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GENERALIZED SOLUTION OF A MIXED PROBLEM FOR LINEAR HYPERBOLIC SYSTEM

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Abstract: In the first part of this article, we will prove an existence-uniqueness result for generalized solutions of a mixed problem for linear hyperbolic system in the Colombeau algebra. In the second part, we apply the result to a wave propagation problem in a discontinuous environment.

AMS Subject Classification: 35L03, 46F30
Key Words: generalized functions, hyperbolic system

1. Introduction

In this paper, we intereste to study the existence and uniqueness of the mixed problem for a linear hyperbolic system in the case of a bounded domain, where the initial data is not smooth.

To complete the work already made in the global case by M. Oberguggenberger [4], we Considere the mixed problem in $\Omega = [0,+\infty[\times[0,+\infty[$ for the linear hyperbolic system in two variables

$$\begin{cases} \Big(\partial_t + \Lambda\left(x,t\right)\partial_x\Big)U = F(x,t)U + A(x,t) & (x,t) \in (\mathbb{R}_+^*)^2, \\ U\left(x,0\right) = U_0\left(x\right) & x \in \mathbb{R}_+, \\ U_i\left(0,t\right) = \sum\limits_{k=r+1}^{n} v_{ik}\left(t\right)U_k\left(0,t\right) + H_i\left(t\right) & i = 1,\dots,r \qquad t \geq 0 \\ + \text{ Compatibility conditions,} \end{cases} \tag{1}$$

where Λ, F and V are $(n\times n)$ matrices whose terms are discontinuous functions.

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The matrix Λ is real and diagonal such that

$$\Lambda_1 > \Lambda_2 > \cdots > \Lambda_r > 0 > \Lambda_{r+1} > \cdots > \Lambda_n$$

In the case where $\Lambda \in L^\infty\left(\mathbb{R}_+^2\right)$ and $F \in W_{\text{loc}}^{-1,\infty}\left(\mathbb{R}_+^2\right)$, multiplicative products of distributions appear in system (1), and so there is no general way of giving a meaning to system (1) in the sense of distribution. This hyperbolic system even when it is in the form of a system of conservation laws does not admit any solutions distributions in general see [3]. Our approach is to study (1) in Colombeau's algebra [1, 2], and under some hypotheses on Λ, F, ν and H, the system (1) admits an unique solution in $\mathcal{G}\left(\mathbb{R}_+^2\right)$.

The second part of this article, we will apply this result to the wave propagation problem in a discontinuous environment. the following system

$$\begin{cases} \Big(\partial_t + c\,(x)\,\partial_x\Big) u\,(x,t) = 0 & (x,t) \in (\mathbb{R}_+^*)^2 \\ \Big(\partial_t - c\,(x)\,\partial_x\Big) v\,(x,t) = 0 & (x,t) \in (\mathbb{R}_+^*)^2 \\ u\,(x,0) = u_0\,(x) & x \geq 0 \\ v\,(x,0) = v_0\,(x) & x \geq 0 \\ u\,(0,t) = h\,(t)\,v\,(0,t) + b\,(t) & t \geq 0 \\ + \text{ Compatibility conditions} \end{cases} \tag{2}$$

with

$$c(x) = \begin{cases} c_R & \text{if } x > x_0 \\ c_L & \text{if } 0 < x < x_0 \end{cases}$$

c_R and c_L are real constants, u_0 and v_0 are continuous almost everywhere.

For this problem one can find a classical solution on $\{0 \leq x < x_0 : t \geq 0\}$ and $\{x > x_0 : t \geq 0\}$, so imposing a transmission condition in $x = x_0$: the continuity of u and v, one will have a classical solution on $\{x \geq 0\ ,\ t \geq 0\}$.

Further if (u_0, v_0) are generalized functions, one can show that the problem (2) has a unique solution $(U,V) \in \mathcal{G}\left(\mathbb{R}_+^2\right) \times \mathcal{G}\left(\mathbb{R}_+^2\right)$, without having us need of the passage conditions, in the same way one shows that this solution admits an associated distribution that is equal to the classical solution by adjusting.

2. Existence and Uniqueness

We recall some definitions from the theory of generalized functions which we need in the sequel.

