cofinite submodule of M. Since M is cofinitely Rad-lifting module, there exist submodules K, K' of M such that $M = K \oplus K'$, $K \leq U$ and $U \cap K' \subseteq Rad(K')$ by Lemma 2.1. It follows that K' is a Rad-supplement of U in M and K' is a direct summand of M. Thus M is a $cgs^{\oplus}$-module. □

The next example shows that a $cgs^{\oplus}$-module doesn't need to be cofinitely Rad-lifting. Firstly we need the following Lemma.

Lemma 2.3. *Let M be a cofinitely Rad-lifting module. If $Rad(M) \ll M$, then M is a cofinitely lifting module.*

Proof. Let M be a cofinitely Rad-lifting module and N be any cofinite submodule of M. Then by Lemma 2.1, there exist submodules K, K' of M such that $M = K \oplus K'$, $K \leq N$ and $N \cap K' \subseteq Rad(K')$. It follows that $N \cap K' \ll M$. Since K' is a direct summand of M, then $N \cap K' \ll K'$. Therefore M is a cofinitely lifting module. □

Example 2.4. (see [17], Example 3.1) Let $M = \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{8\mathbb{Z}}$. Since $\mathbb{Z}$-modules $\frac{\mathbb{Z}}{2\mathbb{Z}}$ and $\frac{\mathbb{Z}}{8\mathbb{Z}}$ are local, M is a $cgs^{\oplus}$-module according to [13, Theorem 2.1]. Note that M is finitely generated. If M is a cofinitely Rad-lifting module, then M is cofinitely lifting by Lemma 2.3. This is a contradiction.

A module M is called *coatomic* if every proper submodule is contained in a maximal submodule of M. Every coatomic module has a small radical. Using Lemma 2.3 we obtain the following Corollaries.

Corollary 2.5. *Every coatomic cofinitely Rad-lifting module is cofinitely lifting.*

Corollary 2.6. *The following statements are equivalent for a finitely generated module M.*

1. *M is a cofinitely Rad-lifting module.*
2. *M is a cofinitely lifting module.*
3. *M is lifting.*
4. *M has the property (P^*).*

Proof. (1) $\Longrightarrow$ (2) follows from Lemma 2.3.
(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1) are clear. □

Recall that a π-projective module M is $\oplus$-supplemented if and only if it is lifting. Now we get the following result:

Theorem 2.7. *Let M be a π-projective module. Then,*

1. *M is a $cgs^{\oplus}$-module if and only if it is cofinitely Rad-lifting.*
2. *M is a $\oplus$-cofinitely supplemented module if and only if it is cofinitely lifting.*

Proof. (1) Let N be any cofinite submodule of M. Since M is a $cgs^{\oplus}$-module, there exists a direct summand T of M such that $M = N + T$ and $N \cap T \subseteq Rad(T)$. Since M is a π-projective module, $M = N + T$ and T is a direct summand of M, then there exists a submodule K of N such that $M = T \oplus K$ by [6, 4.11]. Therefore M is a cofinitely *Rad*-lifting module by Lemma 2.1.

(2) The proof is as in the case of (1). □

It is clear that every cofinitely lifting module is cofinitely *Rad*-lifting. But the converse is not always true as the following example shows.

A module M is said to be *w-local* if M has a unique maximal submodule. It is easy to see that a module M is w-local if and only if $Rad(M)$ is maximal. Clearly, w-local modules are a proper generalization of local modules [5].

Example 2.8. (see [13], Example 2.2) Let M be a biuniform module and $S = End(M)$. Suppose that P is the projective S-module with $dim(P) = (1, 0)$. Then P is an indecomposable w-local module. Since $dim(P) = (1, 0)$, P is not finitely generated. Hence P is a $cgs^{\oplus}$-module but not $\oplus$-cofinitely supplemented. Since P is projective, P is a cofinitely *Rad*-lifting module but not cofinitely lifting by Theorem 2.7.

Proposition 2.9. *Let M be an indecomposable module.*

1. *If M has the property (P^*), M is radical or local.*

2. *If M is a cofinitely Rad-lifting module, M is radical or w-local.*

Proof. (1) If $Rad(M) = M$, M is radical. Suppose that $Rad(M) \neq M$ and N is a proper submodule of M. Since M has the property (P^*), there exist submodules T, V of M such that $T \leq N$, $M = T \oplus V$ and $N \cap V \subseteq Rad(V)$ by [3, Lemma 15]. By the hypothesis, $T = 0$. Then $N \subseteq Rad(M)$. Hence M is local.

