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ON THE HILBERT FUNCTION OF GENERAL UNIONS OF A RATIONAL CURVE AND 2-POINTS

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Abstract: We study the Hilbert function of general unions in $\mathbb{P}^r$, $r = 3, 4$, of a prescribed number of double points and a rational curve with a prescribed degree.

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1. Introduction

For each $P \in \mathbb{P}^r$, $r \geq 3$, let $2P$ be the closed subscheme of $\mathbb{P}^r$ with $(\mathcal{I}_P)^2$ as its ideal sheaf. We will say that $2P$ is a 2-point and that P is the support of $2P$. A scheme $X \subset \mathbb{P}^r$ is said to have *maximal rank* if for every integer $t > 0$ either $h^0(\mathcal{I}_X(t)) = 0$ or $h^1(\mathcal{I}_X(t)) = 0$. We prove the following results.

Theorem 1. *Fix integers $k \geq 3$, $d \geq 0$, $a \geq 0$ such that $(d, a) \notin \{(0, 9), (2, 3)\}$. If $k = 4$, then assume $(d, a) \neq (0, 9)$. Let $X \subset \mathbb{P}^3$ be a general union of a 2-points and one degree d smooth rational curve. Then either $h^0(\mathcal{I}_X(k)) = 0$ or $h^1(\mathcal{I}_X(k)) = 0$.*

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Corollary 1. *Fix* $(d, a) \in \mathbb{N}^2 \setminus \{(0, 0)\}$ *such that* $3d + 1 + 4a \geq 20$ *and* $(d, a) \neq (0, 0)$*. Let* $X \subset \mathbb{P}^3$ *be a general union of* a *2-points and one degree* d *smooth rational curve. Then* X *has maximal rank.*

If $(d, a) = (3, 2)$, then $h^0(\mathcal{I}_X(3)) = 2$ and $h^1(\mathcal{I}_X(3)) = 1$ (see Lemma 1).

Theorem 2. *Fix integers* $k \geq 5$*,* $d \geq 0$*,* $a \geq 0$ *such that* $(d, a) \neq (0, 0)$*. Let* $X \subset \mathbb{P}^4$ *be a general union of* a *2-points and one degree* d *smooth rational curve and* a *2-points. Then either* $h^0(\mathcal{I}_X(k)) = 0$ *or* $h^1(\mathcal{I}_X(k)) = 0$*.*

Corollary 2. *Fix* $(d, a) \in \mathbb{N}^2 \setminus \{(0, 0)\}$ *such that* $5d + 1 + 5a \geq 126$*, i.e.* $d + a \geq 25$*. Let* $X \subset \mathbb{P}^4$ *be a general union of* a *2-points and one degree* d *smooth rational curve. Then* X *has maximal rank.*

For the proofs of the corollaries, see Remark 1.

2. Preliminaries

For any finite subset $S \subset \mathbb{P}^r$ set $2S := \cup_{O \in S} 2O$. If $H \subset \mathbb{P}^r$ is a hyperplane, $P \in S$ and $S \subset H$ is a finite set, then set $\{2P, H\} := 2P \cap H$ and $\{2S, H\} = 2S \cap H$.

Let $X \subset \mathbb{P}^r$ be a closed subscheme. Fix a hypersurface $T \subset \mathbb{P}^r$ and set $y := \deg(T)$. The residual scheme $\mathrm{Res}_T(X)$ of X with respect to t is the closed subscheme of $\mathbb{P}^r$ with $\mathcal{I}_X : \mathcal{I}_T$ as its ideal sheaf. For each $x \in \mathbb{Z}$ we have an exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_T(X)}(x - y) \to \mathcal{I}_X(x) \to \mathcal{I}_{X \cap T, T}(x) \to 0$$

which is often called the Castelnuovo's sequence or the Horace sequence.

Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. For any $P \in Q$, any finite subset $S \subset H$ set $\{2P, Q\} := 2P \cap Q$ and $\{2S, Q\} := 2S \cap Q$. Call $|\mathcal{O}_Q(1, 0)|$ and $|\mathcal{O}_Q(0, 1)|$ the two rulings of Q. Fix $D \in |\mathcal{O}_Q(e, f)|$ and a closed subscheme A of Q. The residual scheme $\mathrm{Res}_D(A)$ of A with respect to D is the closed subscheme of Q with $\mathcal{I}_A : \mathcal{I}_D$ as its ideal sheaf. For all $(a, b) \in \mathbb{Z}^2$ we have an exact sequence (called again a Castelnuovo's sequence):

$$0 \to \mathcal{I}_{\mathrm{Res}_D(A)}(a - e, b - f) \to \mathcal{I}_A(a, b) \to \mathcal{I}_{A \cap D, D}(a, b) \to 0$$

We use that convention that $\emptyset$ is the only rational curve of degree zero. For all $r \geq 3$ and all $(d, a) \in \mathbb{N}^2$ let $F(d, a; r)$ denote the set of all schemes $X \subset \mathbb{P}^r$ which are disjoint unions of a smooth rational curve of degree d and a 2-points (with the convention $F(0, 0; r) = \{\emptyset\}$). Set $F(d, a) := F(d, a; 3)$. For all integers $k \geq 0$ and $d > 0$ set $[d, k] := dk + 1$ and $[0, k] := 0$. With this convention we have $h^0(\mathcal{O}_X(k)) = [d, k] + (r+1)a$ and $h^1(\mathcal{O}_X(k)) = 0$ for all $X \in F(d, a; r)$, $(d, a) \neq (0, 0)$, and all $k \geq 0$.

Remark 1. For each $(d, a) \in \mathbb{N}^2 \setminus \{(0, 0)\}$, each $k \in \mathbb{N}$ and each $X \in F(d, a; r)$ we have $h^0(\mathcal{O}_X(k)) = [d, k] + (r+1)a$ and $h^1(\mathcal{O}_X(k)) = 0$. Hence by the Castelnuovo-Mumford's lemma to prove that $h^1(\mathcal{I}_X(t)) = 0$ for all $t \geq x$ (where x is any positive integer) it is sufficient to prove that $h^1(\mathcal{I}_X(x)) = 0$. Hence Corollaries 1 and 2 are immediate consequences of Theorems 1 and 2 and the equalities $\binom{6}{3} = 20$, $\binom{9}{4} = 126$.

Remark 2. Fix a hyperplane $H \subset \mathbb{P}^r$, $r \geq 3$, and any integer $d > 0$. Fix any $C \in F(d, 0; r)$ and let N_C be the normal bundle of C. Since the vector bundle $T\mathbb{P}^r(-1)$ is spanned (use the Euler's sequence) and the restriction map $T\mathbb{P}^r|C \to N_C$ is surjective, the vector bundle $N_C(-1)$ is spanned. Since C has genus zero, $N_C(-1)$ is a direct sum of $r-1$ line bundles of degree ≥ 0. Hence $h^1(N_C(-1)) = 0$. Hence $X \cap H$ is a general subset of H with cardinality d for a general $X \in F(d, 0; r)$ ([21, Theorem 1.5]. Now assume $r = 3$ and $d \neq 2$. The vector bundle N_C is a direct sum of 2 line bundles of degree $2d - 1$ ([14], [15], [22], [23]). Hence if $d \geq 2$ for a general $X \in F(d, 0; 3)$, $d \neq 2$, the set $Q \cap X$ is a general subset of Q with cardinality $2d$.

Example 1. Take $k = 2$, $a > 0$ and $d > 0$. It is sufficient to do the following cases (d, a): $(1, 1)$, $(2, 1)$, $(1, 2)$. The first one is not defective, because $h^0(\mathcal{I}_{Y(1,1)}(2)) = 3$ and hence $h^1(\mathcal{I}_{Y(1,1)}(2)) = 0$. We have $h^0(\mathcal{I}_{Y(2,1)}(2)) = 1$ (the only element of $|\mathcal{I}_{Y(2,1)}(2)|$ is the obvious irreducible quadric cone) and hence $h^1(\mathcal{I}_{Y(2,1)}(2)) = 0$. Fix any two points $P_1, P_2 \in \mathbb{P}^3$, $P_1 \neq P_2$, and call R the line spanned by $\{P_1, P_2\}$. Fix a line $L \subset \mathbb{P}^3$ such that $L \cap R = \emptyset$. No quadric surface with singular locus containing R may contain L. Hence $h^0(\mathcal{I}_{Y(1,2)}(2)) = 0$. Obviously $h^1(\mathcal{I}_{Y(d,0)}(2)) \cdot h^0(\mathcal{I}_{Y(d,0)}(2)) = 0$ for all $d > 0$, and $h^0(\mathcal{I}_{Y(0,a)}(2)) = 0$, if $a \geq 4$, $h^0(\mathcal{I}_{Y(0,a)}(2)) = \binom{5-a}{2}$ if $1 \leq a \leq 3$.

Lemma 1. *Fix integers $d > 0$ and $a > 0$. If $3d + 4a + 1 \leq 20$ and $(d, a) \neq (2, 3)$, then $h^1(\mathcal{I}_{Y(d,a)}(3)) = 0$. If $3d + 4a + 1 \geq 20$, then $h^0(\mathcal{I}_{Y(d,a)}(3)) = 0$. We have $h^0(\mathcal{I}_{Y(2,3)}(3)) = 2$ and $h^1(\mathcal{I}_{Y(2,3)}(3)) = 1$.*

Proof. Since a general rational curve of degree d has maximal rank, it is sufficient to check the following pairs (d, a): $(1, 4)$, $(2, 3)$, $(2, 4)$, $(3, 2)$, $(3, 3)$, $(4, 1)$, $(4, 2)$, $(5, 1)$, $(6, 1)$. For the case $(d, a) = (1, 4)$, see [4, Corollary 2]. It implies that $h^0(\mathcal{I}_{Y(2,4)}(3)) = 0$, because a reducible conic is a limit of a smooth conic.

(i) Write $Y(2, 3) = D \sqcup 2O \sqcup 2O' \sqcup 2O''$ and let H be the plane spanned by

$\{O, O', O''\}$. Since D is general, we have $h^0(\mathcal{I}_{D\cup\{O,O',O''\}}(2)) = 0$ and $D \cap H$ is the union of two general points of H, we have $h^0(H, \mathcal{I}_{Y(2,1)\cap H}(2)) = 0$. The Castelnuovo's sequence gives $h^0(\mathcal{I}_{Y(2,3)}(3)) = 2$ and $h^1(\mathcal{I}_{Y(2,3)}(3)) = 1$. Taking as D a general element of $F(3, 0)$ the same proof gives $h^0(\mathcal{I}_{Y(3,3)}(3)) = 0$.

(ii) Write $Y(3, 2) = D \sqcup 2O \sqcup 2P$ and take a general plane H containing $\{O, P\}$. Since $h^1(\mathcal{I}_{D\cup\{O,P\}}(2)) = h^1(H, \mathcal{I}_{H\cap Y(3,2)}(3)) = 0$, the Castelnuovo's sequence gives $h^1(\mathcal{I}_{Y(3,2)}(3)) = 0$.

(iii) Write $Y(4, 1) = D \sqcup 2O$ and take a general plane H through O. We have $\mathrm{Res}_H(D \cup 2O) = D \cup O$. Since $h^0(\mathcal{I}_D(2)) = 1$, for general O we have $h^0(\mathcal{I}_{D\cup\{O\}}(2)) = 0$ and hence $h^1(\mathcal{I}_{D\cup\{O\}}(2)) = 0$. The Castelnuovo's sequence gives $h^1(\mathcal{I}_{Y(4,1)}(3)) = 0$.