We define the algebra $\mathcal{G}(\mathbb{R}^m)$ as follows

$$A_q(\mathbb{R}) = \left\{\chi \in \mathcal{D}(\mathbb{R}) : \int_{\mathbb{R}} \chi(x)dx = 1 \text{ and } \int_{\mathbb{R}} x^k \chi(x)dx = 0 \quad \text{for} \quad 1 \leq k \leq q\right\}$$

and

$$A_q(\mathbb{R}^m) = \left\{\varphi(x_1, \ldots, x_m) = \prod_{j=1}^{m} \{\chi(x_j)\}\right\}$$

Let $\mathcal{E}[\mathbb{R}^m]$ be the set of functions on $\mathcal{A}_1(\mathbb{R}^m) \times \mathcal{C}^\infty(\mathbb{R}^m)$ with values in $\mathbb{C}$ witch are $\mathcal{C}^\infty$ to seconde variable. Obviously $\mathcal{E}[\mathbb{R}^m]$ with point wise multiplication is an algebra but $\mathcal{C}^\infty(\mathbb{R}^m)$ is not a subalgebra.

Then given $\varphi \in \mathcal{A}_1(\mathbb{R}^m)$ and $\varepsilon \in]0\,,\,1[$, we define a function φ_ε by

$$\varphi_\varepsilon(x) = \varepsilon^{-m}\varphi\left(\frac{x}{\varepsilon}\right) \quad \text{for} \quad x \in \mathbb{R}^m$$

An element of $\mathcal{E}[\mathbb{R}^m]$ is called "moderate" if for every compact subset K of $\mathbb{R}^m$ and every differential operator $D = \partial_{x_1}^{k_1}, \ldots, \partial_{x_m}^{k_m}$ there is $N \in \mathbb{N}$ such that the following holds

$$\begin{cases} \forall \varphi \in \mathcal{A}_N(\mathbb{R}^m),\ \exists C, \exists \eta > 0 & \text{such that} \\ \sup\limits_{x \in K} |D\, u(\varphi_\varepsilon, x)| \leq C\varepsilon^{-N} & \text{if } 0 < \varepsilon < \eta \end{cases} \tag{3}$$

$\mathcal{E}_M[\mathbb{R}^m]$ denotes the subset of moderate elements where the index M stands for "*Moderate*". We define an ideal $\mathcal{N}[\mathbb{R}^m]$ of $\mathcal{E}_M[\mathbb{R}^m]$ as follows:

$$u \in \mathcal{N}[\mathbb{R}^m]$$

if for every compact subset K of $\mathbb{R}^m$ and every differential operator D, there is $N \in \mathbb{N}$ such that

$$\begin{cases} \forall q \geq N,\ \forall \varphi \in \mathcal{A}_q(\mathbb{R}^m),\ \exists C, \exists \eta > 0 & \text{such that} \\ \sup\limits_{x \in K} |D\, u(\varphi_\varepsilon, x)| \leq C\varepsilon^{q-N} & \text{if } 0 < \varepsilon < \eta \end{cases} \tag{4}$$

Finally the algebra $\mathcal{G}(\mathbb{R}^m)$ is defined as the quotient of $\mathcal{E}_M[\mathbb{R}^m]$ with respect to $\mathcal{N}[\mathbb{R}^m]$.

In what follows, the elements of $\mathcal{G}\left(\mathbb{R}^2\right)$ will be written with capital letters and their representatives in $\mathcal{E}_M\left[\mathbb{R}^2\right]$ with small letters. Furthermore we use the following simplified notations :

$$u(\varphi_\varepsilon, x) = u^\varepsilon(x).$$

In our work we need a subset of $\mathcal{E}_M\left[\mathbb{R}_+^2\right]$ that contains elements u satisfying the following properties:

(a) $\exists\ N \in \mathbb{N}$ such that for all $\varphi \in \mathcal{A}_N\left(\mathbb{R}_+^2\right)$

$$\exists c > 0 \quad \eta > 0\ :\ \sup_{y\in\mathbb{R}_+^2} |u(\varphi_\varepsilon, y)| \le c \quad \text{if } 0 < \varepsilon < \eta$$