(2) Suppose that M is not radical. Then M has a maximal submodule N. Since M is a cofinitely *Rad*-lifting module, M is a $cgs^{\oplus}$-module by Corollary 2.2. Thus there exist submodules K, K' of M such that $M = N + K$, $M = K \oplus K'$ and $N \cap K \subseteq Rad(K)$. By the hypothesis, $K = M$. It follows that $N = Rad(M)$. We obtain that N is a unique maximal submodule of M. Therefore M is w-local. □

Example 2.4 also shows that a class of cofinitely *Rad*-lifting modules is not closed under any direct sum. Now we prove that direct sums of cofinitely *Rad*-lifting modules is cofinitely *Rad*-lifting module, under a certain condition.

Recall from [18, 6.4] that a submodule N of an R-module M is called *fully invariant* if $f(N)$ is contained in N for every R-endomorphism f of M. Let M be an R-module and τ be a preradical for the category of R-modules. Then, $\tau(M)$ is a fully invariant submodule of M. An R-module M is called a *duo module* if every submodule of M is fully invariant [14].

Theorem 2.10. *Let $\{M_i\}_{i\in I}$ be any family of cofinitely Rad-lifting modules on a ring R and $M = \oplus_{i\in I} M_i$. If every cofinite submodule of M is fully invariant, M is a cofinitely Rad-lifting module.*

Proof. Let N be any cofinite submodule of M. Since N is a fully invariant submodule of M, we have $N = \oplus_{i\in I}(N\cap M_i)$. It follows that $N\cap M_i$ is a cofinite submodule of M_i for every $i \in I$. Since M_i is a cofinitely *Rad*-lifting module for every $i \in I$, there exist submodules $K_i, L_i \leq M_i$ such that $K_i \leq (N \cap M_i)$, $N\cap L_i = (N\cap M_i)\cap L_i \subseteq Rad(L_i)$ and $M_i = K_i\oplus L_i$. Then $M_i = (N\cap M_i)+L_i$. Let $K = \oplus_{i\in I} K_i$ and $L = \oplus_{i\in I} L_i$. It is clear that $K \leq N$ and $M = K \oplus L$. It follows that $N \cap L \subseteq Rad(L)$. Hence M is a cofinitely *Rad*-lifting module. □

Corollary 2.11. *For a ring R, let $M = \oplus_{i\in I} M_i$ where each M_i is a cofinitely Rad-lifting R-module. If M is a duo module, M is a cofinitely Rad-lifting module.*

Observe that a factor module of a cofinitely *Rad*-lifting module need not to be cofinitely *Rad*-lifting by [9, Example 2.2], Theorem 2.7 and Lemma 2.3.

Proposition 2.12. *Let M be a cofinitely Rad-lifting module and N be a cofinite direct summand of M. Then N is a cofinitely Rad-lifting module.*

Proof. Let L be any cofinite submodule of N. Since N is direct summand of M, there exists a submodule N' of M such that $M = N \oplus N'$. Then $\frac{M}{L} = \frac{N}{L} \oplus \frac{N'+L}{L}$. Since $\frac{M}{N} \cong \frac{N'+L}{L} \cong N'$ is finitely generated, L is a cofinite submodule of M. Since M is cofinitely *Rad*-lifting, there exist submodules K, K' of M such that $M = K \oplus K'$, $K \leq L$ and $L \cap K' \subseteq Rad(K')$ by Lemma 2.1. It follows that $N = K \oplus (N \cap K')$ and $L \cap (N \cap K') \subseteq Rad(N \cap K')$. Therefore N is a cofinitely *Rad*-lifting module. □

A module M is said to have the *Summand Sum Property* (SSP) if the sum of two direct summands of M is again a direct summand of M.

