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ON LINEAR AND MULTILINEAR MAPPINGS OF NUCLEAR TYPE

B. Martin Cerna Maguiña[1 §], Janet Mamani Ramos[2], Héctor F. Cerna Maguiña[3]

[1]Department of Mathematics
National University of Santiago Antúnez de Mayolo
Campus Shancayan, Av. Centenario 200, Huaraz, PERÚ
[2]Academic Department of Environmental and Natural Resources Engineering
National University of Callao
Av. Juan Pablo II 306, Callao, PERÚ
[3]Departamento Académico de Contabilidad
Universidad Nacional Mayor de San Marcos
Av. Universitaria 306, Lima, Perú

Abstract: In this work, a technique is used in order to demonstrate the following lemmas "If $s \in \mathcal{F}(E, C(K))$, then $N^0_\infty(S) = \|S\|$" and "If $S \in \mathcal{F}(E, L_p(\Omega, \mu))$, then $K^0_p(S) = \|S\|$" and open question is answered in 18.1.16 which belong to Albert Pietsch, at the same time Lemma 2.1 of B. M. Cerna is demonstrated consequently additional results has been got.

Dedicated to the memory of our father: B. Cerna Figueroa

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Following the ideas of Pietsch, I couldn't extend the lemmas "If $s \in \mathcal{F}(E, C(K))$, then $N^0_\infty(S) = \|S\|$" and "If $S \in \mathcal{F}(E, L_p(\Omega, \mu))$, then $K^0_p(S) = \|S\|$" toward the multilinear case. According to my interests fortunately, I could get

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[§]Correspondence author

one technique based If $f \in X^*$ where X is a normed space and $f \neq 0$, then f is surjective and $X = \ker f \oplus M$, where M is a one-dimensional subspace. These result and the theorem of Hahn Banach let me demonstrate the lemmas mentioned before, furthermore respond the question that is formulate in the book of Piestch in the lemma 18.1.16. furthermore this technique must have been extended to the multilinear case which let me get additional results.

A problem arise that It's more natural, There exist some relation between the theory that It served in the show of this results?

We introduce the notations in the present work, for Banach spaces $E_1, \cdots, E_n$ and F over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), we denote $L(E_1, \cdots, E_n; F)$ to the Banach space of all multi-linear and continuous applications of $E_1 \times \cdots \times E_n$ over F with a natural norm given by

$$\|T\| = \sup_{\substack{x_i \in B_{E_i} \\ i=1,\cdots,n}} \|T(x_1, \cdots, x_n)\|$$

where B_{E_i} denote the unitary ball of E_i, centered in 0. E_k^* denote the dual topological of E_k, $k = 1, \cdots, n$.

For $s \in \langle 0, +\infty\rangle$ we denote by $l_s(F)$ (or l_s, $F = \mathbb{K}$), the vector space of all sequences $(y_j)_{j=1}^{\infty}$ of elements that belong to F such that

$$l_s(y_j) = \left\|(y_j)_{j=1}^{\infty}\right\|_s = \left[\sum_{j=1}^{\infty} \|y_j\|^s\right]^{1/s} < +\infty.$$

For $s \geq 1$, $\|.\|_s$ is a norm, and for $s < 1$, is a s-norm. In any case we have a complete metric vector space. We denote by $l_s^w(F)$ the vector space of all sequences $(y_j)_{j=1}^{\infty}$ of elements that belong to F such that

$$\left\|(y_j)_{j=1}^{\infty}\right\|_{w,s} = w_s(y_j) = \sup_{\varphi \in B_{F^*}} \left\|(\varphi(y_j))_{j=1}^{\infty}\right\|_s < +\infty,$$

so, $(l_s^w(F), \|.\|_{w,s})$ is a metric vector space.

For $s = +\infty$, we consider $l_\infty(F) = l_\infty^w(F)$ as a Banach space of all sequences $(y_j)_{j=1}^{\infty}$ of elements of F under the norm

$$w_\infty(y_j) = \left\|(y_j)_{j=1}^{\infty}\right\|_\infty = \left\|(y_j)_{j=1}^{\infty}\right\|_{w,\infty} = \sup_{j \in \mathbb{N}} \|y_j\|$$

Let $0 < r \leq \infty$, $1 \leq p, q \leq \infty$, and $1 + \frac{1}{r} \geq \frac{1}{p} + \frac{1}{q}$. An operator $s \in L(E, F)$ is called (r, p, q)-nuclear if

$$s = \sum_{i=1}^{\infty} \sigma_i a_i \otimes y_i$$

con $(\sigma_j)_{j=1}^{\infty} \in l_r$, $(a_i)_{i=1}^{\infty} \in w_{q'}(E^*)$, and $(y_i)_{i=1}^{\infty} \in w_{p'}(F)$. In the case $r = \infty$ let us suppose that $(\sigma_i) \in c_0$.