(iv) Fix a general $S \subset H$ with $\sharp(S) = 2$ and a general $D \in F(4, 0)$. Since $D \cap H$ is a general union of 4 points, we have $h^i(H, \mathcal{I}_{H\cap(D\cup 2S)}(3)) = 0$, $i = 0, 1$. Since $\mathrm{Res}_H(D \cup 2S) = D \cup S$ and $h^0(\mathcal{I}_{D\cup S}(2)) = 0$, the Castelnuovo's sequence gives $h^0(\mathcal{I}_{D\cup 2S}(3)) = 0$.

(v) Fix a general line $L \subset H$, a general $O \in H$ and a general $C \in F(4, 0)$ containing exactly one point of L. Since $X := C \cup L \cup 2O$ is a flat limit of a family of elements of $F(5, 1)$, it is sufficient to prove that $h^i(\mathcal{I}_X(3)) = 0$, $i = 0, 1$. We have $h^i(H, \mathcal{I}_{X\cap H}(3)) = 0$, $i = 0, 1$, because $C \cap (H \setminus L)$ is a general union of 3 points of H. We have $\mathrm{Res}_H(X) = C \cup \{O\}$. For fixed C and L, the plane H may be taken as a general plane through L. Since O is a general point of H, for fixed C and L the point O is a general point of $\mathbb{P}^3$. Since $h^0(\mathcal{I}_C(2)) = 1$ and O may be taken outside the unique quadric surface containing C, we get $h^0(\mathcal{I}_{C\cup\{O\}}(2)) = 0$. The Castelnuovo's sequence gives $h^0(\mathcal{I}_X(3)) = 0$ and hence $h^1(\mathcal{I}_X(3)) = 0$.

(vi) Since $h^0(\mathcal{I}_{Y(6,0)}(3)) = 1$ ([18]), we have $h^0(\mathcal{I}_{Y(6,1)}(3)) = 0$. □

Lemma 2. *Fix positive integers d, a. If $4d + 1 + 4a \le \binom{7}{3}$, then*

$$h^1(\mathcal{I}_{Y(d,a)}(4)) = 0.$$

If $4d + 1 + 4a \ge \binom{7}{3}$, then $h^0(\mathcal{I}_{Y(d,a)}(4)) = 0$.

Proof. Since $\binom{7}{3} = 35$, it is sufficient to test the following pairs (d, a): $(1, 7)$, $(1, 8)$, $(2, 6)$, $(2, 7)$, $(3, 5)$, $(3, 6)$, $(4, 4)$, $(4, 5)$, $(5, 3)$, $(5, 4)$, $(6, 2)$, $(6, 3)$, $(7, 1)$, $(7, 2)$, $(8, 1)$. If $d+a = 8$, then $h^0(\mathcal{O}_{Y(d,a)}(4)) = \binom{7}{3} - 2$, while if $d+a = 9$, then $h^0(\mathcal{O}_{Y(d,a)}(4)) = \binom{7}{3} + 2$. The cases with $d = 1$ are true by [4, Corollary 2].

(i) Take $(d,a) = (2,6)$. Fix a line $L \in |\mathcal{O}_Q(0,1)|$, a general line $D \subset \mathbb{P}^3$ meeting L and a general $S \subset Q$ with $\sharp(S) = 6$. Since the reducible conic $L \cup D$ is a degeneration of a family of smooth conics, it is sufficient to prove that $h^1(\mathcal{I}_{D\cup L\cup 2S}(4)) = 0$. We have $\mathrm{Res}_Q(D \cup L \cup S) = D \cup S$. Since $h^1(\mathcal{I}_D(2)) = 0$, $h^0(\mathcal{I}_D(2)) = 7$ and $h^0(\mathcal{I}_D) = 0$, [8, Lemma 3] gives $h^1(\mathcal{I}_{D\cup S}(2)) = 0$. By the Castelnuovo's sequence it is sufficient to prove that $h^1(Q, \mathcal{I}_{L\cup(D\cap(Q\setminus L))\cup\{2S,Q\}}(4)) = 0$, i.e. $h^1(Q, \mathcal{I}_{(D\cap(Q\setminus L))\cup\{2S,Q\}}(4,3)) = 0$. We have $h^1(Q, \mathcal{I}_{\{2S,Q\}}(4,3)) = 0$ ([19, Table I, Proposition 5.2 and Theorem 7.2]) and hence $h^0(Q, \mathcal{I}_{\{2S,Q\}}(4,3)) = 2$. Since for a fixed S $D \cap (Q \setminus L)$ is a general point of Q, we get $h^1(Q, \mathcal{I}_{(D\cap(Q\setminus L))\cup\{2S,Q\}}(4,3)) = 0$.

(ii) Take $(d,a) = (2,7)$. Fix a general $S \subset Q$ with $\sharp(S) = 7$ and a general $A \in |\mathcal{O}_Q(1,1)|$. Let $H \subset \mathbb{P}^3$ be the plane spanned by A. We have $h^1(Q, \mathcal{I}_{\{2S,Q\}}(4)) = 0$ and $h^0((Q, \mathcal{I}_{\{2S,Q\}}(3)) = 0$ ([19, Table I, Proposition 5.2 and Theorem 7.2]. Hence $h^i(\mathcal{I}_{\{2S,Q\}\cup B}(4)) = 0$, $i = 0,1$, for a general $B \subset A$ with $\sharp(B) = 4$. Take a smooth conic $D \subset H$ such that $A \cap D = B$ and set $X := D \cup 2S$. We just saw that $h^i(Q, \mathcal{I}_{X\cap Q}(4)) = 0$. We have $\mathrm{Res}_Q(X) = D \cup S$. Since $h^0(\mathcal{I}_D(2)) = 5$, $\mathrm{Res}_Q(D) = D$, $h^0(\mathcal{I}_D) = 0$ and S is general in Q, we get $h^0(\mathcal{I}_{Y\cup S}(2)) = 0$. The Castelnuovo's sequence gives $h^0(\mathcal{I}_X(4)) = 0$.

(iii) We take $3 \le d \le 4$. Fix 3 general lines $L_1, L_2, L_3 \subset H$ and a general $O \in H$. Set $O' := L_1 \cap L_3$ and $O'' := L_2 \cap L_3$. Fix a general $S \subset \mathbb{P}^3$ with $\sharp(S) = a-1$. If $d = 3$, then set $T := \emptyset$. If $d = 4$, then let T be a general line of $\mathbb{P}^3$ intersecting L. Set $X := T \cup (L_1 \cup L_2 \cup L_3 \cup 2O' \cup 2O'') \cup 2S \cup 2O$. X is a flat limit of a family of elements of $F(d,a)$. Hence it is sufficient to prove that X has the expected Hilbert function in degree 4. We have $\mathrm{Res}_H(X) = T \cup \{O', O'', O\} \cup S$ and $h^i(H, \mathcal{I}_{H\cap X}(4)) = 0$, $i = 0,1$. Hence by the Castelnuovo's sequence it is sufficient to prove that $T \cup \{O', O'', O\} \cup 2S$ has the expected Hilbert function in degree 3. If $T = \emptyset$, then $h^1(\mathcal{I}_{2S}(3)) = 0$ by the Alexander-Hirschowitz theorem. If T is a line, then $h^1(\mathcal{I}_{T\cup 2S}(3)) = 0$ by [4, Corollary 2]. Since $\{O, O', O''\}$ may be seen as 3 general points of H, $\mathrm{Res}_H(T \cup 2S) = T \cup 2S$ and $h^0(\mathcal{I}_{T\cup 2S}(2)) = 0$, [4, Lemma 3] gives that $T \cup \{O', O'', O\} \cup 2S$ has the expected Hilbert function in degree 3.

(iv) Take $(d,a) \in \{(5,3), (5,4), (6,3)\}$. Fix a general $D \in F(d-2, 0)$. Hence $D \cap Q$ is a general union of $2d-4$ points of Q (Remark 2). Let $E \subset Q$ be the union of two lines of type $(0,1)$, each of them containing one point of $D \cap Q$. Fix a general $S \subset Q$ such that $\sharp(S) = a$. Set $X := D \cup E \cup 2S$. Since $D \cup E$ is a flat limit of a family of elements of $F(d,0)$ ([18], [24]), it is sufficient to prove that X has the expected Hilbert in degree 4. Since $a \le 4$, we have $h^1(Q, \mathcal{I}_{\{2S,Q\}}(4,2)) = 0$. Since $D \cap (Q \setminus E)$ is general in Q, we get that either $h^0(Q, \mathcal{I}_{X\cap Q}(4)) = 0$ (case $d + a = 9$) or $h^1(Q, \mathcal{I}_{X\cap Q}(4)) = 0$ (case $d + a = 8$). We have $\mathrm{Res}_Q(X) = D \cup S$. By the Castelnuovo's sequence it is sufficient to prove that $h^1(\mathcal{I}_{D\cup S}(2)) = 0$ if $d + a = 8$ and $h^0(\mathcal{I}_{D\cup S}(2)) = 0$ if $d + a = 9$. Set $h := h^0(\mathcal{O}_{D\cup S}(2))$. We have the following triples (d, a, h): $(5, 3, 10)$, $(5, 4, 11)$, $(6, 3, 12)$. We have $h^1(\mathcal{I}_D(2)) = 0$ and $h^0(\mathcal{I}_D(-1)) = 0$. By [8, Lemma 3] and the generality of S we get $h^0(\mathcal{I}_{D\cup S}(2)) = 0$. If $(d, a) = (5, 3)$, then $h^1(\mathcal{I}_{D\cup S}(2)) = h^0(\mathcal{I}_{D\cup S}(2))$, because $h = 10$.

(v) Take $(d, a) \in \{(6, 2), (7, 1), (7, 2)\}$. Fix a general $D \in |\mathcal{O}_Q(1, 3)|$, a general $S \subset Q$ with $\sharp(S) = a$ and general lines $R_i \subset \mathbb{P}^3$, $1 \le i \le d - 4$, each of them containing a point of D. Set $X := D \cup \bigcup_{i=1}^{d-4} R_i \cup 2S$. Since $D \cup \bigcup_{i=1}^{d-4} R_i$ is a flat limit of a family of smooth rational curves of degree d ([18]), it is sufficient to prove that $h^1(\mathcal{I}_X(4)) = 0$. We have $h^1(Q, \mathcal{I}_{\{2S,Q\}}(3, 1)) = 0$ ([19]). Since $(R \cup L) \cap (Q \setminus D)$ is a general union of two points of Q, we get $h^i(Q, \mathcal{I}_{X\cap Q}(4)) = 0$, $i = 0, 1$, if $(d, a) = (6, 2)$, $h^0(Q, \mathcal{I}_{X\cap Q}(4)) = 0$ if $(d, a) = (7, 2)$ and $h^1(Q, \mathcal{I}_{X\cap Q}(4)) = 0$ if $(d, a) = (7, 1)$. We have $\mathrm{Res}_Q(X) = \bigcup_{i=1}^{d-4} R_i \cup S$. We have $h^1(\mathcal{I}_{\mathrm{Res}_Q(X)}(2)) = 0$ if $(d, a) = (6, 2)$, $h^i(\mathcal{I}_{\mathrm{Res}_Q(X)}(2)) = 0$, $i = 0, 1$, if $(d, a) = (7, 1)$ and $h^0(\mathcal{I}_{\mathrm{Res}_Q(X)}(2)) = 0$ if $(d, a) = (7, 2)$. The Castelnuovo's sequence gives $h^1(\mathcal{I}_X(4)) = 0$.