(b) For every compact subset K of $\mathbb{R}_+^2$, $\exists\ N \in \mathbb{N}$ such that $\forall\varphi \in \mathcal{A}_N\left(\mathbb{R}_+^2\right)$

$$\exists c > 0 \quad \exists\eta > 0\ :\ \sup_{y\in K} |u(\varphi_\varepsilon, y)| \le N \log\left(\frac{c}{\varepsilon}\right) \quad \text{if } 0 < \varepsilon < \eta$$

Definition 1. A generalized function $U \in \mathcal{G}\left(\mathbb{R}_+^2\right)$ admitting a representative u with the property (a) (respectively (b)) is called globally bounded (respectively locally logarithmic growth).

Definition 2. the system (1) satisfies the compatibility conditions in $\mathcal{G}\left(\mathbb{R}_+^2\right)$ if there exist u_0^ε, λ^ε, f^ε, h^ε, v^ε et a^ε the representatives of U_0, Λ, F, H, V and A that satisfy to the classic conditions compatibility in order to have a $\mathcal{C}^\infty$ solution for the classic problem.

Theorem 3. *Let F, Λ and A be $n\times n$ matrices with coefficients in $\mathcal{G}\left(\mathbb{R}_+^2\right)$, suppose that: there exists r as :*

$$\Lambda_1 > \Lambda_2 > \cdots > \Lambda_r > 0 > \Lambda_{r+1} > \cdots > \Lambda_n$$

Λ_i $(i = 1, \dots, n)$ are globally bounded, $\partial_x\Lambda_i$ and F_i are locally logarithmic growth, so for an initial data U_0 in $\mathcal{G}(\mathbb{R}_+)$, V_i an element in $\mathcal{G}(\mathbb{R}_+)$ globally bounded and H_i in $\mathcal{G}(\mathbb{R}_+)$, then the problem 1 has an unique solution in $\mathcal{G}\left(\mathbb{R}_+^2\right)$.

Proof. The proof of the theorem is an adaptation to the demonstration of the Theorem 1.2 in [4], therefore one is going to give the big lines rightly.

Let λ a representative of Λ in $\mathcal{G}\left(\mathbb{R}_+^+\right)$ such that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0 > \lambda_{r+1} > \cdots > \lambda_n$$

with λ_i satisfies the property (a) and $\partial_x\lambda_i$ satisfies the property (b).

Let f and a are any representatives of F and A in $\mathcal{G}\left(\mathbb{R}_+^2\right)$ with f satisfies (b).

v, h and u_0 are any representatives of V, H and U_0 in $\mathcal{G}(\mathbb{R}_+)$ with v satisfies (a).

So Let's consider the following problem

$$\begin{cases} \Big(\partial_t + \lambda_i^\varepsilon(x,t)\partial_x\Big)u_i^\varepsilon = \sum\limits_{k=1}^{n} f_{ik}^\varepsilon(x,t)u_k^\varepsilon(x,t) + a_i^\varepsilon(x,t) & (x,t) \in (\mathbb{R}_+^*)^2 \\ u_i^\varepsilon(x,0) = u_{0_i}^\varepsilon(x) & i = 1,\dots,n \quad x \in \mathbb{R}_+ \\ u_i^\varepsilon(0,t) = \sum\limits_{k=r+1}^{n} \nu_{ik}^\varepsilon(t)u_k^\varepsilon(0,t) + h_i^\varepsilon(t) & i = 1,\dots,r \quad t \ge 0 \end{cases} \qquad (\mathbf{I}_\varepsilon)$$

if we denote γ_i^ε the corresponding characteristic curve to λ_i^ε then the problem $\left(\mathbf{I}_\varepsilon\right)$ admits an unique solution u^ε, $u_i^\varepsilon \in \mathcal{C}^\infty\left(\mathbb{R}_+^2\right)$ given by for $i = r+1, ..., n$

$$u_i^\varepsilon(x,t) = u_{0_i}^\varepsilon(\gamma_i^\varepsilon(x,t,0)) + \int_0^t \Big[\sum_{k=1}^n f_{ik}^\varepsilon\Big(\gamma_i^\varepsilon(x,t,\tau),\tau\Big)u_k^\varepsilon\Big(\gamma_i^\varepsilon(x,t,\tau),\tau\Big) + a_i^\varepsilon\Big(\gamma_i^\varepsilon(x,t,\tau),\tau\Big)\Big]d\tau$$