We put

$$N_{(r,p,q)}(s) := \inf l_r(\sigma_i) w_{q'}(a_i) w_{p'}(y_i) \tag{1}$$

where the infimun is taken over all so-called (r, p, q)-nuclear representations described above.

The class of all (r, p, q)-nuclear operators is denoted by $\mathcal{N}_{(r,p,q)}$.
$\mathcal{F}(E, F) :=$ Ideal of finite operators of finite range from E onto F.
For every operators $s \in \mathcal{F}(E, F)$ we put

$$N^0_{(r,p,q)}(s) = N_{f,(r,p,q)}(s) := \inf l_r(\sigma_i) w_{q'}(a_i) w_{p'}(y_i) \tag{2}$$

where the infimum is taken over all finite representations

$$s = \sum_{i=1}^{n} \sigma_i a_i \otimes y_i,$$

for $r \in]0, +\infty]$, $p, q_j \in [1, \infty]$, $k = 1, \cdots, n$, such that $n + \frac{1}{r} \geq \frac{1}{p} + \frac{1}{q_1} + \cdots + \frac{1}{q_n}$, $T \in L(E_1, \cdots, E_n; F)$ is called $(r, p; q_1, \cdots, q_n)$-nuclear type and it takes the form

$$T = \sum_{k=1}^{\infty} \sigma_k x^*_{k,1} \times \cdots \times x^*_{k,n} \otimes y_k \tag{3}$$

with $(\sigma_j)_{j=1}^{\infty} \in l_r$, $(y_k)_{k=1}^{\infty} \in l^w_{p'}(F)$ and $\left(x^*_{k,j}\right)_{k=1}^{\infty} \in l^w_{q'_j}\left(E^*_j\right)$, $j = 1, \cdots, n$.

In the case $r = +\infty$ the condition for $(\sigma_k)_{k=1}^{\infty}$ is to be in c_0.

The set of such applications satisfying such definition is a vector space and is denote by $\mathcal{N}_{r,p,q_1,\cdots,q_n}(E_1, \cdots, E_n; F)$.

Considering that

$$N_{(r,p,q_1,\cdots,q_n)}(T) = \inf l_r(\sigma_k) \prod_{j=1}^{n} w_{q'_j}\left(x^*_{k,j}\right) w_{p'}(y_k)$$

where the infimum is taken over all possible representations of T described in (3).

$L_f\left(E_1, \cdots, E_n; F\right) :=$ multi-linear applications of finite type

For every operator $\psi \in L_f(E_1, \cdots, E_n; F)$ we put

$$N_{f,(r,p,q_1,\cdots,q_n)}(\psi) = \inf \|(\sigma_k)_{k=1}^{m}\|_r \prod_{j=1}^{n} \left\| \left(x^*_{k,j}\right)_{k=1}^{m} \right\|_{w,q'_j} \|(y_k)_{k=1}^{m}\|_{w,p'}$$

where the infimum is taken over all finite representations

$$\psi = \sum_{k=1}^{m} \sigma_k x^*_{k,1} \times \cdots \times x^*_{k,n} \otimes y_k$$

where p' is a dual exponent, $\frac{1}{p} + \frac{1}{p'} = 1$. K_F is the isometric immersion of F into F^{**}.

1. Linear Operators of (r, p, q)-Nuclear Type

In this section using the proposition 1.1 and one consequence of the Hanhn-Banach Theorem see [2] demonstrate the lemmas 19.2.5, 19.3.6 and to get similar result of 18.1.16. See [6].

Proposition 1.1. *Lets X a normed linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), $f_1, f_2 \in X^*$ with f_1, f_2 is linearly independent. Then exist $x \in X$ and $\lambda \in \mathbb{K}$ such that $f_1(x) = a$ and $f_2(x) = \lambda a$ or $f_1(x) = a$ and $f_2(x) = 0$ where $a, \lambda \in \mathbb{K}$ and $a \neq 0$*