(vi) We have $h^0(\mathcal{I}_{Y(7,1)}(4)) = 2$ by step (v). Write $Y(7, 1) = C \sqcup 2O$ and fix a general $P \in \mathbb{P}^3$, so that $h^0(\mathcal{I}_{Y(7,1)\cup\{P\}}(4)) = 1$. Let $T \subset \mathbb{P}^3$ be the only degree 4 surface containing $Y(7, 1) \cup \{P\}$. For a general P the linear projection from P maps birationally C onto a degree 7 plane curve M. Hence for a general line L through P and meeting C, the curve $C \cup L$ is a flat limit of a family of elements of $F(8, 0)$ ([18], [24]). Since the cone, F, with vertex P and base M has degree 7, it is not contained in T. Hence $h^0(\mathcal{I}_{Y(7,1)\cup L}(4)) = 0$ for a general L, proving the case $(d, a) = (8, 1)$. □

Proof of Theorem 1. By Lemmas 1 and 2 we may assume $k \ge 5$. See step (c) for the case $k = 5$, while we assume $k \ge 6$ in steps (a) and (b). By the Alexander-Hirschowitz theorem we may assume $d > 0$. By [4, Corollary 2] we may assume $d \ge 2$.

(a) In this step we assume $dk + 1 + 4a \le \binom{k+3}{3}$. Set $\psi := \binom{k+3}{3} - [k, d] - 4a$. Increasing if necessary a we may assume that $0 \le \psi \le 3$.

(a1) In this step we assume $a \ge 1 + (k + 1)^2/3$. Set $e := \lfloor (k + 1)^2 - 2d)/3 \rfloor$ and $f := (k + 1)^2 - 2d - 3e$. We have $0 \le f \le 2$ and $2d + 3e + f = (k + 1)^2$. Since $d \ge 2$ and $a \ge 1 + \lceil (k + 1)^2/3 \rceil$, we have $a \ge e + f$. Fix a general $Y \in F(d, a - e - f)$ and a general $S \cup S' \subset Q$ such that $\sharp(S) = e$, $\sharp(S') = f$ and $S \cap S' = \emptyset$. Set $U := Y \sqcup 2S$. By the semicontinuity theorem it is sufficient to prove $h^1(\mathcal{I}_W(k)) = 0$ for a general union W of U and f 2-points. We have $\mathrm{Res}_Q(U) = Y \cup S$. Assume for the moment $d \neq 2$. In this case $Y \cap Q$ is a general union of $2d$ points of Q (Remark 2). Hence $h^i(Q, \mathcal{I}_{(U\cap Q)\cup S'}(k)) = 0$, $i = 0, 1$. By the differential Horace lemma for double points ([1], [11, Lemma 5], [2] in characteristic $\neq 2$) to prove that $h^1(\mathcal{I}_W(k)) = 0$ it is sufficient to prove $h^1(\mathcal{I}_{Y\cup S\cup\{2S',Q\}}(k-2)) = 0$.

Claim 1. *We have* $h^1(\mathcal{I}_{Y\cup\{2S',Q\}}(k-2)) = 0$.

Proof of Claim 1. We have $h^0(\mathcal{O}_{Y\cup\{2S',Q\}}(k-2)) = \binom{k+1}{3} - \psi - e$. We first check that $e \geq f$. Assume $e < f$. Since $f \leq 2$, then $3e + f \leq 5$. Hence $2d \geq k^2 + 2k - 4$. Since $k \geq 5$, we get $kd + 1 > \binom{k+3}{3}$, a contradiction. Since $e \geq f$ and $\psi \geq 0$, we have $h^0(\mathcal{O}_{Y\cup 2S'}(k-2)) \leq \binom{k+1}{3}$. Since $f \leq 2$ and any two points of $\mathbb{P}^3$ are contained in a smooth quadric, $Y \cup 2S'$ and $Y(d, a-e)$ have the same Hilbert function in degree $k-2$. The inductive assumption gives $h^1(\mathcal{I}_{Y\cup 2S}(k-2)) = 0$. Since $Y \cup 2S' \supseteq Y \cup \{2S', Q\}$ and the one-dimensional connected components of $Y \cup 2S'$ and $Y \cup \{2S', Q\}$ are the same, we get $h^1(\mathcal{I}_{Y\cup\{2S',Q\}}(k-2)) = 0$.

Claim 1 gives $h^0(\mathcal{I}_{Y\cup\{2S',Q\}}(k-2)) = \psi + e$. Since S is general in Q, to prove that

$$h^1(\mathcal{I}_{Y\cup S\cup\{2S',Q\}}(k-2)) = 0$$

it is sufficient to prove that $h^0(\mathcal{I}_Y(k-4)) \leq \psi$. We have $h^0(\mathcal{O}_Y(k-4)) = h^0(\mathcal{O}_Y(k-2)) - 2d = \binom{k+1}{3} - \psi - e - 3f - 2d$. First assume $k \geq 7$, so that we may use the inductive assumption for the integer $k-4$. In this case it is sufficient to check that $3f + e + 2d \leq \binom{k+1}{3} - \binom{k-1}{3} = (k-1)^2$. Assume $3f + e + 2d \geq (k-1)^2 + 1$. Since $3e + f + 2d = (k+1)^2$, we get $2(e-f) \leq 4k-1$, i.e. $e - f \leq 2k-1$. Since $f \leq 2$, we get $2d \geq k^2 - 4k - 3$. Hence $kd + 1 + 4a \geq k(k^2 - 4k - 3)/2 + 1 + 4(k+1)^2/3$. Set $f(x) = 3x(x^2 - 4x - 3) + 6 + 8(x+1)^2 - (x+3)(x+2)(x+1)$. Since $f(x) > 0$ for all $x \geq 7$, we get a contradiction. Now assume $k = 6$. To get $h^0(\mathcal{I}_Y(2)) = 0$ it is sufficient to use that either $a \geq 1$ and $d \geq 3$ or $a \geq 2$ and $d > 0$ (Remark 1).

Now assume $d = 2$. Since S' is general and $2d + 3e + f = (k+1)^2$, to prove that

$$h^i(Q, \mathcal{I}_{(U\cap Q)\cup S'}(k)) = 0$$

it is sufficient to prove that

$$h^1(Q, \mathcal{I}_{(Y\cap Q)\cup\{2S,Q\}}(k)) = 0.$$

By [19, Theorem 7.2] we have $h^1(Q, \mathcal{I}_{\{2S,Q\}}(k)) = 0$. Hence $h^0(Q, \mathcal{I}_{\{2S,Q\}}(k)) = 4 + f$. Write $Y = D \sqcup 2M$ with $\sharp(M) = a - e - f$. Let H be the plane spanned by the conic D. For a general Y we have $M \cap Q = \emptyset$ and the curve $C := H \cap Q$ is a smooth element of $|\mathcal{O}_Q(1,1)|$. We fix C and then move D among the smooth conics of H. For a general smooth conic $D \subset H$ the set $Q \cap D = C \cap D$ is formed by 4 general points of C. For a general S we have $C \cap S = \emptyset$ and hence $\mathrm{Res}_C(\{2S, Q\}) = \{2S, Q\}$. Since S is general to check that $h^1(Q, \mathcal{I}_{(Y \cap Q) \cup \{2S,Q\} \cup S}(k)) = 0$ it is sufficient to prove that $h^0(Q, \mathcal{I}_{\{2S,Q\}}(k-1)) = 0$ ([8, Lemma 3]). This is true by [19], because $k-1 \geq 3$ and $3e \leq (k+1)^2 - 4 - 2 \geq k^2$.

(a2) Assume $k/3 + 3 \leq a < 1 + (k+1)^2/3$. Hence $d \geq 3$. Let z be the maximal integer such that $2(d-z-1) - 1 \leq k(k+1-z)$ (it exists, because $k > 2$). The definition of the integer z gives

$$0 \leq k(k+1-z) - 2(d-z-1) + 1 \leq k-3 \tag{1}$$

Since $kd + 1 + 4a \leq \binom{k+3}{3}$, we have $2d \leq k(k+1)+1$ and hence $z \geq 0$. First assume $z \geq k$. Since $k > 2$, we get $2(d-k-1) \leq k+1$, i.e. $d \leq 3(k+1)/2$. Since $3(k+1)k/2 + 1 + 4(k-3)/3 + 8 < \binom{k+3}{3}$, we get a contradiction. Hence $z \leq k-1$.

Set $g := \lfloor (k(k+1-z) - 2(d-z-1)+1)/3 \rfloor$ and $h := k(k+1-z) - 2(d-z-1) + 1 - 3g$. We have $0 \leq h \leq 2$ and

$$2(d-z-1) - 1 + 3g + h = k(k+1-z) \tag{2}$$

From (1) we get $0 \leq g \leq \lfloor (k-3)/3 \rfloor$. Since $h \leq 2$ and $\lfloor (k-3)/3 \rfloor + 2 \leq k/3+3$, we get $a \geq g+h$. Since $0 \leq z \leq k-1$, the second inequality in (1) gives $d \geq z+2$. Fix a general $Y \in F(d-z-1, a-g-h)$. Hence $Y \cap Q$ is formed by $2(d-z-1) > 0$ points. If $d-z-1 \neq 2$ we may assume that $Y \cap Q$ is general in Q (Remark 2). If $d-z-1 = 2$ we saw in step (a1) that $Y \cap Q$ is formed by 4 general points on a certain smooth conic $C \in |\mathcal{O}_Q(1,1)|$. In both case there is $F \in |\mathcal{O}_Q(1,z)|$ containing exactly one point of $Y \cap Q$. Moreover, in both cases the set $Y \cap (Q \setminus F)$ is a general subset of Q with cardinality $2(d-z-1)-1$ (use that 3 general points of Q are contained in a smooth element of $|\mathcal{O}_Q(1,1)|$). Fix a general $S \cup S' \subset Q$ such that $\sharp(S) = g$, $\sharp(S') = h$ and $S \cap S' = \emptyset$. Set $U := Y \cup F \cup 2S$. U is a flat limit of a family of elements of $F(d, a-f)$. Hence to prove Theorem 1 for the triple (d,a,k) it is sufficient to prove that $h^1(\mathcal{I}_W(k)) = 0$, where W is the union of U and f general 2-points. We have $h^i(Q, \mathcal{I}_{(U \cap Q) \cup S'}(k)) = h^i(Q, \mathcal{I}_{(Y \cap (Q \setminus F)) \cup \{2S,Q\} \cup S'}(k-1, k-z))$. Since $k-z \geq 1$ and $(Y \cap (Q \setminus F))$ is given by a positive number of general points of Q, (2) and [19, Proposition 5.2 and Theorem 7.2] give $h^i(Q, \mathcal{I}_{(U \cap Q) \cup S'}(k)) = 0$, $i = 0, 1$. By the differential Horace lemma for double points ([1], [11, Lemma 5], [2] in characteristic $\neq 2$) to prove that $h^1(\mathcal{I}_W(k)) = 0$ it is sufficient to prove that $h^1(\mathcal{I}_{U\cup S\cup\{2S',Q\}}(k-2)) = 0$.

(a2.1) Assume $g \geq h$.

Claim 2. *We have* $h^1(\mathcal{I}_{Y\cup\{2S',Q\}}(k-2)) = 0$.