for $i = 1, \ldots, r$

$$\begin{aligned} u_i^\varepsilon(x,t) = &\sum_{k=r+1}^n v_{ik}^\varepsilon(t_0)\int_0^{t_0}\sum_{s=1}^n\Big[f_{ks}^\varepsilon\Big(\gamma_k^\varepsilon(0,t_0,\tau),\tau\Big)u_s^\varepsilon\Big(\gamma_k^\varepsilon(0,t_0,\tau),\tau\Big)\Big]d\tau \\ &+ \int_{t_0}^t\sum_{k=1}^n\Big[f_{ik}^\varepsilon\Big(\gamma_i^\varepsilon(x,t,\tau),\tau\Big)u_k^\varepsilon\Big(\gamma_i^\varepsilon(x,t,\tau),\tau\Big)\Big]d\tau + \int_{t_0}^t a_i^\varepsilon\Big(\gamma_i^\varepsilon(x,t,\tau),\tau\Big)d\tau \\ &+ \sum_{k=r+1}^n v_{ik}^\varepsilon(t_0)\int_0^{t_0} a_k^\varepsilon\Big(\gamma_k^\varepsilon(0,t_0,\tau),\tau\Big)d\tau + \sum_{k=r+1}^n v_{ik}^\varepsilon(t_0)\,u_{0_k}^\varepsilon\Big(\gamma_k^\varepsilon(0,t_0,0)\Big) + h_i^\varepsilon(t_0) \end{aligned}$$

where t_0 is such that the curve γ_i cuts the axis $(0t)$ at a point $P_i(0,t_0)$. u_i^ε is $\mathcal{C}^\infty$ function, so it remains to show therefore that u_i^ε is moderate growth.

From assumptions, we have

$$\exists M > 0 \quad \text{such that :} \quad \left|\frac{d\gamma_i^\varepsilon(x,t,\tau)}{d\tau}\right| < M \quad \forall (x,t) \in \mathbb{R}_+^2 \quad \forall i = 1, \ldots, n$$
$$\exists M_1 > 0 \quad \text{such that :} \quad \max_{i,j}\left|v_{i,j}^\varepsilon(y)\right| < M_1 \quad \forall y \in \mathbb{R}_+$$

Let K_0 be a compact in $\mathbb{R}_+$, we draw the straight line with a slope $-M$, the determination domain K_T of the solution u_i^ε does not depend on ε.

Lemma 4. *Let u^ε a solution of problem (I_ε) then u_i^ε verified*

$$\sup_{(x,t)\in K_T}|u_i^\varepsilon(x,t)| \le M_2\left[\sup_k\sup_{(x,t)\in K_T}|a_k^\varepsilon(x,t)|.T + \sup_k\sup_{x\in K_0}\left|u_{0_k}^\varepsilon(x)\right| + \sup_k\sup_{t\in[0,T]}|h_k^\varepsilon(t)|\right] \times \exp\left(nM_2\sup_{i,k}\sup_{(x,t)\in K_T}|f_{ik}^\varepsilon(x,t)|.T\right)$$

with $M_2 = \max(nM_1, 1)$

Proof. for $i = 1, \ldots, r$, and from the integral equation that verified by u_i^ε we have

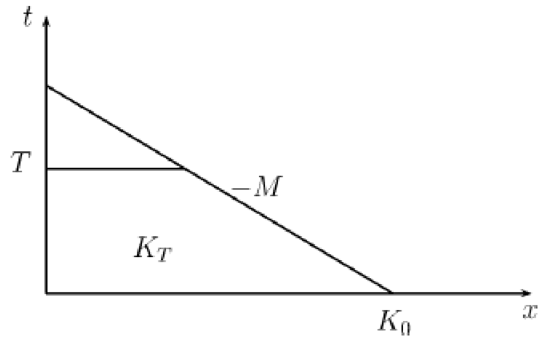

Figure 1

$$\sup_{(x,t)\in K_T} |u_i^\varepsilon(x,t)| \le M_2 \left[T \sup_{(x,t)\in K_T} |a_k^\varepsilon(x,t)| + \sup_k \sup_{x\in K_0} \left|u_{0_k}^\varepsilon(x)\right| \right.$$