Proof. Since f_1, f_2 is surjective there exist $x_1, x_2 \in X$ such that $f_1(x_1) = a$, $f_2(x_2) = \beta a$. We know that $X = \ker f_1 \oplus M_1 = \ker f_2 \oplus M_2$, where M_1 and M_2 are one dimensional subspaces $M_1 = \langle m_1 \rangle$, $M_2 = \langle m_2 \rangle$. Así

$$x_1 = y_1 + \lambda_1 m_2 \quad , \quad y_1 \in \ker f_2, \;\; m_2 \in M_2$$
$$x_2 = y_2 + \lambda_2 m_2 \quad , \quad y_2 \in \ker f_2, \;\; m_2 \in M_2$$

Then

$$f_2(x_1) = \lambda_1 f_2(m_2) \tag{4}$$

$$\beta a = f_2(x_2) = \lambda_2 f_2(m_2) \tag{5}$$

of (4) and (5) we have $f_2(x_1) = \beta\frac{\lambda_1}{\lambda_2}a$, to take $\lambda = \beta\frac{\lambda_1}{\lambda_2}$ for can get $f_2(x_1) = \lambda a$. □

Lemma 1.1. *Let $\psi : E \longrightarrow L_p(\Omega, \mu)$ be defined by*

$$\psi(x) = \sum_{k=1}^{m} \sigma_k x_k^*(x) y_k \tag{6}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p < \infty$, then

$$N_{f,(\infty,p,q)}(\psi) = N_{(\infty,p,q)}(\psi) = \|\psi\|$$

Proof. It is clear that for $\frac{1}{p} + \frac{1}{q} = 1$, one has from (1) and (2)

$$\|\psi\| \leq N_{(\infty,p,q)}(\psi) \leq N_{f,(\infty,p,q)}(\psi) \tag{7}$$

Moreover

$$\|\psi\|\|x\| \geq \left[\int_\Omega \left|\sum_{k=1}^{m} \sigma_k x_k^*(x) y_k(t)\right|^p du(t)\right]^{\frac{1}{p}} \tag{8}$$

Let

$$M = \sup_{\|x\|=1} \left(\sum_{k=1}^{m} |\langle x_k^*, x\rangle|^p\right)^{\frac{1}{p}} \tag{9}$$

it is clear that $M < +\infty$. From the proposition (1.1) there exist $\overline{x} \in E$ such that

$$x_k^*(\overline{x}) = \frac{M}{2^{\frac{k}{p}}}, \quad k = 1, \cdots, m$$

Therefore, from the relation (9) we have that, one has $\epsilon > 0$, $M\|x\| < (1+\epsilon)\left(\sum_{k=1}^{m} \frac{M^p}{2^k}\right)^{\frac{1}{p}}$, which implies that $\|\overline{x}\| < 1+\epsilon$, $\forall \epsilon > 0$.

Therefore $\|\overline{x}\| \leq 1$, from the relation (8) one has

$$\|\psi\| \geq \left[\int_\Omega \left|\sum_{k=1}^{m} \sigma_k \frac{M}{2^{\frac{k}{p}}} y_k(t)\right|^p du(t)\right]^{\frac{1}{p}}$$

$$\|\psi\| \geq M\left[\int_{\Omega}\left|\sum_{k=1}^{m}\sigma_k\frac{1}{2^{\frac{k}{p}}}y_k(t)\right|du(t)\right]^{\frac{1}{p}} \tag{10}$$

Let

$$z(t)=\sum_{k=1}^{m}\sigma_k\frac{y_k(t)}{2^{\frac{k}{p}}}$$

them for $1 \leq p < \infty$ we have

$$|\langle\phi, z\rangle| = \left|\sum_{k=1}^{m}\sigma_k\left\langle\phi, \frac{y_k}{2^{\frac{k}{p}}}\right\rangle\right| \leq \|\phi\|\|z\| \tag{11}$$

In addition, let

$$L=\operatorname{span}_{k\in\{1,\cdots,m\}-\{k_0\}}\left\{\frac{y_k}{2^{\frac{k}{p}}}\right\}$$

Moreover, as consequence of the Hahn-Banach theorem (see [2]) there exists ϕ such that

$$\|\phi\|=\frac{1}{d},\ \ \langle\phi, x\rangle=0,\ \ \text{for all } x\in L \ \text{ and } \ \left\langle\phi, \frac{y_{k_0}}{2^{\frac{k_0}{p}}}\right\rangle=1,$$