Proof of Claim 2. We have $h^0(\mathcal{O}_{Y\cup\{2S',Q\}}(k-2)) = \binom{k+1}{3} - \psi - g$. We continue as in the proof of Claim 1, using $d-z-1$ instead of d.

By Claim 2 we have $h^0(\mathcal{I}_{Y\cup\{2S',Q\}}(k-2)) = \psi + g$. By [8, Lemma 3] to conclude the proof it is sufficient to check that $h^0(\mathcal{I}_Y(k-4)) \leq \psi$. See step (a1).

(a2.2) In this step we assume $g < h$. Hence $(g,h) \in \{(1,2),(0,1),(0,2)\}$. In all cases we have $3g+h \leq 5$. We first check that $z > 0$. Assume $z = 0$. From (1) we get $2d \geq k^2+4$. Hence $kd+1 \geq k(k^2+4)/2+1$. We have $k(k^2+4)/2+1 > \binom{k+3}{3}$ for all $k \geq 6$. Set $z' := z-1$, $g' := \lfloor (k(k+1-z') - 2(d-z'-1))/3 \rfloor$ and $h' := k(k+1-z') - 2(d-z'-1) - 3g'$. Notice that $3g'+h' = 3g+h+k$ and hence $g' \geq 2$. Therefore $g' \geq h'$. Since $3g+h \leq 5$ and $0 \leq h' \leq 2$, we have $g'+h' \leq k/3+3$. Since $a \geq k/3+3$, then $a \geq g'+h'$. Since $z' = z-1$, we have $d-z'-1 \geq 2$. Fix a general $Y' \in F(d-z'-1, a-g'-h')$, a general $F' \in |\mathcal{O}_Q(1,z')|$ containing exactly one point of $Y'\cap Q$ and a general $S_1 \cup S_1' \subset Q$ with $\sharp(S_1) = g'$, $\sharp(S_1') = h'$. We repeat step (a2.1) with Y', S_1, S_1', F' instead of Y, S, S', F.

(a3) Assume $a < k/3+3$, i.e. $a \leq (k+8)/3$. Since $\psi \leq 3$ and $4a \leq 4(k+8)/3$, we have $\lfloor (\binom{k+3}{3}-1)/k \rfloor - 2 \leq d \leq \lfloor (\binom{k+3}{3}-1)/k \rfloor$. Fix a general $B \subset Q$ such that $\sharp(B) = a$. Let w be the minimal integer such that $2(d-w-1)-1+3a \geq k(k+1-w)$ (it exists, because $k > 2$). Since $3a \leq k+8$ and $2d \leq k^2+4$, we have $w \geq 0$. Set $u := 2(d-w-1)+3a-k(k+1-w)$. Since $2(d-w-1)-1+3a \geq k(k+1-w)$, we have $u \geq 1$. The minimality property of w gives $u \leq k-3$.

Claim 3. *We have* $w \leq k-1$.

Proof of Claim 3. Assume $w \geq k$, i.e. assume $2(d-k)+3a \leq 2k$, i.e. $2d+3a \leq 4k$. Since $k \geq 6$, we get $kd+1+4a < kd+1+3ka/2 \leq 2k^2 < \binom{k+3}{3}$, a contradiction.

Claim 4. *We have* $2w \geq u-1$.

Proof of Claim 4. Assume $2w \leq u-2$. Since $u \leq k-3$, we get $w \leq (k-5)/2$ and hence $2d-2-(k-5)+3a \geq k(k+7)/2$. Since $3a \leq k+8$, we get $d \geq (k^2-k-6)/4$. Assume for the moment $k \geq$. Since $a > 0$ we get

If $k \geq 7$, we get $kd+1 \geq k(k^2-k-6)/2+1 > \binom{k+3}{3}$, a contradiction. Now assume $k = 6$ and hence $a \leq 4$. Since $u \leq 3$, it is sufficient to prove that $w \geq 2$.

Assume $w \le 1$. We get $2(d-2) - 42 > 0$, i.e. $d \ge 23$. Since $23 \cdot 6 + 1 > \binom{9}{6}$, we get a contradiction.

Take a general $Y \in F(d-w-u, 0)$ and a general union $J \subset \mathbb{P}^3$ of $u-1$ lines. The set $(Y \cup J) \cap Q$ contains a general subset $V \subset Q$ with cardinality u such that each connected component of $Y \cup J$ contains exactly one point of V. Since $2w \ge u-1$ (Claim 4), there is a smooth $C \in |\mathcal{O}_Q(1, w)|$ with $V \subset C$. For a general $Y \cup J$ we may also assume that $V = (Y \cup J) \cap C$. Hence $U := Y \cup J \cup C$ is a connected nodal curve with arithmetic genus zero. It is a flat limit of a family of elements of $F(d, 0)$. By the semicontinuity theorem it is sufficient to prove that $h^1(\mathcal{I}_{U \cup 2S}(k)) = 0$. We have $h^i(\mathcal{I}_{(U \cup 2S}(k)) = 0$, $i = 0, 1$ (Claim 3 and [19]; we saw it even in the case $\deg(Y) = 2$ in which we may not apply Remark 2). Hence it is sufficient to prove $h^1(\mathcal{I}_{Y \cup J \cup S}(k-2)) = 0$. We have $h^0(\mathcal{O}_{Y \cup J \cup S}(k-2)) = \binom{k+1}{3} - \psi$. Hence $h^0(\mathcal{O}_{Y \cup J}(k-2)) \le \binom{k+1}{3}$. Since $Y \cup J$ has maximal rank ([17]), we have $h^1(\mathcal{I}_{Y \cup J}(k-2)) = 0$ and hence $h^0(\mathcal{I}_{Y \cup J}(k-2)) = \psi + a$. By [8, Lemma 3] to prove that $h^1(\mathcal{I}_{Y \cup J \cup S}(k-2)) = 0$ it is sufficient to prove that $h^0(\mathcal{I}_{Y \cup J}(k-4)) \le \psi$. The curve $Y \cup J$ has maximal rank ([17]). Hence it is sufficient to prove that $h^0(\mathcal{O}_{Y \cup J}(k-4)) \ge \binom{k-1}{3} - \psi$. We have $h^0(\mathcal{O}_{Y \cup J}(k-4)) = h^0(\mathcal{O}_{Y \cup J \cup S}(k-2)) - a - 2(d-w-1)$. Hence it is sufficient to prove that $2(d-w-1) + 3a \le (k-1)^2$. Since $a \le (k+8)/3$ and $w \ge 0$, it is sufficient to check that $d \le (k^2 - 2k)/2$. Since $a > 0$, if $k \ge 8$ it is sufficient to use that $k(k^2 - 2k)/2 + 4 \ge \binom{k+3}{3}$ for all $k \ge 8$. Now assume $k = 7$ and hence $1 \le a \le 5$ and $7d + 1 + 4a + \psi = 120$. We have the following triples (d, a, ψ): $(16, 1, 3)$, $(15, 2, 6)$, $(15, 3, 2)$, $(14, 4, 5)$, $(14, 5, 2)$ and we exclude the triples $(15, 2, 6)$ and $(14, 4, 5)$ which have $\psi > 3$. We have the following quintuples (d, a, ψ, w, u): $(16, 1, 3, 5, 4)$, $(15, 3, 2, 4, 1)$, $(14, 5, 2, 3, 2)$ for which Claim 4 holds. Now assume $k = 6$ and hence $1 \le a \le 4$. We have the following triples (d, a, ψ): $(13, 1, 0)$, $(12, 2, 2)$, $(11, 3, 4)$, $(11, 4, 0)$ and hence it is sufficient to check the following quintuples (d, a, ψ, w, u): $(13, 1, 0, 4, 2)$, $(12, 3, 2, 4, 3)$, $(11, 4, 0, 3, 5)$ for which Claim 4 holds. The case $(k, d, a, \psi) = (11, 4, 0)$ may be done also in the set-up of (a2.2) with $z = 5$, $g = 1$ and $h = 0$.

(b) In this step we assume $kd + 1 + 4a \ge \binom{k+3}{3}$. Set $\phi = kd + 1 - \binom{k+3}{3}$. Decreasing if necessary a we may assume that either $a = 0$ or $\phi \le 3$. Since the case $a = 0$ is true by [16], we may assume $0 \le \phi \le 3$. The case $\phi = 0$, by step (a) (it is the case $\psi = 0$. If $a > 0$ and $1 \le \phi \le 3$, then the case $(d, a-1)$ was checked in step (a) (it has $\psi = 4 - \phi$). Since any point P is contained in the 2-point $2P$, the case $(d, a-1, \psi) = (d, a-1, 1)$ proves the case $(d, a, \phi) = (d, a, 3)$. Hence we may assume $1 \le \phi \le 2$. All cases with $k \ge 8$ are done as in step (a) with the same construction and the same inequalities (b2) Assume $k = 6$. Since $\binom{9}{3} = 84$, $6 \cdot 14 + 1 > 84$, $(13, 1)$, $(11, 4)$ have $\psi = 1$, we need to test the following triples (d, a, ϕ): $(12, 3, 1)$, $(10, 6, 1)$, $(8, 9, 1)$, $(6, 12, 1)$, $(4, 15, 1)$, $(2, 18, 1)$.

(b2.1) Assume $(d, a, \phi) = (12, 3, 1)$. Let $Y \subset \mathbb{P}^3$ be a general union of a rational curve of degree 5 and two lines. We have $h^1(\mathcal{I}_Y(5)) = 0$, $h^0(\mathcal{I}_Y(5)) = 3$ and $h^0(\mathcal{I}_Y(1)) = 0$ ([17]). By [8, Lemma 3] we have $h^i(\mathcal{I}_{Y \cup S}(4)) = 0$, $i = 0, 1$, for a general $S \subset Q$ with $\sharp(S) = 3$. The set $Y \cap Q$ is a general subset with cardinality 14. Fix $V \subset Y \cap Q$ containing exactly one point of each connected component of V. Take a general $C \in |\mathcal{I}_V(1, 4)|$ and use $X := Y \cup C \cup 2S$. We have $\mathrm{Res}_Q(X) = Y \cup S$ and $h^0(Q, \mathcal{I}_{X \cap Q}(6)) = h^0(Q, \mathcal{I}_{\{2S,Q\} \cup ((Y \cap Q) \setminus V)}(5, 2)) = 0$.

(b2.2) Assume $(d, a, \phi) = (10, 6, 1)$. We take a general $S \subset Q$ with $\sharp(S) = 6$, a general $Y \in F(7, 0)$ and a general $C \in |\mathcal{O}_Q(1, 2)|$ containing exactly one point of $Y \cap Q$. We use $X := Y \cup C \cup 2S$. We have $\mathrm{Res}_Q(X) = Y \cup S$. We have $h^1(\mathcal{I}_Y(4)) = 0$, $h^0(\mathcal{I}_Y(4)) = 6$ and $h^0(\mathcal{I}_Y(2)) = 0$ ([18]). By [8, Lemma 3] we have $h^i(\mathcal{I}_{Y \cup S}(4)) = 0$, $i = 0, 1$. We have $h^0(\mathcal{I}_{X \cap Q}(6)) = 0$, because $h^1(\mathcal{I}_{\{2S,Q\}}(5, 4)) = 0$ and $h^0(\mathcal{I}_{\{2S,Q\}}(5, 4)) = 12$, while $Y \cap (Y \setminus C)$ is a general union of 13 points of Q.

