International Journal of Pure and Applied Mathematics
Volume 95 No. 3 2014, 339-345
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v95i3.2
ijpam.eu

RADIO LABELINGS FOR CORONA PRODUCT OF $P_2 \bigodot W_n$, $n \geq 6$

Saima Nazeer[1], Imrana Kousar[2]
[1,2]Lahore College For Women University
Lahore, PAKISTAN

Abstract: For a connected graph G, let $d(x, y)$ denotes the distance between two distinct vertices x, y in G and $\text{diam}(G)$ be the diameter of G. A radio labeling (or multi-level distance labeling) of a graph G is a function f that assigns positive integers to the vertices of G satisfying $|f(x)-f(y)| \geq \text{diam}(G)+1-d(x,y)$. The largest integer in the range of the labeling is its span. The radio number is the minimum possible span taken over all radio labelings of G, denoted by $\text{rn}(G)$.

In this paper we show that $\text{rn}(P_2 \bigodot W_n) = 2n+5$, $n \geq 6$.

AMS Subject Classification: 05C12, 05C15, 05C78
Key Words: graph labeling, radio number, corona product of graphs

1. Introduction

Distance two labeling can be extended as a radio labeling (or multilevel distance labeling) of graphs which are motivated by restrictions inherent in assigning channel frequencies for radio transmitters [3]. For a set of given cities (or stations), the goal is to assign to each city channel, which is a positive integer, so that the interference is avoided, and the span of the channels assigned is

Received: January 1, 2014

minimized. The problem of interferences of geographically closed transmitters could be resolved by assigning channels in which frequency difference is large. Where as the transmitters which are situated further away from each other may be assigned relatively close frequencies. This situation can be physically modeled by considering the transmitters to be the vertices of a graph. Positive integers are assigned to the vertices of the graph with limitation between them and our purpose is to minimize the largest integer used. C. Fernandez et al. proved in [2], that the radio number of the complete graph with n vertices is n. In the same paper $\mathrm{rn}(S_n) = n + 2$ was investigated, where $S_n(n \geq 2)$ is a star graph with $n + 1$ vertices. The radio numbers of the complete bipartite graphs $K_{m,n}$, wheel graph W_n, $(n \geq 3)$ and the family of gear graph of order n, G_n which is a planar graph having vertices $2n + 1$ and edges $3n$ were also investigated in the above mentioned paper [2]. The radio number for paths and cycles were completely determined by D. Liu and X. Zhu in [4]. Lower bound for the generalized gear graph $J_{t,n}$, which is obtained from a wheel graph by introducing t vertices between every pair of adjacent vertices on the cycle was found by M. T. Rahim et al. in [6]. The radio number for square cycles were studied by D. Liu and M. Xie in [5]. Let $G = (V(G), E(G))$ be a simple connected graphs. The distance between two vertices x and y of G is denoted by $d(x, y)$ and $\mathrm{diam}(G)$ indicate the diameter of G. A radio labeling is an injective function $f : V(G) \to \mathbb{Z}^+$ such that $|f(x) - f(y)| \geq \mathrm{diam}(G) + 1 - d(x, y)$ satisfy for every two distinct vertices x and y of G. The span of a labeling f is the greatest integer in the range of f. The minimum span taken over all radio labelings of the graph is called radio number of G, denoted by $\mathrm{rn}(G)$.

The corona product of a graph G with a graph H, denoted by $G \bigodot H$, is a graph obtained by taking one copy of a n-vertex graph G and n copies $H_1, H_2, ..., H_n$ of H, and then joining the i-th vertex of G to every vertex in H_i. In this paper, we investigate the radio number of corona product of path P_2 with a wheel W_n, $n \geq 6$ which is obtained by joining each vertex of the cycle C_{n-1} with the center vertex having $\mathrm{diam}(P_2 \bigodot W_n) = 3$ and prove that $\mathrm{rn}(P_2 \bigodot W_n) = 2n + 5$ for $n \geq 6$.

2. The Radio Number for $P_2 \bigodot W_n$

Let $V(P_2 \bigodot W_n) = \{w, z, z_1, z_2, u_i, v_i : 1 \leq i \leq n - 1\}$ and an edge set is $E(P_2 \bigodot W_n) = \{zw, zz_2, wz_1, wv_i, zu_i : 1 \leq i \leq n - 1\} \cup$ edge set of first copy of $W_n \cup$ edge set of second copy of W_n.