$$\left. + \sup_k \sup_{t\in[0,T]} |h_k^\varepsilon(t)| \right] + nM_2 \int_0^T \sup_{(x,t)\in K_\tau} |f^\varepsilon(x,t)| \sup_k \sup_{(x,t)\in K_\tau} |u_k^\varepsilon(x,t)|\, d\tau$$

and the proof is completed by applying the Gronwall's lemma to the function

$$s \to \max_k \sup_{(x,t)\in K_s} |u_k^\varepsilon(x,t)|$$

For $i = r+1, \ldots, n$ it is the same way with $t_0 = 0$, $v = 0$, $h = 0$. $\square$

the next of the proof of Theorem 1, we have:
$\exists N_1 \in \mathbb{N}$ such that : $\forall \phi \in \mathcal{A}_{N_1}(\mathbb{R}_+)$

$$\exists C_1 > 0 \quad \exists \eta > 0 : \qquad \sup_{(x,t)\in K_T} |a^\varepsilon(x,t)| \le C_1 \varepsilon^{-N_1} \quad \text{if } 0 < \varepsilon < \eta$$

$\exists N_2 \in \mathbb{N}$ such that : $\forall \phi \in \mathcal{A}_{N_2}(\mathbb{R}_+)$

$$\exists C_2 > 0 \quad \exists \eta > 0 : \qquad \sup_{x\in K_0} |u_0^\varepsilon(x)| \le C_2 \varepsilon^{-N_2} \quad \text{if } 0 < \varepsilon < \eta$$

$\exists N_3 \in \mathbb{N}$ such that : $\forall \phi \in \mathcal{A}_{N_3}(\mathbb{R}_+)$

$$\exists C_3 > 0 \quad \exists \eta > 0 : \qquad \sup_{t\in[0,T]} |h^\varepsilon(t)| \le C_3 \varepsilon^{-N_3} \quad \text{if } 0 < \varepsilon < \eta$$

$$\exists N_4 \in \mathbb{N} \text{ such that : } \forall \phi \in \mathcal{A}_{N_4}(\mathbb{R}^2_+)$$

$$\exists C_4 > 0 \quad \exists \eta > 0 : \qquad \sup_{(x,t)\in K_T} |f^\varepsilon(x,t)| \leq N_4 \log\left(\frac{C_4}{\varepsilon}\right) \quad \text{if } 0 < \varepsilon < \eta$$

therefore according to the lemma, we have

$$\forall \phi \in \mathcal{A}_{N_5}, \quad \exists C > 0, \quad \eta > 0 : \quad \sup_{(x,t)\in K_T} |u_i^\varepsilon(x,t)| \leq C_5 \varepsilon^{-N_5} \quad \text{if } 0 < \varepsilon < \eta$$

with

$$N_5 = E\left(N_1 + N_2 + N_3 + NTC_4N_4\right) + 1$$

for the other derivatives, differentiating the system (I_ε) for example with regard to x, one gets a system similar to the first. And because $\partial_x \Lambda$ is locally logarithmic growth one gets the same estimation as before, ..., then one has

$$u_i^\varepsilon \in \mathcal{E}_M(\mathbb{R}^2_+) \qquad i = 1, \ldots, n$$

either the existence of the solution for the problem (1) is in $\mathcal{G}(\mathbb{R}^2_+)$.

Uniqueness. Let U, V two solutions in $\mathcal{G}(\mathbb{R}^2_+)$ of the problem (I_ε), with the same initial data and the same boundary values. One must show that so u^ε is a representative of U and $\mathcal{G}(\mathbb{R}^2_+)$ and if v^ε is a representative of V in $\mathcal{G}(\mathbb{R}^2_+)$ then $u^\varepsilon - v^\varepsilon \in \mathcal{N}(\mathbb{R}^2_+)$ see [2].

