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OSCILLATION RESULTS FOR SECOND ORDER QUASILINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS WITH "MAXIMA"

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Abstract: In this paper, some new oscillation criteria are obtained for the second order quasilinear neutral delay differential equation

$$\left(a(t)\left((x(t)+p(t)x(\tau(t)))'\right)^{\alpha}\right)'+q(t)\max_{[t-\sigma,t]}x^{\beta}(s)=0,\quad t\geq t_0\geq 0.$$

Some of the known results and examples are also provided to illustrate the main results. These results extend some of the known results for the equations without "maxima".

AMS Subject Classification: 34K15
Key Words: oscillation, quasilinear, neutral, delay, differential equations with "maxima"

1. Introduction

This paper concerned with the oscillation problem of the second order quasilinear neutral delay differential equation with "maxima" of the form

$$\left(a(t)\left((x(t)+p(t)x(\tau(t)))'\right)^{\alpha}\right)'+q(t)\max_{[t-\sigma,t]}x^{\beta}(s)=0,\quad t\geq t_0\geq 0. \tag{1.1}$$

Throughout this paper, we will assume that the following conditions hold:

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(C_1) $\tau(t)$ and $\sigma(t)$ are continuous functions in $[t_0, \infty)$ with $\tau(t) \leq t,\ \sigma(t) \leq t$ and α, β are the ratio of odd positive integers;

(C_2) $p(t) \in C([t_0, \infty),\ R)$ with $0 \leq p(t) \leq p < 1$, and $q(t) \in C([t_0, \infty),\ R_+)$;

(C_3) $a(t) \in C([t_0, \infty),\ (0, \infty))$ and $\int_{t_0}^{\infty} \frac{ds}{a^{\frac{1}{\alpha}}(s)} < \infty$.

By a solution of equation (1.1) we mean a continuous function $x(t)$ defined on the interval $[t_0, \infty)$ such that $x(t) + p(t)x(\tau(t))$ and $((x(t) + p(t)x(\tau(t)))')^{\alpha}$ are continuously differentiable and $x(t)$ satisfies the equation (1.1). As is customary, a solution of (1.1) is called oscillatory if it has arbitrary large zeros. Otherwise it is called nonoscillatory. A solution $x(t)$ of equation (1.1) is said to be almost oscillation if $x(t)$ is either oscillatory or $|x(t)| \to 0$ monotonically as $t \to \infty$.

In the last few years, the qualitative theory of differential equations with "maxima" received very little attention even though such equations often arise in the problem of automatic regulation of various real system, see for example [1, 10, 12]. The oscillatory behavior of solutions of differential equations with "maxima" are discussed in [1-6, 11, 15], and the references cited therein. In [2], the authors have established some oscillation criteria for equation (1.1) when $\alpha = 1$ and $a(t) \equiv 1$. Also in [14], the authors discussed oscillatory behavior of equation (1.1) when $\alpha = 1$ and $\int_{t_0}^{\infty} \frac{dt}{a(t)} = \infty$. Motivated by these observations, in this paper, we analyse the oscillatory and asymptotic behavior of solutions of equation (1.1) under the condition $\int_{t_0}^{\infty} \frac{dt}{a^{\frac{1}{\alpha}}(s)} < \infty$.

In Section 2, we establish sufficient conditions for the almost oscillation of all solutions of equation (1.1). In Section 3, we present sufficient conditions for the existence of nonoscillatory solutions for the equation (1.1) using contraction mapping principle. In Section 4, we present some examples to illustrate the main results.

2. Oscillation Results

In this section, we will derive some new sufficient condition for the almost oscillation of equation (1.1). Define

$$z(t) = x(t) + p(t)x(\tau(t)),$$

and

$$A(t) = \int_{t}^{\infty} \frac{ds}{a^{\frac{1}{\alpha}}(s)}.$$

Lemma 2.1. *Let $x(t)$ be an eventually positive solution of equation* (1.1). *Then one of the following holds:*

(*I*) $z(t) > 0,\ z'(t) > 0$ *and* $(a(t)(z'(t))^{\alpha})' \leq 0$;

(*II*) $z(t) > 0,\ z'(t) < 0$ *and* $(a(t)(z'(t))^{\alpha})' \leq 0$.