(b2.2) Assume $a = 9, 12, 15, 19$. We make the construction of step (a1) with $e := \lfloor (49 - 2d)/3 \rfloor$ and $f := 49 - 2d - 3e$. We have $a \geq e + f$ (with $(e, f) = (7, 2)$ if $a = 9$).

(b3) Assume $k = 7$. Since $\binom{10}{4} = 120$, we need to test the following triples (d, a, ϕ) with $1 \leq \phi \leq 2$: $(16, 2, 1)$, $(15, 4, 2)$, $(12, 9, 1)$, $(11, 11, 2)$, $(8, 16, 1)$, $(7, 18, 2)$, $(4, 23, 1)$, $(3, 25, 2)$.

(b3.1) Assume $(d, a) = (16, 2)$. Fix a general $Y \in F(10, 0)$. Let $Y \subset \mathbb{P}^3$ be a general union of a degree 7 smooth rational curve and 3 lines. We have $h^1(\mathcal{I}_Y(5)) = 0$, $h^0(\mathcal{I}_Y(5)) = 2$ and $h^0(\mathcal{I}_Y(3)) = 0$ ([17]). By [8, Lemma 3] we have $h^i(\mathcal{I}_{Y \cup S}(5)) = 0$, $i = 0, 1$, for a general $S \subset Q$ such that $\sharp(S) = 2$. Fix a smooth $C \in |\mathcal{O}_Q(1, 5)|$ containing exactly one point for each connected component of Y (it exists for a general Y and $Y \cap (Q \setminus Y)$ is a general subset with cardinality 16). Use $X := Y \cup C \cup 2S$. We have $\mathrm{Res}_Q(X) = Y \cup S$ and $h^0(Q, \mathcal{I}_{Q \cap X}(7)) = h^0(Q, \mathcal{I}_{\{2S,Q\} \cup (Y \cap (Q \setminus C))}(6, 2)) = 0$.

(b3.2) Assume $(d, a) = (15, 4)$. Fix a general union $Y \subset Q$ of a smooth rational curve of degree 9 and one line. We have $h^1(\mathcal{I}_Y(5)) = 0$, $h^0(\mathcal{I}_Y(5)) = 4$ and $h^0(\mathcal{I}_Y(3)) = 0$ ([17]). By [8, Lemma 3] we have $h^i(\mathcal{I}_{Y \cup S}(5)) = 0$, $i = 0, 1$, for a general $S \subset Q$ such that $\sharp(S) = 4$. Fix a smooth $C \in |\mathcal{O}_Q(1, 4)|$ containing exactly one point of each connected component of Y (it exists for a general Y

and $Y \cap (Q \setminus Y)$ is a general subset with cardinality 18). Use $X := Y \cup C \cup 2S$. We have $\mathrm{Res}_Q(X) = Y \cup S$ and $h^0(Q, \mathcal{I}_{Q \cap X}(7)) = h^0(Q, \mathcal{I}_{\{2S,Q\} \cup (Y \cap (Q \setminus C))}(6,3)) = 0$.

(b3.3) Assume $a = 9, 11$. Take the set-up of step (a1) taking $z := 6$ and, respectively, $(g, h) = (2, 0)$ and $(g, h) = (3, 1)$. In the first case in Q we have $h^0(Q, \mathcal{I}_{X \cap Q}(7)) = 0$ and $h^1(Q, \mathcal{I}_{X \cap Q}(7)) = 1$.

(b3.4) Assume $a = 16, 18, 23, 25$. Take the set-up of step (a1) with $e := \lfloor (64 - 2d)/3 \rfloor$ and $f = 64 - 2d - 3e$. We have $a \geq e + f$ (with $(e, f) = (16, 2)$ if $(d, a) = (7, 18)$ and $(e, f) = (16, 0)$ if $(d, a) = (8, 16)$).

In characteristic zero even the case $(d, a, \phi) = (d, a, 2)$ with $a > 0$ follows from the case $(d, a-1, \psi) = (d, a-1, 2)$ ([13], [10, Lemma 1.8]).

(c) Take $k = 5$. We have $\binom{8}{3} = 56$. We use ψ as in step (a) and ϕ as in step (b). It is sufficient to test all pairs (d, a) with $d > 0$, $a > 0$, and either $0 \leq \psi \leq 3$ or $1 \leq \phi \leq 2$. If $5d + 1 + 4a \leq 56$ we need to test the following triples (d, a, ψ): $(10,1,1)$, $(9,2,2)$, $(8,3,3)$, $(7,5,0)$, $(6,6,1)$, $(5,7,2)$, $(4,8,3)$, $(3,10,0)$, $(2,11,1)$, $(1,12,2)$. If $5d + 1 + 54 > 56$ we need to test the following triples (d, a, ϕ) with $1 \leq \phi \leq 2$: $(9,3,2)$, $(8,4,1)$, $(5,8,2)$, $(4,9,1)$, $(1,12,2)$.

(c1) Take $(d, a, \psi) = (10, 1, 1)$. Fix a general union $Y \subset \mathbb{P}^3$ of 4 lines and one 2-points. We have $h^i(\mathcal{I}_Y(3)) = 0$, $i = 0, 1$ ([4]). Take a general $C \in |\mathcal{O}_Q(1, 5)|$ containing exactly one point of each line of Y. Use $Y \cup C$ (we have $h^1(Q, \mathcal{I}_{Y \cap (Q \setminus C)}(4, 0)) = 0$).

(c2) Take $d = 8, 9$. Fix a general $S \subset Q$ with $\sharp(S) = a$. Let $Y \subset \mathbb{P}^3$ be a general scheme of degree 5 with exactly $5 - a$ connected components, all but ones being a line, while the remaining one is a smooth rational curve. We have $h^1(\mathcal{I}_Y(3)) = 0$, $h^0(\mathcal{I}_Y(3)) = a$ and $h^0(\mathcal{I}_Y(1)) = 0$ ([17]). Hence $h^i(\mathcal{I}_{Y \cup S}(3)) = 0$, $i = 0, 1$ ([8, Lemma 3]). Fix a general $C \in |\mathcal{O}_Q(1, d-6)|$ containing exactly one point of each connected component of Y and with $C \cap S = \emptyset$. Use $X := Y \cup C \cup 2S$ (in all cases either $h^0(Q, \mathcal{I}_{X \cap Q}(5)) = 0$ or $h^1(Q, \mathcal{I}_{X \cap Q}(5)) = 0$).

(c3) Take $(d, a, \psi) = (7, 5, 0)$. Fix a line $L \subset H$, a general $D \in F(6, 0)$ containing exactly one point of L, a general $O \in \mathbb{P}^3$, a general $S \subset H$ with $\sharp(S) = 3$ and a general $P \in H$. Set $U := L \cup D \cup 2O \cup 2S$. To prove this case it is sufficient to prove that $h^i(\mathcal{I}_W(5)) = 0$, $i = 0, 1$, for a general union W of U and one 2-point. Since $K := D \cap (H \setminus L)$ is formed by 5 general points of H, we have $h^i(H, \mathcal{I}_{\{2S,H\} \cup K \cup \{P\} \cup L}(5)) = 0$, $i = 0, 1$. Hence by the differential Horace lemma for 2-points ([1], [11, Lemma 5]), it is sufficient to prove that $h^i(\mathcal{I}_{D \cup 2O \cup S \cup \{2P,H\}}(4)) = 0$, $i = 0, 1$. Since $\sharp(S) \geq 1$, Lemma 2 gives $h^1(\mathcal{I}_{D \cup 2O \cup 2P}(4)) = 0$. Hence we have $h^1(\mathcal{I}_{D \cup 2O \cup \{2P,H\}}(4)) = 0$. Therefore we have $h^0(\mathcal{I}_{D\cup 2O\cup\{2P,H\}}(4)) = 3$. Since $h^0(\mathcal{I}_{D\cup 2O}(3)) = 0$ (Lemma 1), [8, Lemma 3] gives $h^i(\mathcal{I}_{D\cup 2O\cup S\cup\{2P,H\}}(4)) = 0$, $i = 0, 1$.

(c4) Take $2 \le d \le 6$. Take the set-up of step (a2) with $z = 1$ (without the maximality condition), i.e. take $C \in |\mathcal{O}_Q(1,1)|$. Set $g := \lfloor(30-2d)/3\rfloor$ and $h := 30 - 2d - 3g$. In all cases, except $(d, a, \psi) = (5, 7, 2)$, we have $a \ge g + h$. Now assume $(d, a, \psi) = (5, 7, 2)$. In this case we take $S' = \emptyset$, instead of $\sharp(S') = 2$. Calling X the total scheme, we get $h^i(\mathcal{I}_{\mathrm{Res}_Q(X)}(3)) = 0$, $i = 0, 1$, $h^1(Q, \mathcal{I}_{X\cap Q}(5)) = 0$ and $h^0(Q, \mathcal{I}_{X\cap Q}(5)) = 2$.

(c5) Take $d = 1$. Fix a a line $L \subset Q$ of type $(1, 0)$ and a general $S \subset Q$ with $\sharp(S) = 10$. Use a general union of $L \cup 2S$ and $a - 10$ 2-points. □

3. r=4

Let $H \subset \mathbb{P}^r$, $r \ge 4$, be a hyperplane. For any $(x, y) \in \mathbb{N}^2 \setminus \{(0, 0)\}$ let $Y(x, y; r)$ denote a general element of $F(x, y; r)$.

Remark 3. Fix integers $n \ge 3$ and $d > 0$. First assume $d < n$. Hence a general $D \in F(d, 0; n)$ spans a d-dimensional linear subspace $\langle D\rangle$ in which it is a rational normal curve (and the converse holds). We get that for any linearly independent set $S \subset \mathbb{P}^n$ with $\sharp(S) = d + 1$ there is $C \in F(d, 0; n)$ with $C \supset S$. Now assume $d \ge n$. Remark 2 gives $h^1(N_D(-1)) = 0$. By [21] for a general $B \subset \mathbb{P}^n$ with $\sharp(B) = d$ there is $C \in F(d, 0; n)$ such that $C \supset B$. In the case $d = n$ it is classical that the same is true if $\sharp(B) = n + 3$ (and in this case C is unique).

Lemma 3. *Take the following pairs* (d, a)*:* $(1, 8)$, $(2, 8)$, $(3, 8)$, $(4, 8)$, $(5, 5)$, $(6, 5)$, $(7, 5)$, $(8, 5)$, $(9, 2)$, $(10, 2)$, $(11, 2)$, $(12, 2)$. *Then* $h^0(\mathcal{I}_{Y(d,a;4)}(3)) = 0$.

Proof. If $a \ge 8$, then it is sufficient to use the Alexander-Hirschowitz theorem, even ignoring the rational curve, i.e. to use that $h^0(\mathcal{I}_{Y(0,8;4)}(3)) = 0$. Now take $(d, a) = (5, 5)$. Fix a general $Y \in F(5, 1; 4)$ and a general $S \subset H$ with $\sharp(S) = 4$. It is sufficient to prove that $h^0(\mathcal{I}_{Y\cup 2S}(3)) = 0$. The set $Y \cap H$ is a general union of 5 points. By the Alexander-Hirschowitz theorem we have $h^1(H, \mathcal{I}_{\{2S,H\}}(3)) = 0$ and $h^0(H, \mathcal{I}_{\{2S,H\}}(3)) = 4$. Hence $h^0(H, \mathcal{I}_{H\cap(Y\cup 2S)}(3)) = 0$. Since a general projection from a general point of a general element of $F(5, 0; 4)$ is a general element of $F(5, 0; 3)$ and $h^0(\mathcal{I}_{Y(5,0;3)}(2)) = 0$ ([18]), we have $h^0(\mathcal{I}_Y(2)) = 0$ and hence $h^0(\mathcal{I}_{Y\cup S}(2)) = 0$. Use the Castelnuovo's sequence.

