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EXACT SOLUTION OF THE CLASSICAL $SU(2)$ YANG-MILLS FIELD EQUATIONS

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Abstract: In this paper we find a new representation for self-duality equations. In addition exact solution class of the classical $SU(2)$ Yang-Mills field equations in four-dimensional Euclidean space and two exact solution classes for $SU(2)$ Yang- Mills equations when is ρ a complex analytic function are also obtained.

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1. Introduction

The self-dual Yang-Mills equations (a system of equations for Lie algebra valued functions of C^4) play a central role in the field of integrable systems and also play a fundamental role in several other areas of mathematics and physics, see [1]-[4]. In addition the self-dual Yang-Mills equations are of great importance in

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their own right and have found a remarkable number of applications in physics and mathematics as well. These equations arise in the context of gauge theory (see [5]), in classical general relativity (see [6], [7]), and can be used as a powerful tool in the analysis of 4-manifolds, see [8]. The Yang-Mills equations are a set of coupled, second-order partial differential equations in four dimensions for the Lie algebra-valued gauge potential functions A_μ, and are extremely difficult to solve in general. The self-dual Yang-Mills equations describe a connection for a bundle over the Grassmannian of two-dimensional subspaces of the twistor space, see [9], [10].

In this paper we found a new representation for self-duality equations. In addition exact solution class of the classical $SU(2)$ Yang-Mills field equations in four-dimensional Euclidean space and two exact solution classes for $SU(2)$ Yang-Mills equations when ρ is a complex analytic function are also obtained. This paper is organized as follows: This introduction followed by the new representation of the self-duality equations in Section 2. In Section 3 we found an exact solution class of the classical $SU(2)$ Yang-Mills field equations. Moreover two exact solution classes for self-dual $SU(2)$ gauge fields on Euclidean space when ρ is a complex analytic function are given in Section 4. Finally, we give some conclusions in Section 5.

2. New Representation of the Self-Duality Equations

The essential idea of Yang and Mills (1954) [11] is to consider an analytic continua-tion of the gauge potential A_μ into complex space where x_1, x_2, $x_3$and x_4 are complex. The self-duality equations $F_{\mu\nu} = {}^*F_{\mu\nu}$ are then valid also in complex space , in a region containing real space where the x's are real. Now consider four new complex variables y, $\bar{y}$,z and $\bar{z}$ defined by

$$\begin{aligned} \sqrt{2}\,y &= x_1 + i\,x_2, \qquad \sqrt{2}\,\bar{y} = x_1 - i\,x_2, \\ \sqrt{2}\,z &= x_3 - i\,x_4, \qquad \sqrt{2}\,\bar{z} = x_3 + i\,x_4, \end{aligned} \tag{1}$$

it is simple to check that the self-duality equations $F_{\mu\nu} = {}^*F_{\mu\nu}$ reduces to

$$F_{yz} = 0, \quad F_{\bar{y}\bar{z}} = 0, \quad F_{y\bar{y}} + F_{z\bar{z}} = 0. \tag{2}$$

Equations (2) can be immediately integrated, since they are pure gauge, to give [12]-[14]

$$A_y = D^{-1}\,D_y,\; A_z = D^{-1}\,D_z,\; A_{\bar{y}} = \bar{D}^{-1}\,\bar{D}_{\bar{y}},\; A_{\bar{z}} = \bar{D}^{-1}\,\bar{D}_{\bar{z}}, \tag{3}$$

where D and $\bar{D}$ are arbitrary 2×2 complex matrix functions of y, $\bar{y}$,z and $\bar{z}$, and with determinant $= 1$ (for SU(2) gauge group) and $D_y = \partial_y D$, etc. For real gauge fields $A_\mu \doteq -A_\mu^+$ (the symbol $\doteq$ is used for equations valid only for real values of x_1, x_2, x_3 and x_4), we require

$$\bar{D} \doteq (D^+)^{-1}. \tag{4}$$

Gauge transformations are the transformations

$$D \rightarrow DU, \;\; \bar{D} \rightarrow \bar{D}U, \;\; U^+ U \doteq I, \tag{5}$$