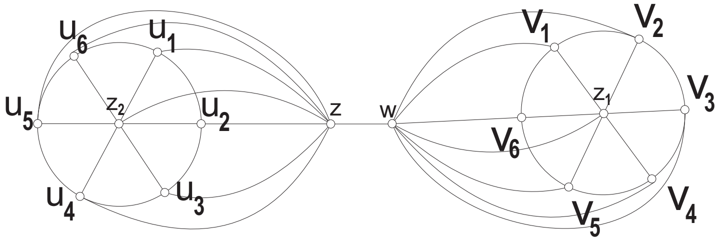
To establish the radio number of corona product of path P_2 with W_n, $n \geq 6$.

First of all we determine the lower bound by examining the minimum necessary differences between labels. Then we determine an upper bound, for this we use a specific radio labeling, as the labeling provides span equal to the lower bound their common value is the radio number.

2.1. The Standard Labeling of the $P_2 \odot W_n$

The vertices of one copy of W_n are labeled by $\{u_1, u_2, ..., u_{n-1}\}$ and other copy by $\{v_1, v_2, ..., v_{n-1}\}$ sequentially. The center of these two copies are labeled by z_1 and z_2. The vertex of P_2 which is adjacent to $\{z_2, u_1, u_2, ..., u_{n-1}\}$ is labeled as z and the vertex of P_2 which is adjacent to $\{z_1, v_1, v_2, ..., v_{n-1}\}$ is labeled as w.

The standard labeling of the $P_2 \odot W_6$ and of the $P_2 \odot W_7$ are shown in figure 1.



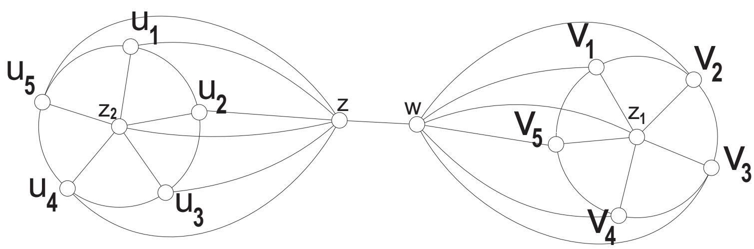


Figure 1

Theorem 2.2. *For any $n \geq 6$, $rn(P_2 \odot W_n) \geq 2n + 5$.*

Proof. Consider $n \geq 6$, $\text{diam}(P_2 \odot W_n) = 3$, so the radio condition $|f(u) - f(v)| \geq 4 - d(u, v)$ for all distinct vertices $u, v \in V(P_2 \odot W_n)$ holds for any radio labeling f of $P_2 \odot W_n$. For this, we count the number of values required to labels and add the minimum number of restricted values associated with the vertices of $P_2 \bigodot W_n$. For example, if we label the center vertex z_1 which is adjacent to the set of vertices $\{v_1, v_2, ..., v_{n-1}\}$ by a (i.e. $f(z_1) = a$), then as $d(z_1, x) \leq 3$ for all vertices $x \neq z_1$, so there is no restricted value. Thus, we can assigns the consecutive integers $1, 2$ with z_1 and z_2 respectively. After labeled z_1 and z_2, next we move to w. Since $d(z_2, w) = 2$, so there is one restricted value associated with w (i.e. we can not assign 3 with w or we label w with 4). Similarly, if $d(w, u_i) = 2$ for all i, thus there is one restricted value associated with u_i for all i. However, as $d(u_i, v_i) = 3$ for $1 \leq i \leq n-1$, it is possible to use consecutive labels on u_i and v_i for all i. That is there is no restricted values associated with the vertices of the set $\{u_i, v_i : 1 \leq i \leq n-1\}$. Now, $d(z, v_i) = 2$, so there is one restricted value associated with z. Without loss of generality, assume that $f(z_1) = 1$ and $f(z)$ is the span of f. It yields a total of $1 + 1 + 1 = 3$ restricted values. Summing in the $2n + 2$ values desired to label the $2n + 2$ vertices gives a total of $2n + 5$, hence $\mathrm{rn}(P_2 \bigodot W_n) \geq 2n + 5$. □

Theorem 2.3. *For any $n \geq 6$, $rn(P_2 \bigodot W_n) \leq 2n + 5$.*

Proof. If f is any radio labeling of $(P_2 \bigodot W_n)$, then span of this labeling will give an upper bound for the radio number of $(P_2 \bigodot W_n)$. In order to find an upper bound, firstly we define a position function that use the set $\{x_0, x_1, ..., x_{2n+1}\}$ to rename the vertices of $P_2 \bigodot W_n$. Then we specify the labels $f(x_i)$ so that $i < j$ if and only if $f(x_i) < f(x_j)$. Throughout this proof, $n \geq 6$.