Adding to $Y(5,5;4)$ $d-5$ general lines intersecting the curve of $Y(5,5;4)$ we get the cases $a = 5$ and $d = 6, 7, 8$ ([24]).

Now assume $(d, a) = (9, 2)$. Fix a general $Y \in Y(7, 0; 4)$, a general smooth conic $T \subset H$ containing a point of $Y \cap H$ and a general $S \subset H$ with $\sharp(S) = 2$. By [24] it is sufficient to prove that $h^0(\mathcal{I}_{Y\cup T\cup 2S}(3)) = 0$. We have

$$h^1(H, \mathcal{I}_{\{2S,H\}\cup T}(3)) = 0$$

and

$$h^0(H, \mathcal{I}_{\{2S,H\}\cup T}(3)) = 5.$$

Since $Y \cap (H\backslash T)$ is formed by 5 general points, we have $h^i(H, \mathcal{I}_{H\cap(Y\cup T\cup 2S)}(3)) = 0$, $i = 0, 1$. By the Castelnuovo's sequence it is sufficient to prove that $h^0(\mathcal{I}_{Y\cup S}(2)) = 0$. We have $h^i(\mathcal{I}_Y(2)) = 0$, $i = 0, 1$, because Y has maximal rank ([9]).

Adding to $Y(9,2;4)$ $d-a$ suitable lines we get all cases with $d \geq 10$. □

Lemma 4. *Take the following pairs* (d, a)*:* $(1, 11)$, $(2, 10)$, $(3, 9)$, $(4, 8)$, $(5, 8)$, $(6, 7)$, $(7, 6)$, $(8, 5)$, $(9, 5)$, $(10, 4)$, $(11, 3)$, $(12, 2)$. *Then* $h^1(\mathcal{I}_{Y(d,a;4)}(4)) = 0$.

Proof. (i) Assume $7 \leq d \leq 12$. Fix a general $Y \in F(d-1, 0; 4)$, a general line $L \subset H$ containing one point of $Y \cap H$ and a general $S \subset H$ with $\sharp(S) = a$. By semicontinuity it is sufficient to prove that $h^1(\mathcal{I}_{Y\cup L\cup S}(4)) = 0$. We have $h^1(H, \mathcal{I}_{L\cup\{2S,H\}}(4)) = 0$ and $h^0(\mathcal{I}_{L\cup\{2S,H\}}(4)) = 35-4a \geq d-1$ ([4]) and hence $h^1(\mathcal{I}_{H\cap(Y\cup L\cup 2S)}(4)) = 0$. Since $h^0(\mathcal{I}_Y(2)) = 0$, $h^1(\mathcal{I}_Y(3)) = 0$ and $h^0(\mathcal{I}_Y(3)) = 34 - 3d \geq a$ ([9]) and S is general in H, we get $h^1(\mathcal{I}_{Y\cup S}(4)) = 0$.

(ii) Assume $4 \leq d \leq 6$. Fix a general $Y \in F(d, 0; 4)$ and a general $S \subset H$ with $\sharp(S) = a$. By the Alexander-Hirschowitz theorem we have $h^1(H, \mathcal{I}_{\{2S,H\}}(4)) = 0$ and $h^0(\mathcal{I}_{\{2S,H\}}(4)) = 35 - 4a \geq d$. Hence

$$h^1(H, \mathcal{I}_{H\cap(Y\cup 2S)}(4)) = 0.$$

Since Y has maximal rank ([9]) we have $h^1(\mathcal{I}_Y(3)) = 0$ and $h^0(\mathcal{I}_Y(3)) \geq a + h^0(\mathcal{I}_Y(2))$. Use [8, Lemma 3] and the Castelnuovo's sequence.

(iii) Assume $1 \leq d \leq 3$. Fix a general $Y \in F(d, a-8; 4)$ and a general $S \subset H$ with $\sharp(S) = 8$. Since $h^1(H, \mathcal{I}_{\{2S,H\}}(4)) = 0$ and $h^0(H, \mathcal{I}_{\{2S,H\}}(4)) = 4 \geq d$, we have $h^1(H, \mathcal{I}_{H\cap(Y\cup 2S)}(4)) = 0$. In all cases Lemma 3 gives $h^1(\mathcal{I}_Y(3)) = 0$ and hence $h^0(\mathcal{I}_Y(2)) + 8 \leq h^0(\mathcal{I}_Y(3))$. Use [8, Lemma 3] and the Castelnuovo's sequence. □

Lemma 5. *For all* $(d, a) \in \mathbb{N}^2\backslash\{(0, 0)\}$ $h^0(\mathcal{I}_{Y(d,a;4)}(5))\cdot h^1(\mathcal{I}_{Y(d,a;4)}(5)) = 0$.

Proof. Since $125 \equiv 0 \pmod 5$ we reduce to the case $a = 25 - d$ in which $5d + 1 + 5a = \binom{9}{4}$. Since we excluded the case $d = 0$ and the case $a = 0$ is covered by [9], we may assume $1 \le d \le 24$.

(i) Assume $16 \le d \le 24$. Consider the following triples (d, z, m): $(16, 1, 1)$, $(17, 2, 2)$, $(18, 3, 3)$, $(19, 4, 4)$, $(20, 4, 1)$, $(21, 5, 2)$, $(22, 6, 3)$, $(23, 6, 4)$, $(24, 6, 1)$. In all cases we have $(d-z-m)+5z+1+4a = \binom{8}{3}$ and $a+m+4(d-z) = \binom{8}{4}$. In all cases we have $d - z \ge m$ and $z \ge m > 0$. Fix a general union Y of a general element of $F(d - z - m + 1, 0)$ and $m - 1$ lines. Hence m has m connected components and $H \cap Y$ is a general subset of H with cardinality $d - z$ (Remark 2). Fix a general $S \subset H$ with $\sharp(S) = a$ and a general smooth rational curve $C \subset H$ of degree z containing exactly one point of each connected component of Y (this is possible by Remark 3 and Since Y has maximal rank ([9])), we get $h^1(\mathcal{I}_Y(4)) = 0$, $h^0(\mathcal{I}_Y(4)) = a$ and $h^0(\mathcal{I}_Y(3)) = 0$. Since S is general in H we get $h^i(\mathcal{I}_{Y\cup S}(4)) = 0$, $i = 0, 1$. We have $h^1(H, \mathcal{I}_{C\cup\{2S,H\}}(5)) = 0$ by the case $(d, a) = (z, a)$ of Theorem 1. Since $Y \cap (H \setminus C)$ is general in H and $(d - z - m) + 5z + 1 + 4a = \binom{8}{3}$, we get $h^i(\mathcal{I}_{(Y\cap(H\setminus C)\cup\{2S,H\}}(5)) = 0$, $i = 0, 1$. The Castelnuovo's sequence gives $h^i(\mathcal{I}_{Y\cup C\cup 2S}(5)) = 0$, $i = 0, 1$. Since $Y \cup C$ is a flat limit of a family of rational curves of degree d ([24]), we are done.

(ii) Assume $13 \le d \le 15$. Set $w := d - 11 = 14 - a$. We have $2 \le w \le 4$. Let $Y \subset \mathbb{P}^4$ be a general union of w lines and a 2-points. By [6] we have $h^i(\mathcal{I}_Y(4)) = 0$, $i = 0, 1$. The set $Y \cap H$ is formed by w general points and the general rational curve $C \subset H$ of degree 11 containing $Y \cap H$ has the Hilbert function of a general rational curve of degree 11 of H. Hence $h^i(H, \mathcal{I}_C(5)) = 0$, $i = 0, 1$ ([18]). Use $Y \cup C$ and the Castelnuovo's sequence.

(iii) Now assume $1 \le d \le 12$. Consider the following triples (d, e, f): $(1, 13, 3)$, $(2, 13, 2)$, $(3, 13, 1)$, $(4, 13, 0)$, $(5, 12, 3)$, $(6, 12, 2)$, $(7, 12, 1)$, $(8, 12, 0)$, $(9, 11, 3)$,
$(10, 11, 2)$, $(11, 11, 1)$, $(12, 11, 0)$. In all cases we have $a = 25 - d \ge e + f$. In all cases we have $d + 4e + f = 56$ and $4d + 5(a - e - f) + 1 + e + 4f = 70$. Fix a general $Y \in F(d, a - e - f; 4)$ and a general $S \cup S' \subset H$ such that $\sharp(S) = e$, $\sharp(S') = f$ and $S \cap S' = \emptyset$. It is sufficient to prove that $h^i(\mathcal{I}_W(5)) = 0$, $i = 0, 1$, for a general union W of $Y \cup S$ and f 2-points. Since $h^1(H, \mathcal{I}_{\{2S,H\}}(5)) = 0$ by the Alexander-Hirschowitz theorem and $Y \cap H$ is a general union of d points, we have $h^i(H, \mathcal{I}_{(Y\cap H)\cup S'\cup\{2S,H\}}(5)) = 0$, $i = 0, 1$. By the differential Horace lemma for 2-points ([1], [11, Lemma 5]) to prove that $h^i(\mathcal{I}_W(5)) = 0$, $i = 0, 1$, it is sufficient to prove that $h^i(\mathcal{I}_{Y\cup S\cup\{2S',H\}}(4)) = 0$. In all cases we have $e \ge f$. In all cases Lemma 4 gives $h^1(\mathcal{I}_{Y\cup 2S'}(4)) = 0$. Hence $h^1(\mathcal{I}_{Y\cup\{2S',H\}}(4)) = 0$. Therefore $h^0(\mathcal{I}_{Y\cup\{2S',H\}}(4)) = e$. Since S is general in H and $h^0(\mathcal{I}_Y(3)) = 0$ (Lemma

3 for the pair $(d, 25-d-e-f)$), [8, Lemma 3] gives $h^i(\mathcal{I}_{Y\cup S\cup\{2S',H\}}(4)) = 0$. □

Lemma 6. *We have* $h^0(\mathcal{I}_{Y(d,c;4)}(4)) = 0$ *for the following pairs* (d,c)*:* $(1,17)$, $(2,17)$, $(3,17)$, $(4,17)$, $(5,13)$, $(6,13)$, $(7,13)$, $(8,13)$, $(9,10)$, $(11,9)$, $(12,9)$, $(13,6)$, $(14,6)$, $(15,5)$, $(16,5)$, $(17,3)$, $(9,8)$.

Proof. If $c \geq 17$, then it is sufficient to use that $h^0(\mathcal{I}_{Y(0,15;4)}(4)) = 0$ by the Alexander-Hirschowitz theorem ([11]) or because $h^0(\mathcal{I}_{Y(0,14;4)}(4)) = 1$ by [12].

(i) Assume $c = 13$ and $5 \leq d \leq 8$. The Alexander-Hirschowitz theorem gives $h^0(\mathcal{I}_{Y(0,13;4)}(4)) = 5$. Since any degree $d \geq 5$ rational curve of $\mathbb{P}^4$ pass through 5 general points (Remark 2), we have $h^0(\mathcal{I}_{Y(d,13;4)}(4)) = 0$ for all $d \geq 5$.