where

$$d=\inf_{x\in L}\left\|x-\frac{y_{k_0}}{2^{\frac{k_0}{p}}}\right\|$$

and further can choose σ_{k_0} such that

$$|\sigma_{k_0}|=\max_{k=\overline{1,n}}|\sigma_k|=l_\infty(\sigma_k)$$

Taking into account these last relations in the equation (11) we can get

$$\|z\| \geq |\sigma_{k_0}|\,d \tag{12}$$

Since

$$x=\sum_{k=1,\ k\neq k_0}^{m}-\frac{y_k}{2^{\frac{k}{p}}}\in L$$

then for a given $\epsilon > 0$ one has

$$(1+\epsilon)d > \left\|\sum_{k=1}^{m}\frac{y_k}{2^{\frac{k}{p}}}\right\|$$

therefore, from the relations (12) one has:

$$(1+\epsilon)\|z\| > l_\infty(\sigma_k)\left\|\sum_{k=1}^{m}\frac{y_k}{2^{\frac{k}{p}}}\right\| \tag{13}$$

We know that

$$w_{p'}(y_k) = \sup_{a\in B^m_{l_p}}\left\|\sum_{k\le m} a_k y_k\right\|$$

and since

$$a = \left(\frac{1}{2^{\frac{k}{p}}}\right)_{k=1}^{m} \in B^m_{l_p}$$

for $\widetilde{\epsilon} > 0$ we have

$$(1+\widetilde{\epsilon})\left\|\sum_{k\le m}\frac{1}{2^{\frac{k}{p}}}y_k\right\| > w_{p'(y_k)}$$

From the last relation an the equation (13) one obtains

$$\|z\|(1+\epsilon)(1+\widetilde{\epsilon}) > l_\infty(\sigma_k)w_{p'}(y_k),\ \text{for all } \epsilon > 0,\ \text{and } \widetilde{\epsilon} > 0 \tag{14}$$

Therefore, from the relations (10) y (14) one has

$$\|\psi\| \ge l_\infty(\sigma_k)w_{p'}(y_k)M = N_{f,(\infty,p,q)} \tag{15}$$

From equations (7) and (15) one has required result.

For $p = +\infty$, we have

$$\|\psi\| = \sup_{\|x\|_E=1}\|\psi(x)\|_{L_\infty(\Omega,\mu)}$$

where

$$\|\psi(x)\|_{L_\infty(\Omega,\mu)} = \inf_{A>0}\{A\ /\ |\psi(x(t))| \le A,\ \text{except for a set of measure zero }\}$$

also we have:

$$\|\psi\|\|x\|_E \ge \|\psi(x)\|_{L_\infty(\Omega,\mu)},$$

for $\epsilon > 0$, there exists $A > 0$ and $N \subset \Omega$ with $\mu(N) = 0$ such that

$$|\psi(x(t))|(1-\epsilon) \le (1-\epsilon)A < \|\psi(x)\|_{L_\infty(\Omega,\mu)}$$

$$\|\psi\|\|x\|_E > (1-\epsilon)|\psi(x(t))|,\ \forall t \in (\Omega - N) \tag{16}$$

From the equations (4) and (16) one can get

$$\|\psi\|\|x\|_E > (1-\epsilon)\left|\sum_{k=1}^{m}\sigma_k x_k^*(x)y_k(t)\right|, \quad \forall t\in(\Omega-N) \tag{17}$$

From the proposition (1.1) there exists $\overline{x}\in E$ such that

$$\|\overline{x}\|_E \leq 1 \ \text{ y } \ x_k^*(\overline{x}) = \frac{M}{2^{\frac{k}{p}}} = M, \ \text{ with } p=\infty$$

where

$$M = \sup_{\|x\|_E=1}\ \sup_{k\in\{1,\cdots,m\}} |\langle x_k^*, x\rangle|$$

Then it is clear that $M<+\infty$ and of (17) we have

$$\|\psi\| > (1-\epsilon)M\left|\sum_{k=1}^{m}\sigma_k y_k(t)\right|, \ \forall t\in(\Omega-N) \tag{18}$$

Let

$$z(t) = \sum_{k=1}^{m}\sigma_k y_k(t), \quad \forall t\in(\Omega-N)$$

we have

$$|\langle\varphi, z\rangle| = \left|\sum_{k=1}^{m}\sigma_k\langle\varphi, y_k\rangle\right| \leq \|\varphi\|\|z\| \tag{19}$$