Proof. Let $x(t)$ be an eventually positive solution of equation (1.1). Then we may assume that $x(\sigma(\tau(t))) > 0,\ x(\tau(t)) > 0$ for all $t \geq T$. Then inview of (C_2), we have $z(t) > 0$ for $t \geq T$. From the equation (1.1) we obtain

$$\left(a(t)\left(z'(t)\right)^{\alpha}\right)' = -q(t) \max_{[t-\sigma,t]} x^{\beta}(s) \leq 0.$$

Hence, $a(t)\left(z'(t)\right)^{\alpha}$ and $z(t)$ are of eventually of one sign. This completes the proof. □

Lemma 2.2. *Let $x(t)$ be an eventually negative solution of equation* (1.1). *Then one of the following holds:*

(*I*) $z(t) < 0,\ z'(t) < 0$ *and* $(a(t)(z'(t))^{\alpha})' \geq 0$;

(*II*) $z(t) < 0,\ z'(t) > 0$ *and* $(a(t)(z'(t))^{\alpha})' \geq 0$.

The proof of Lemma 2.2 is analogous to that of Lemma 2.1.

Lemma 2.3. *The function $x(t)$ is a negative solutions of equation* (1.1) *if and only if $-x(t)$ is a positive solution of the equation*

$$\left(a(t)\left((x(t) + p(t)x(\tau(t)))'\right)^{\alpha}\right)' + q(t) \min_{[t-\sigma,t]} x^{\beta}(s) = 0.$$

Proof. The assertion of Lemma 2.3 can be verified easly. □

Lemma 2.4. *Let $x(t)$ be a positive solution of equation* (1.1) *and let the corresponding $z(t)$ satisfy Lemma 2.1 (II). If*

$$\int_T^t \left(\frac{1}{a(s)} \int_s^{\infty} q(u)du\right)^{\frac{1}{\alpha}} ds = \infty \tag{2.1}$$

then

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0.$$

Proof. Let $x(t)$ be a positive solution of equation (1.1). Then by Lemma 2.1(II), we have $z(t) > 0$ and $z'(t) < 0$ for all $t \geq T$. Therefore $z(t) \to L \geq 0$ as $t \to \infty$. If $L > 0$, then for $\epsilon = \frac{L(1-p)}{2p} > 0$, there exists $T \geq t_0$ such that $L < z(t) < L + \epsilon$ for $t \geq T$. Then for $t \geq T$, we have

$$x(t) = z(t) - p(t)x(\tau(t)) > L - p(t)z(t) > L - p(L+\epsilon) = L_1.$$

From equation (1.1), we have

$$\left(a(t)\left(z(t))'\right)^{\alpha}\right)' \leq -q(t) \max_{[t-\sigma,t]} x^{\beta}(s).$$

Integrating from T to ∞ and using the fact that $a(t)\left(z'(t)\right)^{\alpha}$ is positive and decreasing we obtain

$$a(t)\left(z'(t)\right)^{\alpha} \geq \int_t^{\infty} q(s) \max_{[s-\sigma,s]} x^{\beta}(s)ds \geq L_1^{\beta} \int_t^{\infty} q(s)ds.$$

Divide the last inequality by $a(t)$ and then integrating the resulting inequality we obtain

$$z^{\alpha}(t) - z^{\alpha}(T) \geq L_1^{\beta} \int_T^t \left(\frac{1}{a(s)}\left(\int_s^{\infty} q(u)du\right)\right) ds$$

or

$$\infty > z(t) \geq L_1^{\beta/\alpha} \int_T^t \left(\frac{1}{a(s)}\left(\int_s^{\infty} q(u)du\right)\right)^{1/\alpha} ds$$

or

$$L_1^{\beta/\alpha} \int_T^t \left(\frac{1}{a(s)}\left(\int_s^{\infty} q(u)du\right)\right)^{1/\alpha} ds < \infty$$

which is contradiction to (2.1) and shows that $L = 0$, that is, $z(t) \to 0$. Since, $z(t) > x(t) > 0$ we have, $x(t) \to 0$ as $t \to \infty$. □