(ii) We have $h^0(\mathcal{I}_{Y(0,8;4)}(3)) = 0$ by the Alexander-Hirschowitz theorem. Fix a general smooth rational curve $C \subset H$ of degree $x \geq 9$. We have $h^0(H, \mathcal{I}_C(4)) = 0$ ([18]). Hence $h^0(\mathcal{I}_{Y(x,c)} = 0$ for all (x,c) with $x \geq 9$ and $c \geq 8$.

(iii) Assume $c = 6$ and $13 \leq d \leq 14$. Fix a general $Y \in F(d,0;4)$ and a general $S \subset H$ with $\sharp(S) = 6$. We have $h^0(\mathcal{I}_Y(3)) = 0$ ([9]), because $3\cdot 13+1 \geq \binom{7}{3}$. We have $h^0(\mathcal{I}_{(Y\cap H)\cup\{2S,H\}}(4)) = 0$. Use $Y \cup 2S$ and the Castelnuovo's sequence.

(iv) Assume $(d,c) = (15,5)$. Fix a general $Y \in F(15,0;4)$ and a general $S \subset H$ with $\sharp(S) = 5$. We have $h^0(\mathcal{I}_Y(3)) = 0$ ([9]) and $h^i(H, \mathcal{I}_{(Y\cap H)\cup\{2S,H\}}(4)) = 0$, $i = 0,1$. Use $Y \cup 2S$ and the Castelnuovo's sequence.

(v) Since $h^0(\mathcal{I}_{Y(16,0;4)}(4)) = 5$([9]), we have $h^0(\mathcal{I}_{Y(16,5;4)}(4)) = 0$. Since we have $h^0(\mathcal{I}_{Y(17,0;4)}(4)) = 1$, then $h^0(\mathcal{I}_{Y(17,x;4)}(4)) = 0$ for all $x > 0$. □

Proof of Theorem 2: We use induction on k, the case $k = 5$ being true by Lemma 5. Hence we may assume $k \geq 6$ and that Theorem 2 is true for all integers k' with $5 \leq k' < k$.

(a) Assume $kd+1+5a \leq \binom{k+4}{4}$ and set $\psi := \binom{k+4}{4} - kd - 1 - 5a$. Increasing if necessary a we reduce to check all cases with $0 \leq \psi \leq 3$.

(a1) Assume $a \leq k$. Let z be the minimal integer such that $(d-z)+kz+1+4a \geq \binom{k+3}{3}$. Set $w := (d-z)+kz-\binom{k+3}{3}$. We have $w \geq 1$. The minimality property of the integer z gives $w \leq k-1$.

Claim 1. *We have* $z \geq k-1$.

Proof of Claim 1. Assume $z \le k-2$, i.e. assume $d+k^2-3k+3+4a \ge \binom{k+3}{3}$. Hence $kd+4ka+k^3-3k^2+3k \ge k\binom{k+3}{3}$. In the set-up of step (a) we have $kd+1+5a \le \binom{k+4}{4}$, but since we need the computation in the set-up of step (b), too, we only assume $kd+1+5a \le \binom{k+4}{4}+3$. Since $a \le k$ we get $k\binom{k+3}{3}-\binom{k+4}{4} \le k(4k-5)+3+3+k^3-3k^2+3k$, which is false for all $k \ge 6$.

Claim 2. *We have $d-z \ge k-1$.]*

Proof of Claim 2. Assume $d-z \le k-2$, i.e. $z \ge d-k+2$. The minimality property of the integer z gives $k-3+k(d-k+3)+4a \le \binom{k+3}{3}$. Since $kd = \binom{k+4}{4}-\psi-5a-1$, we get $\binom{k+4}{4}-\psi-a-1-k^2+4k-3 \le \binom{k+3}{3}$. We are assuming $a \le k$; in step (a) we have $\psi \ge 0$; in the set-up of step (b) we would use that $\psi \ge -3$. We get $\binom{k+4}{4}-k^2+3k-4 \le \binom{k+3}{3}$, which is false for all $k \ge 6$.

Since $w \le k-1$, Claim 2 gives $d-z \ge w$. Fix a general $S \subset H$ with $\sharp(S) = a$ and a general union $Y \subset \mathbb{P}^4$ of a general smooth rational curve of degree $d-z-w+1$ and $w-1$ lines. The set $Y \cap H$ is a general subset of H with cardinality $d-z$. Let $C \subset H$ be a general degree z smooth rational curve containing exactly one point of each connected component of Y; it exists and $C \cup \{2S, H\}$ and $Y(z,a;3)$ have the same Hilbert function by Remark 2. Since $Y \cap (H \setminus C)$ is a general subset with cardinality $d-z-w$, Theorem 1 gives $h^i(H, \mathcal{I}_{(Y\cap(H\setminus C))\cup C\cup\{2S,H\}}(k)) = 0$. By the Castelnuovo's sequence it is sufficient to prove that $h^1(\mathcal{I}_{Y\cup S}(k-1)) = 0$. We have $h^0(\mathcal{O}_{Y\cup S}(k-1)) = \binom{k+3}{4}-\psi$ and $h^0(\mathcal{O}_Y(k-2)) = h^0(\mathcal{O}_Y(k-1))-(d-z) = \binom{k+2}{4}-\psi-a-(d-z)$. By [9] Y has maximal rank and hence $h^1(\mathcal{I}_Y(k-1)) = 0$, $h^0(\mathcal{I}_Y(k-1)) = \psi+a$ and $h^0(\mathcal{I}_Y(k-2)) = \max\{0, \binom{k+2}{4}-h^0(\mathcal{I}_Y(k-2))\}$. Since S is general, [8, Lemma 3] shows that to prove $h^1(\mathcal{I}_{Y\cup S}(k-1)) = 0$ it is sufficient to prove that $d-z+a \le \binom{k+2}{3}$. Since $a \le k$ and $z \ge k-1$, it is sufficient to use that $d+1 \le \binom{k+2}{3}$ for all $k \ge 6$ (see the proof of Claim 2).

(a2) Assume $d+4a \ge \binom{k+3}{3}+12$ and write $d+4e+f = \binom{k+3}{3}$ with $0 \le f \le 3$. By assumption we have $a \ge e+f$. Fix a general $Y \in F(d, a-e-f; 4)$ and a general $S \cup S' \subset H$ with $\sharp(S) = e$, $\sharp(S') = f$ and $S \cap S' = \emptyset$. Since $Y \cap H$ is a general union of d points of H and $d+4e+f = \binom{k+3}{3}$, the Alexander-Hirschowitz theorem gives $h^i(H, \mathcal{I}_{S'\cup(H\cap(Y\cup 2S))}(k)) = 0$, $i = 0,1$. By the differential Horace lemma ([1], [11]) to prove $h^1(\mathcal{I}_W(k)) = 0$ for a general union W of $Y \cup 2S$ and f 2-points (and hence prove Theorem 2 in this case) it is sufficient to prove that $h^1(\mathcal{I}_{Y\cup S\cup\{2S',H\}}(k-1)) = 0$.

Claim 3. *We have $e \ge f$.*

Proof of Claim 3. Assume $e \le f-1$. Since $f \le 3$ we get $4e+f \le 11$. Hence $d \ge \binom{k+3}{3}-11$. Hence $kd+1 \ge k\binom{k+3}{3}-11k+1$. Since a general rational curve of fixed degree has maximal rank in $\mathbb{P}^4$, we may assume $kd \le \binom{k+4}{4}$. We get $\binom{k+4}{4} \ge k\binom{k+3}{3}-11k+2$, which is false for all $k \ge 6$.

We have $h^0(\mathcal{O}_{Y\cup S\cup\{2S',H\}}(k-1)) = \binom{k+3}{4}-\psi$ and hence $h^0(\mathcal{O}_{Y\cup 2S'}(k-1)) = \binom{k+3}{4}-\psi-e+f$. Claim 3 gives $h^0(\mathcal{O}_{Y\cup 2S'}(k-1)) \le \binom{k+3}{4}$. Since $f \le 3$ and any 3 points of $\mathbb{P}^4$ are contained in a hyperplane, $Y\cup 2S'$ and $Y(d,a-e;4)$ have the same Hilbert function. The inductive assumption on k gives $h^1(\mathcal{I}_{Y\cup 2S'}(k-1)) = 0$ and hence $h^1(\mathcal{I}_{Y\cup\{2S',H\}}(k-1)) = 0$. Therefore $h^0(\mathcal{I}_{Y\cup\{2S',H\}}(k-1)) = \psi+e$. Since S is general in H, to apply [8, Lemma 3] and get $h^1(\mathcal{I}_{Y\cup S\cup\{2S',H\}}(k-1)) = 0$ is is sufficient to prove that $h^0(\mathcal{I}_Y(k-2)) \le \psi$. Assume for the moment $k=7$, so that we may apply the inductive assumption to Y in degree $k-2$. To get $h^0(\mathcal{I}_Y(k-2)) \le \psi$ it is sufficient to check that $h^0(\mathcal{I}_Y(k-2)) \ge \binom{k+2}{4}-\psi$. We have $h^0(\mathcal{I}_Y(k-2)) = h^0(\mathcal{I}_Y(k-1))-d = \binom{k+3}{4}-\psi-e-4f$. Hence it is sufficient to prove that $d+e+4f \le \binom{k+2}{3}$. Since $d+4e+f = \binom{k+3}{3}$, it is sufficient to prove that $3(e-f) \ge \binom{k+2}{2}$. Assume $3e-3f \le \binom{k+2}{2}-1$. Since $f \le 3$ and $d+4e+f = \binom{k+3}{3}$, we first get $3e \le (k^2+3k+2)/2+3f$ and then $d \ge \binom{k+3}{3}-2(k^2+3k+2)/3-12$. Since a general rational curve of fixed degree has maximal rank in $\mathbb{P}^4$, we may assume $kd \le \binom{k+4}{4}$. Hence $\binom{k+4}{4} \ge k\binom{k+3}{3}-2(k^2+3k+2)/3-12k$, which is false for all $k \ge 7$. Now assume $k=6$. By [9] we have $h^0(\mathcal{I}_{Y(18,0;4)}(4)) = 0$. Hence we may assume $1 \le d \le 17$. Since $d+4e+f = 84$ and $6d+1+5a = 210-\phi$ we have the following quintuples (d,a,ϕ,e,f): $(1,40,3,20,3)$, $(2,39,2,20,2)$, $(3,38,1,20,1)$, $(4,37,0,20,0)$, $(5,35,4,19,3)$, $(6,34,3,19,2)$, $(7,33,2,19,1)$, $(8,32,1,19,0)$, $(9,31,0,18,3)$, $(10,29,4,18,2)$, $(11,28,3,18,1)$, $(12,27,2,18,0)$, $(13,26,1,17,3)$, $(14,25,0,17,2)$, $(15,23,4,17,1)$, $(16,22,3,17,0)$, $(17,22,2,16,3)$. In each of these cases Lemma 6 gives $h^0(\mathcal{I}_{Y(d,a-e-f;4)}(4)) = 0$.