where U is a 2×2 matrix function of y, $\bar{y}$,z , $\bar{z}$ with determined $= 1$. Under transformation (5), equation (4) remains unchanged. We now define the hermitian matrix $\jmath$ [15]-[17] as

$$\jmath \equiv D\,\bar{D}^{-1} \doteq D\,D^+, \tag{6}$$

$\jmath$ has the very important property of being invariant under the gauge transformation (5). The only non vanishing field strengths in terms of $\jmath$ becomes

$$F_{u\,\bar{v}} = -\bar{D}^{-1}\,(\jmath^{-1}\,\jmath_u)_{\bar{v}}\,\bar{D}, \tag{7}$$

$(u\,, v = y\,, z)$ and the remaining self-duality equation (2) takes the form

$$(\jmath^{-1}\,\jmath_y)_{\bar{y}} + (\jmath^{-1}\,\jmath_z)_{\bar{z}} = 0. \tag{8}$$

The action density in terms of $\jmath$ [18]is

$$\begin{aligned} \phi(\jmath) &= -\frac{1}{2}\,Tr\,F_{\mu\,\nu}\,F_{\mu\,\nu} \\ &= -2\,Tr\Big(F_{y\,\bar{y}}\,F_{z\,\bar{z}} + F_{y\,\bar{z}}\,F_{\bar{y}\,z}\Big), \end{aligned} \tag{9}$$

where

$$F_{\mu\nu} = -[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu], \tag{10}$$

where $F_{\mu\nu}$ are the gauge field strengths. Our construction begins by explicit parametrization of the matrix $\jmath$

$$\jmath = \begin{pmatrix} \frac{1}{\phi} & \rho \\ \bar{\rho} & \phi(1+\rho\bar{\rho}) \end{pmatrix}, \tag{11}$$

and for real gauge fields $A_\mu \doteq -A_\mu^+$,we require $\phi \doteq real$,$\rho \doteq \rho^*$ ($\rho^* \equiv$ *complex conjugate of* ρ). The self-duality equations (8) take the form

$$\frac{1}{2}(1+\rho\bar{\rho})\partial_\mu\partial_\mu \ln\phi + \frac{1}{2}\rho\partial_\mu\partial_\mu\bar{\rho} + \frac{\bar{\rho}}{\phi}(\phi_y\rho_{\bar{y}} + \phi_z\rho_{\bar{z}})$$

$$+\frac{\rho}{\phi}(\phi_y\bar{\rho}_{\bar{y}}+\phi_z\bar{\rho}_{\bar{z}})+(\rho_{\bar{y}}\bar{\rho}_y+\rho_{\bar{z}}\bar{\rho}_z)=0, \tag{12}$$

$$\begin{aligned}&\frac{\phi}{2}\partial_\mu\partial_\mu\bar{\rho}+\frac{\bar{\rho}}{2}\partial_\mu\partial_\mu\phi-\frac{2\bar{\rho}}{\phi}(\phi_y\phi_{\bar{y}}+\phi_z\phi_{\bar{z}})\\&+(\phi_y\bar{\rho}_{\bar{y}}+\phi_z\bar{\rho}_{\bar{z}}-\phi_{\bar{y}}\rho_{\bar{y}}-\phi_{\bar{z}}\rho_{\bar{z}}),\end{aligned} \tag{13}$$

$$\begin{aligned}&\frac{\phi}{2}\partial_\mu\partial_\mu\rho+\frac{\rho}{2}\partial_\mu\partial_\mu\phi-\frac{2\rho}{\phi}(\phi_y\phi_{\bar{y}}+\phi_z\phi_{\bar{z}})\\&+(\phi_{\bar{y}}\rho_y+\phi_{\bar{z}}\rho_z-\phi_y\rho_{\bar{y}}-\phi_z\rho_{\bar{z}}),\end{aligned} \tag{14}$$