Theorem 2.5. *Let $\alpha = \beta = 1$ in the equation* (1.1). *If* (2.1) *and*

$$\lim_{t\to\infty} \frac{1}{t^m} \int_T^t (t-s)^m \left[q(s) \max_{[s-\sigma,s]} (1-p(u)) - \frac{m^2 a(s)}{4(t-s)^2}\right] ds = \infty \qquad (2.2)$$

where $m \geq 1$ is an integer, then every solution of equation (1.1) *is almost oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Then either $x(t) > 0$ eventually or $x(t) < 0$ eventually. We shall consider the case when $x(t) > 0$. Since the other case can be investigated analogously.

Let $x(\sigma(t)) > 0$, $x(\tau(t)) > 0$ for all $t \geq T$, where T is chosen so large enough that the conclusions of Lemma 2.1 hold for all $t \geq T$.

First we assume Lemma 2.1(I) holds. Then $z(t) = x(t)+p(t)x(\tau(t))$ implies that

$$x(t) \geq (1 - p(t))z(t) \tag{2.3}$$

and

$$\max_{[t-\sigma,t]} x(s) \geq \max_{[t-\sigma,t]} (1 - p(s))z(s) = z(t) \max_{[t-\sigma,t]} (1 - p(t)). \tag{2.4}$$

From the equation (1.1) and (2.4) we have

$$\left(a(t)(z'(t))^{\alpha}\right)' + q(t)z(t) \max_{[t-\sigma,t]} (1 - p(s)) \leq 0, \quad t \geq T. \tag{2.5}$$

Define

$$w(t) = \frac{a(t)z'(t)}{z(t)}, \quad t \geq T. \tag{2.6}$$

Then from (2.5) and (2.6) we have

$$w'(t) + \frac{w^2(t)}{a(t)} + q(t) \max_{[t-\sigma,t]} (1 - p(s)) \leq 0.$$

Multiply by $(t - s)^m$ and then integrating from T to t, we obtain

$$\begin{aligned} -w(T)(t-T)^m + \int_T^t m(t-s)^{m-1}w(s)ds + \int_T^t (t-s)^m \frac{w^2(s)}{a(s)}dt \\ + \int_T^t (t-s)^m q(s) \max_{[s-\sigma,s]} (1-p(u))ds \leq 0 \end{aligned}$$

or

$$\begin{aligned} &\int_T^t (t-s)^m q(s) \max_{[s-\sigma,s]} (1-p(u))ds \\ &\quad \leq w(T)(t-T)^m - \int_T^t \left[(t-s)^m \frac{w^2(s)}{a(s)} + m(t-s)^{m-1}w(s)\right] ds \\ &\quad \leq w(T)(t-T)^m - \int_T^t \frac{(t-s)^m}{a(s)} \left[w(s) + \frac{1}{2}\frac{ma(s)}{(t-s)}\right]^2 ds \\ &\qquad + \frac{1}{4}\int_T^t m^2 a(s)(t-s)^{m-2}ds \\ &\quad \leq w(T)(t-T)^m + \int_T^t \frac{m^2}{4} a(s)(t-T)^{m-2}ds \end{aligned}$$

or

$$\int_T^t (t-s)^m \left[q(s) \max_{[s-\sigma,s]} (1-p(u)) - \frac{m^2 a(s)}{4(t-s)^2} \right] ds \le w(T)(t-T)^m.$$

Dividing the last inequality by t^m and then taking limit supremum, we obtain

$$\lim_{t\to\infty} \frac{1}{t^m} \int_T^t (t-s)^m \left[q(s) \max_{[s-\sigma,s]} (1-p(u)) - \frac{m^2 a(s)}{4(t-s)^2} \right] ds \le w(T) < \infty$$

which is a contradiction to (2.2).