(a3) Assume $kd \ge \binom{k+3}{3}+k$ and $a \ge k$. Let z be the maximal integer such that $d-z+kz \le \binom{k+3}{3}$. The assumption on d gives $z < d$. The maximality of the integer z gives $(d-z-1)+kz+1+w = \binom{r+k-1}{r-1}$ with $0 \le w \le k-2$. Write $w = 4x+y$ with $0 \le y \le 3$ and $x = \lfloor w/4 \rfloor$. Since $a \ge k$, we have $a \ge x+y$. Since $kd \ge \binom{k+3}{3}+k$, we have $z > 0$. Fix a general $Y \in F(d-z,a-x-y;0)$, a general smooth rational curve $C \subset H$ of degree z containing exactly one of the $d-z$ points of $Y\cap H$ and a general $S\cup S' \subset H$ with $\sharp(S) = x$, $\sharp(S') = y$ and $S\cap S' = \emptyset$.

(a4) Assume $kd \le \binom{k+3}{3}+k-1$. Call c the maximal integer such that $kd+1+4c \le \binom{k+3}{3}$. Hence $kd+1+4c+y = \binom{k+3}{3}$ with $0 \le y \le 3$. Our *assumption gives* $c \ge 0$*. Since* $kd+1+5a = \binom{k+4}{4} - \psi$*, we have* $5a-4c-y = \binom{k+3}{4} - \psi$*.*

Claim 4. *We have* $a \ge c+y$*.*

Proof of Claim 4. Assume $a \le c+y-1$. We get $5+c+4y \ge \binom{k+3}{4} - \phi$. If $c < y$, then $a \le 2y-2 \le 4$, a contradiction. Hence $5+4c+y \ge \binom{k+3}{4} - \phi$. Since $\psi \le 4$ we get $4c+y \ge \binom{k+3}{4} \ge \binom{k+3}{4}$. Since $k \ge 6$ and $kd+1+4c+y = \binom{k+3}{3}$, we get a contradiction.

(a4.1) Assume $c \ge y$*. By Claim 4 we have* $a-c-y \ge 0$*. Fix a general* $Y \in F(0, a-y-c; 4)$*, a general smooth rational curve* $C \subset H$ *of degree* d *and a general* $S \cup S' \subset H$ *with* $\sharp(S) = c$*,* $\sharp(S') = y$ *and* $S \cap S' = \emptyset$*. It is sufficient to prove that* $h^1(\mathcal{I}_W(k)) = 0$ *for a general union* W *of* $Y \cup C \cup 2S$ *and* y *general 2-points. Since* Y *is general, we have* $Y \cap H = \emptyset$*. Since* $kd+1+4c+y = \binom{k+3}{3}$ *and* S' *is general, Theorem 1 gives* $h^i(\mathcal{I}_{C \cup \{2S,H\} \cup S'}(k)) = 0$*. By the differential Horace lemma to prove that* $h^1(\mathcal{I}_W(k)) = 0$ *it is sufficient to prove that* $h^1(\mathcal{I}_{Y \cup S \cup \{2S',H\}}(k-1)) = 0$*. We have* $h^0(\mathcal{O}_{Y \cup S \cup \{2S',H\}}(k-1)) = \binom{k+3}{4} - \psi$*. Hence* $h^0(\mathcal{O}_{Y \cup 2S'}(k-1)) = \binom{k+3}{4} - \psi - c + y$*. Since* $y \le c$*, we have* $h^0(\mathcal{O}_{Y \cup 2S'}(k-1)) \le \binom{k+3}{4}$*. Hence the Alexander-Hirschowitz theorem gives* $h^1(\mathcal{O}_{Y \cup 2S'}(k-1)) = 0$*. Hence* $h^1(\mathcal{I}_{Y \cup \{2S',H\}}(k-1)) = 0$*. Hence* $h^0(\mathcal{I}_{Y \cup \{2S',H\}}(k-1)) = \psi + c$*. By [8, Lemma 3] to get* $h^1(\mathcal{I}_{Y \cup S \cup \{2S',H\}}(k-1)) = 0$ *it is sufficient to prove that* $h^0(\mathcal{I}_Y(k-2)) = 0$*. Assume for the moment that either* $k \ne 6$ *or* $a-y-c \ne 9$*. By the Alexander-Hirschowitz theorem* Y *has the expected Hilbert function in degree* $k-2$*. Hence it is sufficient to prove that* $h^0(\mathcal{O}_Y(k-2)) \ge \binom{k+2}{4} - \psi$*. We have* $h^0(\mathcal{O}_Y(k-2)) = h^0(\mathcal{O}_Y(k-1)) = \binom{k+3}{4} - \psi - c - 4y$*. Hence it is sufficient to check that* $c+4y \le \binom{k+2}{3}$*. Since* $c \ge y$ *and* $kd+1+4c+y = \binom{k+3}{3}$*, it is sufficient to have* $kd+1 \ge \binom{k+2}{2}$*, i.e.* $d \ge (k+1)/2$*. This inequality is obvious (since* $\psi \le 3$*), because by step (a2) we may assume* $d+4a \le \binom{k+3}{3} + 11$*.*

Now assume $k=6$ *and* $a = 9+c+y$*. We have* $6d+1+5(9+c+y) = 210-\psi$ *and* $6d+1+4c+y = 84$*. We get* $c+4y = 71-\psi$*. Since* $\psi \le 3$*,* $y \le 3$ *and* $a = 9+c+y$*, then* $a \ge 59$*. Hence* $5a \ge \binom{10}{4} + 4$*, a contradiction.*

(a4.2) Assume $c < y$*. Since* $y \le 3$*, we get* $0 \le c \le 2$ *and* $y > 0$*. Write* $k+4c+y = 4c'+y'$ *with* $0 \le y' \le 3$ *and* c' *an integer. We may assume that we are not in step (a1) and in particular we may assume* $d \ge 2$*. Our assumption on* d *implies* $a \ge k+2$ *and hence* $a \ge y'+c'$*. Fix a general* $Y_1 \in F(1, a-y'-c'; 4)$*, a general smooth rational curve* $C_1 \subset H$ *containing the only point of* $Y \cap H$ *and a general* $S_1 \cup S'_1 \subset H$ *with* $\sharp(S_1) = c'$*,* $\sharp(S'_1) = y'$ *and* $S_1 \cap S'_1 = \emptyset$*. It is sufficient to prove* $h^1(\mathcal{I}_W(k)) = 0$ *for a general union* W *of* $Y_1 \cup C_1 \cup 2S_1$ *and* y' *general 2-points. Since* $d-1 > 0$*, we have* $h^0(\mathcal{O}_{C_1 \cup \{2S_1,H\} \cup S'_1}(k)) = \binom{k+3}{3}$*. Since*

S_1' is general in H, Theorem 1 gives $h^i(H, \mathcal{I}_{C_1 \cup \{2S_1, H\} \cup S_1'}(k)) = 0$, $i = 0, 1$. Since $C_1 \supset Y_1 \cap H$, the differential Horace lemma for 2-points show that to prove $h^1(\mathcal{I}_W(k)) = 0$ it is sufficient prove $h^1(\mathcal{I}_{Y_1 \cup S_1 \cup \{2S_1', H\}}(k-1)) = 0$. We have $h^0(\mathcal{O}_{Y_1 \cup S_1 \cup \{2S_1', H\}}(k-1)) = \binom{k+3}{4} - \psi$. Since $k + 4c + y = 4c' + y'$ with $0 \le y' \le 3$, $k \ge 6$ and $y > 0$, we have $c' \ge y'$. Hence $h^0(\mathcal{O}_{Y_1 \cup 2S_1'}(k-1)) = \binom{k+3}{4} - \psi + y' - c' \le \binom{k+3}{3}$. Since any 3 points of $\mathbb{P}^4$ are contained in a hyperplane, $Y_1 \cup 2S_1'$ have the same Hilbert function of $Y(1, a-c'; 4)$. Since $h^0(\mathcal{O}_{Y_1 \cup 2S_1'}(k-1)) \le \binom{k+3}{4}$, [6] gives $h^1(\mathcal{I}_{Y_1 \cup 2S_1'}(k-1)) = 0$. Hence $h^1(\mathcal{I}_{Y_1 \cup \{2S_1', H\}}(k-1)) = 0$. Therefore $h^0(\mathcal{I}_{Y_1 \cup \{2S_1', H\}}(k-1)) = c' + \psi$. To conclude applying [8, Lemma 3] it is sufficient to prove that $h^0(\mathcal{I}_{Y_1}(k-2)) \le \phi$. By [6] Y_1 has the expected Hilbert function in degree $k-2$. Hence it is sufficient to check that $h^0(\mathcal{O}_{Y_1}(k-2)) \ge \binom{k+2}{4} - \psi$. We have $h^0(\mathcal{O}_{Y_1}(k-2)) = h^0(\mathcal{O}_{Y_1}(k-1)) - 1 = h^0(\mathcal{O}_{Y_1 \cup S_1 \cup \{2S_1', H\}}(k-1)) - 1 - c' - 3y' = \binom{k+3}{4} - \psi - 1 - c' - 3y'$. Hence it is sufficient to check that $\binom{k+2}{3} \ge 1 + c' + 3y'$. This inequality is obvious, because $c' \ge y'$, $4c' + y' = k + 4c + y$ and $4c + y \le 11$.

(b) Assume $kd + 1 + 5a \ge \binom{k+4}{4}$ and set $\phi := kd + 1 + 5a - \binom{k+4}{4}$. Decreasing if necessary a we see that is sufficient to test all pairs (d, a) with eith $0 \le \phi \le 4$ or $a = 0$. Since a general degree rational curve has maximal rank ([9]), it it sufficient to test all (d, a) with $0 \le \phi \le 4$. The case $\phi = 0$ was tested in step (a) (it has $\psi = 0$). Now assume $a > 0$ and $\phi = 4$. Step (a) gives $h^0(\mathcal{I}_{Y(d,a-1;4)}(k)) = 1$. Hence $h^0(\mathcal{I}_{Y(d,a-1;4) \cup 2P}(k)) \le h^0(\mathcal{I}_{Y(d,a-1;4) \cup \{P\}}(k)) = 0$ for a general $P \in \mathbb{P}^4$. Hence it is sufficient to test all (d, a) with $1 \le \phi \le 3$. It is easy to adapt step (a) (in many cases we even did explicitly the numerical check needed for step (b)). We only point out an infinitesimal difference. There are (as for instance in step (a2)) some Y and a general $S \cup S' \subset H$ with, say, $\sharp(S) = e$, $\sharp(S') = f$, $S \cap S' = \emptyset$, $e \ge f$, and we need to prove that $h^0(\mathcal{I}_{Y \cup S \cup \{2S', H\}}(k-1)) = 0$. We have $h^0(\prime_{Y \cup S \cup \{2S', H\}}(k-1)) = \binom{k+3}{4} + \phi$. If $e \ge f + \phi$, then we get by the inductive assumption first $h^1(\mathcal{I}_{Y \cup 2S'}(k-1)) = 0$ and then $h^1(\mathcal{I}_{Y \cup \{2S', H\}}(k-1)) = 0$. The condition $h^0(\mathcal{I}_Y(k-2)) = 0$ is at least as easy as in step (a). So it would be sufficient to check a stronger inequality in Claim 3. However, this is not necessary. If $h^1(\mathcal{I}_{Y \cup 2S'}(k-1)) > 0$, then the inductive assumption gives $h^0(\mathcal{I}_{Y \cup 2S'}(k-1)) = 0$. Hence $h^0(\mathcal{I}_{Y \cup \{2S', H\}}(k-1)) \le f$. Since $e \ge f$ and $h^0(\mathcal{I}_Y(k-2)) = 0$, [8, Lemma 3] gives $h^0(\mathcal{I}_{Y \cup S \cup \{2S', H\}}(k-1)) = 0$. □

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