Proposition 2.13. *Let M be a cofinitely Rad-lifting module with the property SSP. Suppose that N is a direct summand of M. Then $\frac{M}{N}$ is a cofinitely Rad-lifting module.*

Proof. Let $\frac{L}{N}$ be any cofinite submodule of $\frac{M}{N}$. Then L is a cofinite submodule of M. Since M is a cofinitely Rad-lifting module, there exist submodules $K, K^{'}$ of M such that $M = K \oplus K^{'}$, $K \leq L$ and $L \cap K^{'} \subseteq Rad(K^{'})$ by Lemma 2.1. Thus $\frac{K^{'}+N}{N}$ is a Rad-supplement of $\frac{L}{N}$ in $\frac{M}{N}$. Since M has SSP, there exists a submodule X of M such that $M = (K+N) \oplus X$. We have $\frac{M}{N} = \frac{K+N}{N} \oplus \frac{X+N}{N}$. It is clear that $\frac{K+N}{N} \leq \frac{L}{N}$. Furthermore $\frac{L}{N} \cap \frac{X+N}{N} \subseteq Rad(\frac{X+N}{N})$. By Lemma 2.1, $\frac{M}{N}$ is cofinitely *Rad*-lifting. □

Theorem 2.14. *Let M be a cofinitely Rad-lifting module. If N is a fully invariant submodule of M, $\frac{M}{N}$ is a cofinitely Rad-lifting module.*

Proof. Let $\frac{T}{N}$ be any cofinite submodule of $\frac{M}{N}$. Then T is a cofinite submodule of M. Since M is cofinitely Rad-lifting module, there exist submodules $K, K^{'}$ of M such that $M = K \oplus K^{'}$, $K \leq T$ and $T \cap K^{'} \subseteq Rad(K^{'})$ by Lemma 2.1. Thus $\frac{K^{'}+N}{N}$ is a Rad-supplement of $\frac{T}{N}$ in $\frac{M}{N}$. Since N is fully invariant submodule of M, $N = (N \cap K) \oplus (N \cap K^{'})$. It is clear that $\frac{K+N}{N} \leq \frac{T}{N}$ and $\frac{M}{N} = \frac{K+N}{N} \oplus \frac{K^{'}+N}{N}$. Therefore $\frac{M}{N}$ is cofinitely *Rad*-lifting by Lemma 2.1. □

Corollary 2.15. *Let M be a cofinitely Rad-lifting module. Then $\frac{M}{\tau(M)}$ is a cofinitely Rad-lifting module.In particular, every cofinite submodule of $\frac{M}{Rad(M)}$ is a direct summand of $\frac{M}{Rad(M)}$.*

Proposition 2.16. *Let M be a module such that every cofinite submodule of M has a coclosure in M. Then M is cofinitely Rad-lifting if and only if every cofinitely coclosed submodule of M is a direct summand of M.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Let N be any cofinite submodule of M. By hypothesis, there exists a submodule K of N such that $\frac{N}{K} \ll \frac{M}{K}$ and K is coclosed in M. It follows that K is a cofinite submodule of M. Since K is coclosed in M, K is a direct summand of M and $\frac{N}{K} \subseteq Rad(\frac{M}{K})$. Hence M is a cofinitely *Rad*-lifting module. □

In the next Theorem, we characterize semiperfect rings via cofinitely *Rad*-lifting modules. This result generalizes [17, Theorem 3.18].

Theorem 2.17. *The following statements on a ring R are equivalent.*

1. *R is semiperfect.*
2. *Every free left R-module is $\oplus$-cofinitely supplemented.*
3. *Every free left R-module is cofinitely lifting.*

4. *Every free left R-module is cofinitely Rad-lifting.*

5. *Every free left R-module is $cgs^{\oplus}$.*

Proof. (1) $\Longrightarrow$ (2) [7, Theorem 2.9].
(2) $\Longrightarrow$ (3) Clear by Theorem 2.7(2).
(3) $\Longrightarrow$ (4) is clear.
(4) $\Longrightarrow$ (5) By Corollary 2.2.
(5) $\Longrightarrow$ (1) It follows from [13, Theorem 2.4].

□

Finally, we give an example of a module, which is cofinitely Rad-lifting but doesn't have the property (P^*).

Example 2.18. Let M be the left $\mathbb{Z}$-module $\prod_{\mathfrak{p}\in\Lambda}(\frac{\mathbb{Z}}{\mathfrak{p}})$, where Λ is a collection of maximal ideals of $\mathbb{Z}$. Then $Rad(M) = 0$. By [2, Lemma 2.9], for some submodule N of M, we have $\frac{N}{Tor(M)} \cong \mathbb{Q}$, where $Tor(M)$ is the torsion submodule of M. According to [16, Example 2.2], N is a cofinitely Rad-lifting but it doesn't have the property (P^*).

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