Next assume that Lemma 2.1(II) holds. Then by Lemma 2.4 we obtain that $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Theorem 2.6. *Assume that* $\beta > 1$. *If* (2.1) *and*

$$\int_{t_0}^{\infty} q(t) A^\beta (t-\sigma) \max_{[t-\sigma,t]} (1-p(s))^\beta dt = \infty \tag{2.7}$$

hold then every solution of equation (1.1) *is almost oscillatory.*

Proof. Assume that there exists a nonoscillatory solution $x(t)$ of equation (1.1) such that $x(\sigma(t)) > 0$, $x(\tau(t)) > 0$ for all $t \le T$ where T is chosen large enough that the conclusion of Lemma 2.1 hold for all $t \ge T$. Integrating (1.1) from T to t yields,

$$a(t)(z'(t))^\alpha - a(T)(z'(T))^\alpha + \int_T^t q(s) \max_{[s-\sigma,s]} x^\beta(u) ds = 0. \tag{2.8}$$

Letting $t \to \infty$, we have

$$\int_T^\infty q(s) \max_{[s-\sigma,s]} x^\beta(u) ds < \infty. \tag{2.9}$$

In this case $z(t)$ is increasing, so there exists a positive number C such that $z(t) > C$ for $t \ge T$. This, together with (2.3) yields.

$$x^\beta(t) \ge C^\beta (1-p(t))^\beta \quad \text{for} \quad t \ge T.$$

Now $A^\beta(t) \to 0$ as $t \to \infty$. So there exists $T_1 \ge T$ such that

$$x^\beta(t) \ge C^\beta (1-p(t))^\beta A^\beta(t) \quad \text{for} \quad t \ge T_1.$$

Then

$$\begin{aligned}\max_{[t-\sigma,t]} x^\beta(s) &\geq C^\beta \max_{[t-\sigma,t]}(1-p(s))^\beta \max_{[t-\sigma,t]} A^\beta(s)\\ &\geq C^\beta \max_{[t-\sigma,t]}(1-p(s))^\beta A^\beta(t-\sigma). \end{aligned}\tag{2.10}$$

Combining (2.9) and (2.10) we have

$$C^\beta \int_T^\infty q(s)A^\beta(s-\sigma) \max_{[s-\sigma,s]}(1-p(u))^\beta ds < \infty \tag{2.11}$$

which contradicts (2.11). □

Next assume that Lemma 2.1(II) holds. Then by Lemma 2.4, we obtain that $\lim_{t\to\infty} x(t) = 0$. This completes the proof.

Theorem 2.7. *Assume that* $0 < \beta < 1$. *If* (2.1) *and*

$$\int_{t_0}^\infty q(t)A(\sigma(t))dt = \infty \tag{2.12}$$

hold then every solution of equation (1.1) *is almost oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.2, we have that Lemma 2.1 holds. For Case(I), we have (2.9) and (2.10). For large t, we have $A(t) \leq 1$ and $A^\beta(t) \geq A(t)$. So (2.11) implies

$$\int_T^\infty \max_{[t-\sigma,t]}(1-p(s))^\beta A(s)ds < \infty.$$

This contradicts (2.10). Next assume that Lemma 2.1(II) holds. Then by Lemma 2.4, we obtain that $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

3. Existence of Nonoscillatory Solutions

In this section, we provide sufficient conditions for the existence of nonoscillatory solutions of equation (1.1) in case $\alpha > \beta > 1$ or $\beta < \alpha < 1$. Note that in this section we do not require $p(t) \equiv p$.

Theorem 3.1. *Assume that $\alpha > \beta > 1$. If*

$$\int_{t_0}^{\infty} \left(\frac{1}{a(t)} \int_t^{\infty} q(s)ds \right)^{1/\alpha} dt < \infty$$

and

$$\int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(t)} \left(\int_t^{\infty} q(s)ds \right) dt < \infty$$

then the equation (1.1) *has a bounded nonoscillatory solution.*

Proof. Choose $T \geq t_0$ sufficiently large so that

$$\int_{T}^{\infty} \left(\frac{1}{a(t)} \int_t^{\infty} q(s)ds \right) dt \leq \frac{1-p}{8},$$

and

$$\int_{T}^{\infty} \frac{1}{a^{1/\alpha}(t)} \left(\int_t^{\infty} q(s)ds \right) dt \leq \frac{(1-p)^2}{16}.$$