where $\partial_\mu\partial_\mu = 2(\partial_y\partial_{\bar{y}}+\partial_z\partial_{\bar{z}})$. The positive definite Hermitian matrix $\jmath = DD^+$ can be factored into a product upper and lower (or vice versa) triangular matrices as follows

$$\jmath = RR^+ = R^I R^{I+},\ R=\begin{pmatrix}\frac{1}{\sqrt{\phi}} & 0\\ \bar{\rho}\sqrt{\phi} & \sqrt{\phi}\end{pmatrix}, R^I=\begin{pmatrix}\sqrt{\phi^I} & \bar{\rho}^I\sqrt{\phi^I}\\ 0 & \frac{1}{\sqrt{\phi^I}}\end{pmatrix}, \tag{15}$$

$$\phi \doteq real, \qquad \bar{\rho}\doteq\rho^*, \qquad \bar{\rho}^I\doteq\rho^{I*}. \tag{16}$$

It is evident from (15) that one can choose a gauge so that $D = R$ or $D = R^I$ and it is easy to check that in both gauges the self-duality equations (12)-(14) (in the case of $D = R^I$ all the $\phi,\rho,\bar{\rho}$ are replaced by $\phi^I,\rho^I,\bar{\rho}^I$).

From equation (15) we see that $R^{-1}R^I$ is a unitary matrix so that we can always make a gauge transformation from the gauge R to the R^I gauge.

Theorem 1. *If $(\phi,\rho,\bar{\rho})$ satisfy equations (12)- (14) then so do $(\phi^I,\rho^I,\bar{\rho}^I)$ defined by (see [19])*

$$\phi^I=\frac{1}{\phi}(1+\rho\bar{\rho}), \qquad \rho^I=\bar{\rho}, \qquad \bar{\rho}^I=\rho. \tag{17}$$

Proof. By equating equation (15) ,we obtain the following equations

$$\frac{1}{\phi}=\phi^I(1+\rho^I\bar{\rho}^I),\quad \rho=\bar{\rho}^I,\quad \bar{\rho}=\rho^I,\quad \phi(1+\rho^I\bar{\rho}^I)=\frac{1}{\phi^I}. \tag{18}$$

We solve the system of equation (18) we obtain the relations in equation (17).

3. Exact Solution Class of the Classical SU(2) Yang-Mills Field Equations

To obtain an exact solution class of the classical SU(2) Yang-Mills field equations in four-dimensional Euclidean space, consider the system.

$$\frac{1}{2}(1+\rho\bar{\rho})\partial_\mu\partial_\mu \ln\phi + \frac{1}{2}\rho\partial_\mu\partial_\mu\bar{\rho} + \frac{\bar{\rho}}{\phi}(\phi_y\rho_{\bar{y}} + \phi_z\rho_{\bar{z}}) + \frac{\rho}{\phi}(\phi_y\bar{\rho}_{\bar{y}} + \phi_z\bar{\rho}_{\bar{z}}) + (\rho_{\bar{y}}\bar{\rho}_y + \rho_{\bar{z}}\bar{\rho}_z) = 0, \tag{19}$$

$$\frac{\phi}{2}\partial_\mu\partial_\mu\rho + \frac{\rho}{2}\partial_\mu\partial_\mu\phi - \frac{2\rho}{\phi}(\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) + (\phi_{\bar{y}}\rho_y + \phi_{\bar{z}}\rho_z - \phi_y\rho_{\bar{y}} - \phi_z\rho_{\bar{z}}). \tag{20}$$

Let us make the ansatz [20]

$$\phi = \phi(g), \qquad \rho = e^{ia}\sigma(g). \tag{21}$$

Where $g = g(x_1, x_2, x_3, x_4)$ is a real function of $x_\mu, \mu = 1,2,3,4$, ϕ *and* $\sigma$are real functions of g and a is a real constant. Then equations (19),(20) give the relations

$$(g_{y\bar{y}} + g_{z\bar{z}})((1+\sigma^2)\phi^2)' - 2(g_y g_{\bar{y}} + g_z g_{\bar{z}})\phi^2[(1+\sigma^2)\frac{\phi'}{\phi} + \sigma\sigma']' = 0, \tag{22}$$

$$(g_{y\bar{y}} + g_{z\bar{z}})(\phi\sigma)' + (g_y g_{\bar{y}} + g_z g_{\bar{z}})\phi^2[\frac{(\phi\sigma)'}{\phi^2}]' = 0, \tag{23}$$