The position function $p : V(G) \to \{x_0, x_1, ..., x_{2n+1}\}$ is defined as follows. Let n be even:

$$\begin{aligned}
p(z_1) &= x_0, \\
p(z_2) &= x_1, \\
p(w) &= x_2, \\
p(u_{2i-1}) &= x_{2i+1}, \ \ i = 1, 2, 3, ..., \frac{n}{2}, \\
p(v_{2i-1}) &= x_{2i+2}, \ \ i = 1, 2, 3, ..., \frac{n}{2}, \\
p(u_{2i}) &= x_{n+2i}, \ \ i = 1, 2, 3, ..., \frac{n-2}{2}, \\
p(u_{2i}) &= x_{n+2i+1}, \ \ i = 1, 2, 3, ..., \frac{n-2}{2}, \\
p(z) &= x_{2n+1}.
\end{aligned}$$

When n is odd. The value of i changes slightly and satisfy

$$\begin{array}{rcl} p(z_1) &=& x_0, \\ p(z_2) &=& x_1, \\ p(w) &=& x_2, \\ p(u_{2i-1}) &=& x_{2i+1},\ \ i = 1,2,3,...,\dfrac{n-1}{2}, \\ p(v_{2i-1}) &=& x_{2i+2},\ \ i = 1,2,3,...,\dfrac{n-1}{2}, \\ p(u_{2i}) &=& x_{n+2i},\ \ i = 1,2,3,...,\dfrac{n-1}{2}, \\ p(u_{2i}) &=& x_{n+2i+1},\ \ i = 1,2,3,...,\dfrac{n-1}{2}, \\ p(z) &=& x_{2n+1}. \end{array}$$

The above defined position function orders the sets

$$\{z_1, z_2, w, u_1, v_1, u_3, v_3, ..., u_{n-1}, v_{n-1}, u_2, v_2, u_4, v_4, ..., u_{n-2}, v_{n-2}, z\},$$

$$\{z_1, z_2, w, u_1, v_1, u_3, v_3, ..., u_{n-2}, v_{n-2}, u_2, v_2, u_4, v_4, ..., u_{n-1}, v_{n-1}, z\}$$

of vertices for even and odd n respectively to the set $\{x_0, x_1, ..., x_{2n+1}\}$.

Figure 2 shows the rename vertices of $P_2 \bigodot W_6$ and $P_2 \bigodot W_7$. Now, the radio labeling $f : V(P_2 \bigodot W_n) \to \mathbb{Z}^+$ is defined by

$$f(x_i) = \begin{cases} 1, & \text{if } i = 0; \\ 2i, & \text{if } i = 1,2; \\ 3+i, & \text{if } 3 \le i \le 2n; \\ 2n+5, & \text{if } i = 2n+1. \end{cases}$$

Claim: The labeling f is a radio labeling by showing that the radio condition $|f(u) - f(v)| \ge 4 - d(u,v)$ satisfy for all pair (u,v), where $u \ne v$.

Case 1: Consider (z_1, z_2). Recall $p(z_1) = x_0$ and $p(z_2) = x_1$. Then radio condition

$$f(x_1) - f(x_0) = 1 \ge \text{diam}(P_2 \bigodot W_n) + 1 - 3$$

is satisfied because $d(z_1, z_2) = 3$.