Let ψ be the set of all bounded continuous function on $[t_0, \infty)$ with norm

$$||x|| = \max_{t \geq t_0} \{x(t)\}$$

and let

$$S = \{x \in \psi : \frac{1-p}{4} \leq x(t) \leq 1,\ t \geq t_0\}.$$

Define the operator $T : S \to \psi$ by

$$(Tx)(t) = \begin{cases} \frac{5p+3}{8} - p(t)x(t-\tau) \\ \quad + \int_T^t \left(\frac{1}{a(s)} \int_s^{\infty} q(u) \max_{[u-\sigma,u]} x^{\beta}(v)du \right)^{1/\alpha} ds, & t \geq T \\ (Tx)(t), & t_0 \leq t < T. \end{cases}$$

Clearly, T is continuous, now for every $x \in S$ and $t \geq T$

$$\begin{aligned} (Tx)(t) &\geq \frac{5p+3}{8} - p(t)x(t-\tau) \\ &\geq \frac{5p+3}{8} - p \\ &\geq \frac{3(1-p)}{8} > \frac{1-p}{4}. \end{aligned}$$

Also,

$$\begin{aligned}(Tx)(t) &\le \frac{5p+3}{8}+\int_T^t\left(\frac{1}{a(s)}\int_s^\infty q(u)\max_{[u-\sigma,u]}x^\beta(v)du\right)^{1/\alpha}ds\\ &\le \frac{5p+3}{8}+\int_T^t\left(\frac{1}{a(s)}\int_s^\infty q(u)du\right)^{1/\alpha}ds\\ &\le \frac{5p+3}{8}+\frac{1-p}{8}\\ &\le \frac{p+1}{2}<1.\end{aligned}$$

Thus we have that $TS\subset S$. Since S is bounded closed and convex subset of ψ. We only need to show that T is contraction mapping on S, in order to apply contraction principle. For $x,y\in S$ and $t\ge T$, we have

$$\begin{aligned}&|(Tx)(t)-(Ty)(t)|\\ &\le p(t)\max_{[t-\sigma,t]}\mid x(s-\tau)-y(s-\tau)\mid\\ &\quad+|\int_T^t\left(\frac{1}{a(s)}\int_s^\infty q(u)\max_{[u-\sigma,u]}x^\beta(v)du\right)^{1/\alpha}ds\\ &\quad-\int_T^t\left(\frac{1}{a(s)}\int_s^\infty q(u)\max_{[u-\sigma,u]}y^\beta(v)du\right)^{1/\alpha}ds|\\ &\le p||x-y||+\int_T^\infty\frac{1}{a^{1/\alpha}}(t)|\left(\int_t^\infty q(s)\max_{[s-\sigma,s]}x^\beta(s)ds\right)^{1/\alpha}\\ &\quad-\left(\int_t^\infty q(t)\max_{[t-\sigma,t]}y^\beta(t)dt\right)^{1/\alpha}|dt\end{aligned}$$

By the Mean Value Theorem for derivative applied to the function $V(u)=u^\beta,\ \alpha>\beta>1$, we see that for any $x,y\in S$, we have

$$\left|x^{1/\alpha}-y^{1/\alpha}\right|\le\frac{4}{\alpha(1-p)}|x-y|$$

and

$$\left|x^\beta-y^\beta\right|\le 2\beta||x-y||.$$

Thus,

$$||Tx-Ty|| \le p||x-y||+\frac{4}{\alpha(1-p)}\int_T^\infty\frac{1}{a^{1/\alpha}(t)}$$

$$\begin{aligned}
&\left(\int_t^\infty q(s)\max_{[s-\sigma,s]}\left|x^\beta(v)-y^\beta(v)\right|ds\right)dt\\
\leq\ & p||x-y||+\frac{4}{\alpha(1-p)}2\beta\\
&\int_T^\infty\left(\frac{1}{a^{1/\alpha}(t)}\int_t^\infty q(s)ds\right)dt||x-y||\\
\leq\ & \left(p+\frac{8\beta}{\alpha(1-p)}\frac{(1-p)^2}{16}\right)||x-y||\\
\leq\ & \left(\frac{p+1}{2}\right)||x-y||<||x-y||
\end{aligned}$$

and we see that T is a contraction on S. Hence, T has a unique fixed point which is clearly a positive solution of equation (1.1). This completes the proof. □