In addition, let

$$L = \text{span}_{k\in\{1,\cdots,m\}-\{k_0\}}\{y_k\}$$

Moreover, as a consequence of the Hahn-Banach theorem (see [2]), there exists φ such that

$$\|\varphi\| = \frac{1}{d}, \ \langle\varphi, x\rangle = 0, \ \text{ for all } x\in L \ \text{ and } \ \langle\varphi, y_{k_0}\rangle = 1,$$

where

$$d = \inf_{x\in L}\|x - y_{k_0}\|$$

and further one can choose σ_{k_0} such that

$$|\sigma_{k_0}| = \max_{k=\overline{1,m}}\sigma_k = l_\infty(\sigma_k)$$

Taking into account these last relations in equation (19) one can get

$$\|z\| \geq |\sigma_{k_0}|\, d \tag{20}$$

since

$$x = \sum_{\substack{k=1 \\ k \neq k_0}}^{m} -y_k \ \text{ and } \ x \in L$$

then

$$(1+\epsilon)d > \left\| \sum_{k=1}^{m} y_k \right\|$$

Therefore, from the relations (20) one has:

$$(1+\epsilon)\|z\| > l_\infty\left(\sigma_k\right) \left\| \sum_{k=1}^{m} y_k \right\| \tag{21}$$

we know that

$$w_1\left(y_k\right) = \sup_{a \in B^m_{l_\infty}} \left\| \sum_{k \leq m} a_k y_k \right\|$$

since

$$a = \{1\}_{k=1}^{m} \ \text{ y } \ a \in B^m_{l_\infty}$$

For $\widetilde{\epsilon} > 0$ we have that

$$(1+\widetilde{\epsilon}) \left\| \sum_{k \leq m} y_k \right\| > w_1\left(y_k\right) \tag{22}$$

From the relations (21) and (22) we have:

$$(1+\epsilon)\left(1+\widetilde{\epsilon}\right)\|z\| > l_\infty\left(\sigma_k\right) w_1\left(y_k\right), \ \text{ for all } \epsilon > 0, \text{ and } \widetilde{\epsilon} > 0$$

Therefore, from the last relations and (18) we have

$$\|\psi\| \geq M l_\infty\left(\sigma_k\right) w_1\left(y_k\right) = N_{f,(\infty,\infty,1)}(\psi) \tag{23}$$

From equations (5) and (23) one has the required result.

$$N_{f,(\infty,\infty,1)}(\psi) = N_{(\infty,\infty,1)}(\psi) = \|\psi\|$$

□

Therefore, for $p=\infty$, one has the following

Lemma 1.2. *Let K be any compact Hausdorff space. Then for $S \in \mathcal{F}(E, C(K))$, we have $N_{(\infty,\infty,1)}(\psi) = N_{f,(\infty,\infty,1)}(\psi) = \|\psi\|$.*

In the Subsection 18.1.16 Pietsch say for Special exponents the above result holds without any assumption on the underlying Banach space.

Proposition 1.2. *[6] $N^0_{(r,2,q)}(s) = N_{(r,2,q)}(s)$ for all $s \in \mathcal{F}(E,F)$.*

Proof. (See [6]) □

Proposition 1.3. *$N_{f,(\infty,p,q)}(s) = N_{(\infty,p,q)}(s)$ for all $s \in \mathcal{F}(E,F)$.*

Proof. See lemma (1.1) on this article. □

2. Multilinear Mappings of Nuclear Type

The next results is slightly different from the one given in lemma (1.1) and its proof can be performed following the lines of this reference.

Lemma 2.1. *[4] Let $\psi: E_1 \times \cdots \times E_n \longrightarrow L_p(\Omega,\mu)$ defined by*

$$\psi(x_1,\cdots,x_n) = \sum_{k=1}^{m} \sigma_k x^*_{1,k}(x_1)\cdots x^*_{n,k}(x_n)\, y_k$$

where $\frac{1}{p} = \frac{1}{q'_1} + \cdots + \frac{1}{q'_n}$.
Then $N_{f,(\infty,p,q_1,\cdots,q_n)}(\psi) = N_{(\infty,p,q_1,q_n)}(\psi) = \|\psi\|$.

Proof. (See [4]) □

Lemma 2.2. *Let $\psi: E_1 \times \cdots \times E_n \longrightarrow F$. If $\psi \in L_f(E_1,\cdots,E_n;F)$, then*

$$N_{f,(\infty,p,q_1,\cdots,q_n)}(\psi) = N_{(\infty,p,q_1,\cdots,q_n)}(\psi) = \|\psi\|$$

Lemma 2.3. *Let $\psi: E_1 \times \cdots \times E_n \longrightarrow F$. If $\psi \in L_f(E_1,\cdots,E_n;F)$, then*

$$N_{(\infty,p;q_1,\cdots,q_n)}(\psi) = N_{f,(\infty,p;q_1,\cdots,q_n)}(K_F\psi)$$

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