Where the prime means differentiation with respect to g. The above relations imply that the determinant of the coefficients of $(g_{y\bar{y}} + g_{z\bar{z}})$ and $(g_y g_{\bar{y}} + g_z g_{\bar{z}})$ is zero i.e.

$$((1+\sigma^2)\phi^2)'[\frac{(\phi\sigma)'}{\phi^2}]' + 2[(1+\sigma^2)\frac{\phi'}{\phi} + \sigma\sigma']'(\phi\sigma)' = 0. \tag{24}$$

We shall determine ϕ *and* σ from the above equation (24), let $(\phi\sigma) = c$,where c is a constant, then $(\phi\sigma)' = 0$,

$$((1+\sigma^2)\phi^2)'[\frac{(\phi\sigma)'}{\phi^2}]' = 0, [(1+\sigma^2)\frac{\phi'}{\phi} + \sigma\sigma']'(\phi\sigma)' = 0. \tag{25}$$

We suppose

$$\phi = \sqrt{c}e^{-g}, \ \sigma = \sqrt{c}e^{g}, then \ \rho = \sqrt{c}e^{g+ia}. \tag{26}$$

Applying theorem (1) to ϕ and ρ of equation (26), then we get

$$\phi^I = \frac{e^g}{\sqrt{c}(1+ce^{2g})}, \quad \rho^I = \sqrt{c}e^{g-ia}, \quad \bar{\rho}^I = \sqrt{c}e^{g+ia}. \tag{27}$$

Equations (26) and (27) is a new class of solutions of Yang-Mills equations for self-dual SU(2)gauge fields.

4. Exact Solutions for Self-Dual $SU(2)$ Gauge Fields on Euclidean Space, when ρ is a Complex Analytic Function, see [21]

We reduce the equations for self-dual $SU(2)$gauge fields on Euclidean space to the following equations

$$\begin{aligned} &\frac{1}{2}(1+\rho\bar{\rho})\partial_\mu\partial_\mu \ln\phi + \frac{1}{2}\rho\partial_\mu\partial_\mu\bar{\rho} + \frac{\bar{\rho}}{\phi}(\phi_y\rho_{\bar{y}} + \phi_z\rho_{\bar{z}}) \\ &+\frac{\rho}{\phi}(\phi_y\bar{\rho}_{\bar{y}} + \phi_z\bar{\rho}_{\bar{z}}) + (\rho_{\bar{y}}\bar{\rho}_y + \rho_{\bar{z}}\bar{\rho}_z) = 0, \end{aligned} \tag{28}$$

$$\begin{aligned} &\frac{\phi}{2}\partial_\mu\partial_\mu\rho + \frac{\rho}{2}\partial_\mu\partial_\mu\phi - \frac{2\rho}{\phi}(\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) \\ &+(\phi_{\bar{y}}\rho_y + \phi_{\bar{z}}\rho_z - \phi_y\rho_{\bar{y}} - \phi_z\rho_{\bar{z}}). \end{aligned} \tag{29}$$

When ρ is a complex analytic function of y and z, then we have

$$\rho_{\bar{y}} = \bar{\rho}_z = 0 \quad , \quad \rho_y\rho_{\bar{y}} + \rho_z\rho_{\bar{z}} = 0. \tag{30}$$

Then, the self-dual Yang-Mills equations(28),(29)takes the form

$$\phi(\phi_{y\bar{y}} + \phi_{z\bar{z}}) - (\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) = 0, \tag{31}$$

$$\rho(\phi_{y\bar{y}} + \phi_{z\bar{z}}) - 2\frac{\rho}{\phi}(\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) + (\rho_y\phi_{\bar{y}} + \rho_z\phi_{\bar{z}}) = 0. \tag{32}$$