Theorem 3.2. *Assume that $\beta < \alpha < 1$. If*

$$\int_{t_0}^\infty\left(\frac{1}{a(t)}\int_t^\infty q(s)ds\right)^{1/\alpha}dt<\infty$$

then the equation (1.1) *has a bounded nonoscillatory.*

Proof. Choose $T \geq t_0$ sufficiently large so that,

$$\int_T^\infty\left(\frac{1}{a(t)}\int_t^\infty q(s)ds\right)^{1/\alpha}dt<\frac{1-p}{9},$$

and

$$\int_T^\infty\left(\frac{1}{a^{1/\alpha}(t)}\int_t^\infty q(s)ds\right)dt\leq\frac{5(1-p)^2}{27}.$$

Let, ψ be the set of all bounded continuous function on $[t_0,\infty)$ with norm

$$||x||=\max_{t\geq t_0}\{x(t)\}$$

and let

$$S=\{x\in\psi:\frac{5(1-p)}{9}\leq x(t)\leq 1,\ t\geq t_0\}.$$

Define the operator $T: S\to\psi$ by

$$(Tx)(t)=\begin{cases}\frac{2p+7}{9}-p(t)x(t-\tau)\\ \quad+\int_T^t\left(\frac{1}{a(s)}\int_s^\infty q(u)\max_{[u-\sigma,u]}x^\beta(v)du\right)^{1/\alpha}ds, & t\geq T\\ \\ (Tx)(t), & t_0\leq t<T.\end{cases}$$

Clearly, T is continuous. Now for every $x \in S$ and $t \geq T$.

$$Tx(t) \geq \frac{2p+7}{9} - p(t)x(t-\tau) \geq \frac{2p+7}{9} - p > \frac{5(1-p)}{9}.$$

Also,

$$\begin{aligned} Tx(t) &\leq \frac{2p+7}{9} + \int_T^t \left(\frac{1}{a(s)} \int_s^\infty q(u) \max_{[u-\sigma,u]} x^\beta(v)du \right)^{1/\alpha} ds \\ &\leq \frac{2p+7}{9} + \int_T^t \left(\frac{1}{a(s)} \int_s^\infty q(u)du \right)^{1/\alpha} ds \\ &\leq \frac{2p+7}{9} + \frac{1-p}{9} \leq \frac{p+8}{9} \leq 1. \end{aligned}$$

Thus, we have that $TS \subset S$. Since S is bounded closed and convex subset of ψ. We only need to show that T is contraction mapping on S in order to apply contraction principle. For $x, y \in S$ and $t \geq T$ we have

$$\begin{aligned} &|(Tx)(t) - (Ty)(t)| \\ &\leq p(t) \max_{[t-\sigma,t]} |x(s-\tau) - y(s-\tau)| \\ &\quad + | \int_T^t \left(\frac{1}{a(s)} \int_s^\infty q(u) \max_{[u-\sigma,u]} x^\beta(v)du \right)^{1/\alpha} ds \\ &\quad - \int_T^t \left(\frac{1}{a(s)} \int_s^\infty q(u) \max_{[u-\sigma,u]} y^\beta(v)du \right)^{1/\alpha} ds| \\ &\leq p||x-y|| + \int_T^\infty \frac{1}{a^{1/\alpha}(t)} | \left(\int_s^\infty q(u) \max_{[u-\sigma,u]} x^\beta(v)du \right)^{1/\alpha} \\ &\qquad - \left(\int_s^\infty q(u) \max_{[u-\sigma,u]} y^\beta(v)du \right)^{1/\alpha} |dt. \end{aligned}$$