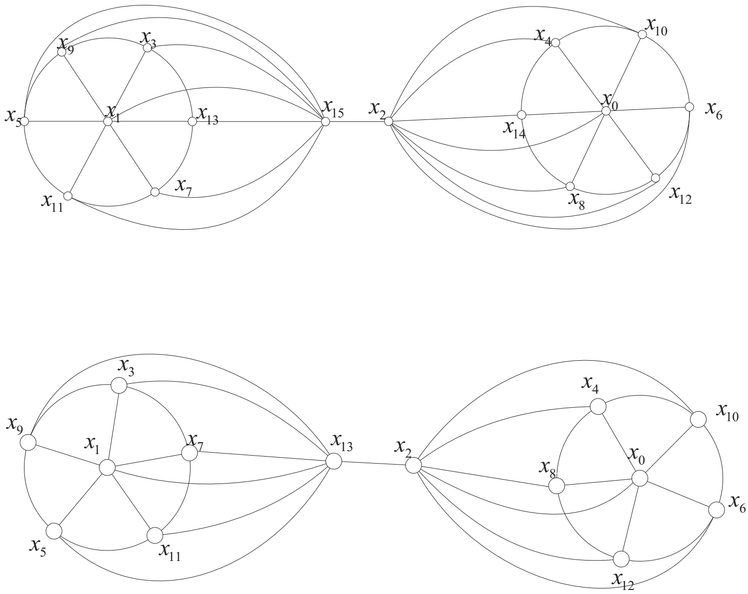


Figure 2

Case 2: Consider the pair (z_2, w). Recall $p(z_2) = x_1$ and $p(w) = x_2$. Then the radio condition

$$f(x_2) - f(x_1) = 2 \geq \text{diam}(P_2 \bigodot W_n) + 1 - 2,$$

as $d(z_2, w) = 2$.

Case 3: Consider (w, r) for any vertex $r \neq w$. Recall $p(w) = x_2 = 4$. As $f(x_i) \geq 6$ for any $i \geq 3$, we have

$$|f(w) - f(x_i)| + d(w, x_i) \geq 2 + |4 - 6| \geq 4 = \text{diam}(P_2 \bigodot W_n) + 1.$$

Case 4: Consider (u_i, v_j) with $i = j$, their label differ by 1 as $d(u_i, v_j) = 3$ for all i, j. Thus $|f(u_i) - f(v_j)| + d(u_i, v_j) = 1 + 3 = 4$.

Case 5: Consider (u_i, u_j) and (v_i, v_j). If $(j \neq i + 1)$, $1 \leq i, j \leq 2n$. Then their label differ by 2 as $d(u_i, u_j) = 2$ and $d(v_i, v_j) = 2$. Thus

$$|f(u_i) - f(u_j))| + d(u_i, u_j) = 1 + 3 = 4$$

and

$$|f(v_i) - f(v_j))| + d(v_i, v_j) = 1 + 3 = 4.$$

Case 6: Consider (z, r) for any vertex $r \neq z$. Recall $p(z) = x_{2n+1}$. As $f(x_i) \geq 6$ for any $i \geq 3$, we have $|f(z) - f(x_i)| + d(z, x_i) \geq 2 + |2n+5-4| \geq 4 = \text{diam}(P_2 \bigodot W_n) + 1$. all the above cases establish the claim that f is a radio labeling of $P_2 \bigodot W_n$.

Thus $\text{rn}(P_2 \bigodot W_n) \leq span(f) = f(2n+1) = 1 + 2 + 2 + 2(n-1) + 2 = 2n + 5$. □

Theorem 2.4. *For any $n \geq 6$, $rn(P_2 \bigodot W_n) = 2n + 5$.*

Proof. Theorem 2.1 shows that for any $n \geq 6$, $\text{rn}(P_2 \bigodot W_n) \geq 2n + 5$, and Theorem 2.2 shows that for $n \geq 6$, $\text{rn}(P_2 \bigodot W_n) \leq 2n + 5$. Which completes the proof of Theorem 2.3. □

References

[1] G. Chartrand , D. Erwin, and P. Zhang, and F. Harary, Radio labelings of graphs, *Bull. Inst. Combin. appl.*, **33**, 2001, 77-85.

[2] C. Fernandez, A. Flores, M. Tomova, C. Wyels, The radio number of gear graphs, arxiv:0809.2623, September 15, 2008.

[3] W. K. Hale, Frequency assignment, Theory and application, Proc;IEEE, **68**, 1980.

[4] D. Liu and X. Zhu, Multi-level distance labelings for paths and cycles, *Siam J. Disc. Math.*, **19**, 2005, 610-621.

[5] D. Liu and M. Xie, Radio number for square cycles, *Congr. Numerantium*, **169**, 2004, 105-125.

[6] M. T. Rahim, M. Farooq, M. Ali and S. Jan, Multi-level distance labelings for generalized gear graphs, *International Journal of Mathematics and Soft Computing*, **Vol. 2,** No. 1 (2012), 57-63.