We consider now two cases:

(a) Let $\rho = \rho(\phi)$, then we find

$$\rho_y = \rho'\phi_y \quad , \quad \rho_z = \rho'\phi_z. \tag{33}$$

By using equation(33),then the two equations(31)and(32)become

$$\phi(\phi_{y\bar{y}} + \phi_{z\bar{z}}) - (\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) = 0, \tag{34}$$

$$\rho(\phi_{y\bar{y}} + \phi_{z\bar{z}}) - (\rho^{'} - 2\frac{\rho}{\phi})(\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) = 0. \quad (35)$$

If we do not consider the cases $(\phi_{y\bar{y}} + \phi_{z\bar{z}}) = 0$ and $(\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) = 0$, then we have

$$\phi\rho^{'} - \rho = 0, \quad (36)$$

by integration we obtain

$$\rho = c\phi, \quad (37)$$

where c is a complex constant both equations (34) and (35) reduce to the same equation. A solution is given by

$$\phi_y = \phi_z \quad , \quad \phi_{\bar{y}} = -\phi_{\bar{z}}. \quad (38)$$

The solution class is given by

$$\phi = F(y + z, \bar{y} - \bar{z}), \quad (39)$$

where F is an arbitrary function, equations (37) and (39) gives a new class of solutions of Yang-Mills equations for self-dual $SU(2)$ gauge fields. Applying theorem (1) to ϕ and ρ of equations (37) and (39) , then we get

$$\phi^I = \frac{1}{F(1 + c\bar{c}F^2)}, \quad \rho^I = \bar{c}F, \quad \bar{\rho}^I = cF. \quad (40)$$

(b) Let us make the ansatz [20]

$$\phi = \phi(g), \quad \rho = e^{ia}\sigma(g). \quad (41)$$

Where $g = g(x_1, x_2, x_3, x_4)$ is a real function of $x_\mu, \mu = 1, 2, 3, 4$, ϕ *and* $\sigma$are real functions of g and a is a real constant. Then equations (31),(32) give the relations

$$\phi\phi^{'}(g_{y\bar{y}} + g_{z\bar{z}}) + (g_y g_{\bar{y}} + g_z g_{\bar{z}})[\phi\phi^{''} - (\phi^{'})^2] = 0, \quad (42)$$

$$\sigma\phi^{'}(g_{y\bar{y}} + g_{z\bar{z}}) + (g_y g_{\bar{y}} + g_z g_{\bar{z}})[\sigma\phi^{''} - 2\frac{\sigma(\phi^{'})^2}{\phi} + \phi^{'}\sigma^{'})] = 0. \quad (43)$$

Where the prime means differentiation with respect to g. The above relations imply that the determinant of the coefficients of $(g_{y\bar{y}} + g_{z\bar{z}})$ and $(g_y g_{\bar{y}} + g_z g_{\bar{z}})$; is zero i.e.

$$\frac{\sigma^{'}}{\sigma} = \frac{\phi^{'}}{\phi}, \quad (44)$$

by integration (44) we obtain

$$\sigma(g) = c\phi(g), \qquad \rho = ce^{ia}\phi(g). \tag{45}$$

Applying theorem (1) to ϕ and ρ of equation (45), then we get

$$\phi^I = \frac{1}{\phi(g)(1 + c^2\phi^2(g))}, \quad \rho^I = ce^{-ia}\phi(g), \quad \bar{\rho}^I = ce^{ia}\phi(g). \tag{46}$$

Equations (45) and (46) is a new class of solutions of Yang-Mills equations for self-dual $SU(2)$ gauge fields.

5. Conclusions

A new class of solutions of Yang-Mills equations for self-dual $SU(2)$gauge fields are investigated. In this paper we found a new representation for self-duality equations. In addition exact solution class of the classical $SU(2)$ Yang-Mills field equations in four-dimensional Euclidean space and two exact solution classes for $SU(2)$ Yang-Mills equations when ρ is a complex analytic function are also obtained.

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