By the Mean Value Theorem of derivatives applied to the function $V(u) = u^\beta$, $\beta < \alpha < 1$. We see that for any $x, y \in S$, we have

$$\left| x^{1/\alpha} - y^{1/\alpha} \right| \leq \frac{2}{\alpha} |x-y|$$

and

$$\left| x^\beta - y^\beta \right| \leq \frac{9\beta}{5(1-p)} ||x-y||.$$

Hence

$$||Tx - Ty|| \leq p||x-y|| + \frac{9\beta}{5(1-p)} \int_T^\infty \frac{1}{a^{1/\alpha}(t)}$$

$$
\begin{aligned}
& \left(\int_t^\infty q(s) \max_{[s-\sigma,s]} \left|x^\beta(v) - y^\beta(v)\right| ds\right) dt \\
\leq\ & p||x-y|| + \frac{9\beta}{5(1-p)} \frac{2}{\alpha} \\
& \qquad \int_T^\infty \left(\frac{1}{a^{1/\alpha}(t)} \int_t^\infty q(s)ds\right) dt||x-y|| \\
\leq\ & p\left(1 + \frac{18}{5(1-p)} \frac{5(1-p)^2}{27}\right) ||x-y|| \\
\leq\ & \left(\frac{p+2}{3}\right) ||x-y|| < ||x-y||.
\end{aligned}
$$

Thus, T is a contraction mapping. So T has a unique fixed point x, that is clearly a positive solution of equation (1.1). This completes the proof. □

4. Examples

In this section we present some examples to illustrate the main results.

Example 4.1. Consider the differential equation

$$
\left(e^{2t}(x(t) + \frac{1}{2}x(\tau-1))'\right)' + \frac{e^{2t}(2+e)}{2e} \max_{[t-1,t]} x(s) = 0. \tag{4.1}
$$

One can easily verify that all conditions of Theorem 2.5 are satisfied and hence every solution of equation (1.1) is almost oscillatory. Infact $x(t) = e^{-t}$ is one such solution of equation (4.1).

Example 4.2. Consider the differential equation

$$
\left(e^{2t}\left((x(t) + \frac{1}{2}x(\tau-1))'\right)^3\right)' + e^{4t-3}\left(1 + \frac{e}{2}\right)^3 \max_{[t-1,t]} x^3(s) = 0. \tag{4.2}
$$

We can easily check that all conditions of Theorem 2.6 are satisfying and hence every solution of equation (1.1) is almost oscillatory.

Example 4.3. Consider the differential equation

$$
\left(e^{5t}\left((x(t) + \frac{1}{3}x(\tau-1))'\right)^3\right)' + 2e^{\frac{7t+1}{3}}\left(1 + \frac{e}{3}\right)^3 \max_{[t-1,t]} x^{1/3}(s) = 0. \tag{4.3}
$$

It is easy to prove that all conditions of Theorem 2.7 are satisfies and hence every solution of equation (1.1) is almost oscillatory.

Example 4.4. Consider the differential equation

$$\left(e^{7t}\left((x(t)+\frac{1}{2}x(\tau-2))'\right)^5\right)'+2e^{5t-6}\left(1+\frac{e^2}{3}\right)^5\max_{[t-2,t]}x^3(s)=0. \qquad (4.4)$$

It is easily verified that all conditions of Theorem 3.1 are satisfies and hence every nonoscillatory solution of equation (4.4) tends to zero as $t\to\infty$. Infact $x(t)=e^{-t}$ is one such solution of equation (4.4).

Example 4.5. Consider the differential equation

$$\left(e^{t}\left((x(t)+x(\tau-1))'\right)^{1/5}\right)'+4e^{\frac{29t}{30}}(1+e)^{1/5}\max_{[t-1,t]}x^{1/6}(s)=0,\ t\geq 1. \quad (4.5)$$

It is easy to see that all conditions of Theorem 3.2 are satisfies and hence every bounded nonoscillatory solution of equation (4.5) tends to zero as $t\to\infty$. Infact $x(t)=e^{-t}$ is one such solution of equation (4.5).

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