Indeed: $u^\varepsilon - v^\varepsilon$ verifies the same problem that previously and therefore the demonstration is the same. Then one has

$$u^\varepsilon - v^\varepsilon = O\left(\varepsilon^q\right) \quad \forall q$$

□

Remark 5. To get the solution in the case where $\Lambda \in \mathbf{L}^\infty(\mathbb{R}^2_+)$, $F \in \mathbf{W}^{-1,\infty}(\mathbb{R}^2_+)$, one uses the following result. see [4, Proposition 2]

Proposition 6. ***a)*** *Let* $\omega \in \mathbf{W}^{-1,\infty}_{loc}(\mathbb{R}^2_+)$ *then there exist* $U \in \mathcal{G}(\mathbb{R}^2)$ *such that:* U *is associated to* ω *and* U *is locally logarithmic growth.*

b) *Let* $\omega \in \mathbf{L}^\infty(\mathbb{R}^2)$ *then there exist* $U \in \mathcal{G}(\mathbb{R}^2)$ *such that:* U *is associated to* ω *and* U *is globally bounded, and* $\partial^\alpha U$ *is locally logarithmic growth.* $\alpha = (\alpha_1, \alpha_2)$ *such that* $|\alpha| = \alpha_1 + \alpha_2 = 1$

Remark 7. For $g \in \mathbf{L}^\infty(\mathbb{R}_+)$ one can find $G \in \mathcal{G}(\mathbb{R}_+)$ such that $G \approx g$, and there exist a representative g^ε of G such that g^ε is nil at the neighborhood of 0 for all ε.

3. Application

Consider the problem (2)

$$\begin{cases} \Big(\partial_t + c\,(x)\,\partial_x\Big)u\,(x,t) = 0 & (x,t) \in (\mathbb{R}_+^*)^2 \\ \Big(\partial_t - c\,(x)\,\partial_x\Big)v\,(x,t) = 0 & (x,t) \in (\mathbb{R}_+^*)^2 \\ u\,(x,0) = u_0\,(x) & x \geq 0 \\ v\,(x,0) = v_0\,(x) & x \geq 0 \\ u\,(0,t) = v\,(0,t) & t \geq 0 \\ + \text{ Compatibility conditions} & \end{cases}$$

with

$$c\,(x) = \begin{cases} c_R & \text{if } x > x_0 \\ c_L & \text{if } 0 < x < x_0 \end{cases}$$

For the initials data u_0, v_0 continuous almost everywhere, and nil at neighborhood of 0.

The problem (2) admits a classic solution for

$$\big\{0 < x < x_0 : t \geq 0\big\} \quad \text{and} \quad \big\{x > x_0 : t \geq 0\big\}$$

and while imposing a passage condition on the x_0 (continuity of u and v at the point x_0) then one will have a solution on

$$\big\{x \geq 0 : t \geq 0\big\}$$

defined by

$$\begin{aligned} v(x,t) &= v_0\,(\gamma_2(x,t,0)) \\ u(x,t) &= \begin{cases} u_0\,(\gamma_1(x,t,0)) & \text{on (I)} \\ v(0,t) & \text{on (II)} \end{cases} \end{aligned}$$

so one designates by Γ the characteristic curve comes from of $(0,0)$ the part (I) designates the set of $(x,t) \in \mathbb{R}_+^2$ below Γ. and the part (II) the set the points (x,t) over Γ (see the figure (2)).

γ_1 the connected curve characteristic corresponding to c.

γ_2 the connected curve characteristic corresponding to $-c$.

Proposition 8. *given u_0, v_0 two continuous functions nearly everywhere, bounded and nil at the neighborhood of* 0 *then the problem (2) admit an unique solution U, V in $\mathcal{G}(\mathbb{R}_+^2)$ besides one has:*

$$U \approx u \quad \textit{and} \quad V \approx v$$

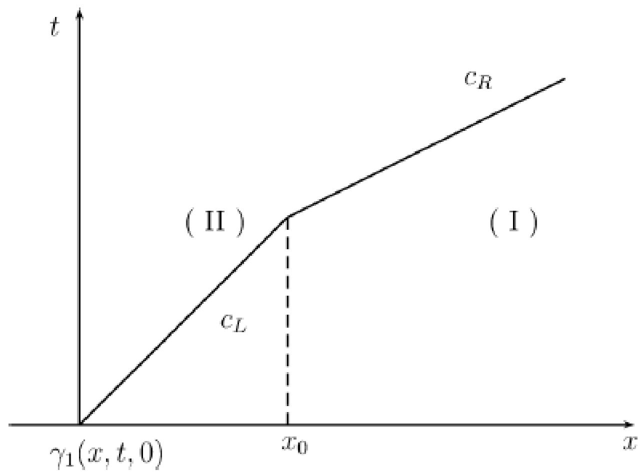

Figure 2

with u et v are the distributions solutions of the same problem obtained by imposing a passage condition.

Proof. $c \in \mathbf{L}^{\infty}(\mathbb{R}_+)$, from the proposition (1) there exists $C \in \mathcal{G}(\mathbb{R}_+)$ such that $C \approx c$.
c is globally bounded and $\partial_x C$ is locally logarithmic growth. And so, from the theorem 1, there exists an unique solution U, V in $\mathcal{G}(\mathbb{R}^2_+)$ of the problem (2).

To show that

$$U \approx u$$

we suppose that (x,t) belongs to the region limited by the broken characteristic curve Γ comes from the origin and the axis (ox) which we note (region I).

If (x,t) is over of this curve, the demonstration is identical but with reflection (region II) and for $(x,t) \in \Gamma$ (the characteristic curve comes from the origin) this set is negligible.

Let:

c^{ε} a representative of C in $\mathcal{G}(\mathbb{R}_+)$,
u_0^{ε} a representative of U_0 in $\mathcal{G}(\mathbb{R}_+)$,
v_0^{ε} a representative of V_0 in $\mathcal{G}(\mathbb{R}_+)$.

Considering then the following problem

$$\begin{cases} (\partial_t + c^\varepsilon \partial_x)\, u^\varepsilon = 0 & (x,t) \in (\mathbb{R}_+^*)^2 \\ (\partial_t - c^\varepsilon \partial_x)\, v^\varepsilon = 0 & (x,t) \in (\mathbb{R}_+^*)^2 \\ u^\varepsilon(x,0) = u_0^\varepsilon(x) & x \in \mathbb{R}_+ \\ v^\varepsilon(x,0) = v_0^\varepsilon(x) & x \in \mathbb{R}_+ \\ u^\varepsilon\,(0,t) = v^\varepsilon\,(0,t) & t \in \mathbb{R}_+ \end{cases}$$

This problem admits an unique solution u^ε, v^ε in $\mathcal{C}^\infty(\mathbb{R}_+^2)$.

Taking

$$\gamma_1^\varepsilon = \gamma_1 * \phi_{\eta_\varepsilon}$$

with $\phi \in \mathcal{D}(\mathbb{R}^+)$ such that

$$\int_{\mathbb{R}^+} \phi(\lambda) d\lambda = 1 \qquad \text{supp } \phi_{\eta_\varepsilon} \subset \,]x_0 - \eta_\varepsilon, x_0 + \eta_\varepsilon[\qquad \eta_\varepsilon = |\log \varepsilon|^{-1}$$

it is evident that for all (x,t) in (region I)

$$u^\varepsilon(x,t) = u_0^\varepsilon\big(\gamma_1^\varepsilon(x,t,0)\big)$$

then to show that $U \approx u$ it is necessary and sufficient to show that :
$\forall \psi \in \mathcal{D}(\mathbb{R}_+^2)$

$$\lim_{\varepsilon \to 0} \int_{\text{region I}} \Big(u_0^\varepsilon\big(\gamma_1^\varepsilon(x,t,0)\big) - u_0\big(\gamma_1(x,t,0)\big) \Big) \psi(x,t) dx dt = 0$$

we have

$$\int \Big(u_0^\varepsilon\big(\gamma_1^\varepsilon(x,t,0)\big) - u_0\big(\gamma_1(x,t,0)\big) \Big) \psi(x,t) dx\, dt =$$

$$\int \Big(u_0^\varepsilon\big(\gamma_1^\varepsilon(x,t,0)\big) - u_0\big(\gamma_1^\varepsilon(x,t,0)\big) \Big) \psi(x,t) dx\, dt$$

$$+ \int \Big(u_0\big(\gamma_1^\varepsilon(x,t,0)\big) - u_0\big(\gamma_1(x,t,0)\big) \Big) \psi(x,t) dx\, dt$$

but

$$\int \Big(u_0^\varepsilon\big(\gamma_1^\varepsilon(x,t,0)\big) - u_0\big(\gamma_1^\varepsilon(x,t,0)\big) \Big) \psi(x,t) dx\, dt$$

$$= \int \big(u_0^\varepsilon - u_0 \big)\big(\gamma_1^\varepsilon(x,t,0)\big) \psi(x,t) dx\, dt$$

$$\leq \sup_{x \in \mathbb{R}_+} |u_0 * \phi_\varepsilon - u_0| \left| \int_{\mathbb{R}_+^2} \psi(x,t) dx\, dt \right|$$

so $\lim\limits_{\varepsilon\to 0}\displaystyle\int \Big(u_0^\varepsilon\big(\gamma_1^\varepsilon(x,t,0)\big)-u_0\big(\gamma_1^\varepsilon(x,t,0)\big)\Big)\psi(x,t)dx\,dt=0$

to show that $\lim\limits_{\varepsilon\to 0}\displaystyle\int \Big(u_0\big(\gamma_1^\varepsilon(x,t,0)\big)-u_0\big(\gamma_1(x,t,0)\big)\Big)\psi(x,t)dx\,dt=0$

it is sufficient to show that $\lim\limits_{\varepsilon\to 0}\Big(\gamma_1^\varepsilon(x,t,0)-\gamma_1(x,t,0)\Big)=0$

or c is globally bounded, then $\exists M>0 \quad \sup\limits_{x\in\mathbb{R}_+}|c^\varepsilon(x)|<M.$

So we can to surround the curve γ_1^ε between two broken curves, (see Figure 3) and taking the intersection of these two curves with the axis $(0x)$, it gives us two points

$$\begin{aligned} x_1 &= c_L\Big(-\frac{2\eta_\varepsilon}{M}-\frac{x_0+\eta_\varepsilon-x}{c_R}-t\Big)-\eta_\varepsilon+x_0\\ x_2 &= -c_L\Big(-\frac{2\eta_\varepsilon}{M}+\frac{x_0+\eta_\varepsilon-x}{c_R}+t\Big)-\eta_\varepsilon+x_0\end{aligned}$$

such that

$$x_1\le\gamma_1^\varepsilon(x,t,0)\le x_2$$

hence

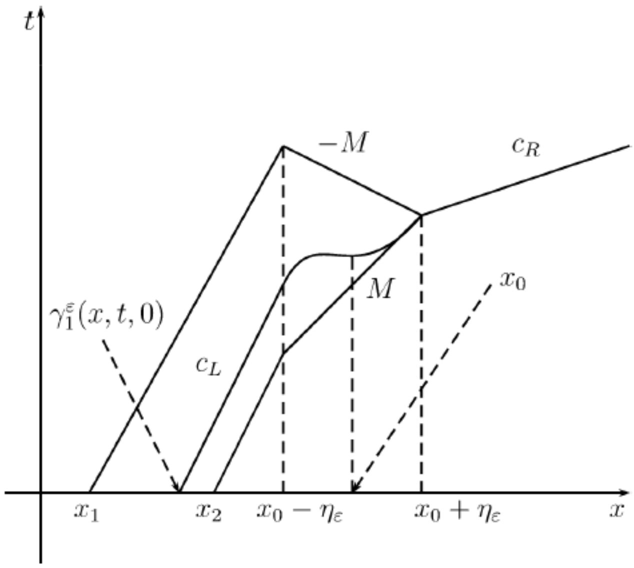

Figure 3

$$\lim_{\varepsilon\to 0}\gamma_1^\varepsilon(x,t,0) \quad = \quad -c_Lt+\frac{c_L}{c_R}\big(x-x_0\big)+x_0$$

$$= \gamma_1(x,t,0)$$

then

$$U \approx u$$

for v, the demonstration is the same. □

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