

**SOME COMMON FIXED POINT THEOREMS FOR THREE  
PAIRS OF MAPPINGS VIA WEAKLY COMMUTING AND  
WEAKLY COMPATIBILITY IN  $G$ -METRIC SPACES**

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**Abstract:** The purpose of this paper is to establish the existence and uniqueness of fixed point theorems for three pairs of self mappings in the frame work of complete  $G$ -metric spaces. Our results are obtained by using the contractive conditions weakly commuting for one pair and weakly compatibility for other two pairs. These results generalize and partially extend some recent results in existing literature.

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## 1. Introduction

The concept of  $D$ -metric spaces was introduced by Dhage[1],[2],[3],[4]. et al. as generalization of ordinary metric functions and went on to present several fixed point results for single and multivalued mappings. Mustafa and Sims[12] and Naidu et al. [8],[9],[10] demonstrated that most of the claims concerning the fundamental topological structure of  $D$  - metric space are incorrect, alternatively, Mustafa and Sims introduced in [12] more appropriate notion of generalized metric space which they called  $G$  - metric spaces, and obtained some topological properties. Later Zead Mustafa, Hamed Obiedat and Fadi Awawdeh[13], Mustafa, Shatanawi and Bataineh [14], Mustafa and Sims [15] Shatanawi [11] and Renu Chugh, Tamanna Kadian, Anju Rani and B.E. Rhoades[7] et al. obtained some fixed point theorems for a single map in  $G$ - metric spaces.

The purpose of this paper is to use the concept of weakly commuting mappings and weakly compatible mappings to discuss some new common fixed point problems for six self-mappings in  $G$ -metric spaces. First, we present some known definitions and propositions in  $G$  - metric spaces .

## 2. Preliminaries

Now we give basic definitions and some basic results which are helpful for proving our main result.

In 2006, Mustafa and Sims [12] introduced the concept of  $G$ -metric spaces as follows:

**Definition 2.1.** [12] Let  $X$  be a nonempty set, and let  $G : X \times X \times X \rightarrow R^+$  be a function satisfying the following properties:

1. ( $G-1$ )  $G(x, y, z) = 0$  if  $x = y = z$ ;
2. ( $G-2$ )  $0 < G(x, x, y)$ , for all  $x, y \in X$  with  $x \neq y$ .
3. ( $G-3$ )  $G(x, x, y) \leq G(x, y, z)$  for all  $x, y, z \in X$  with  $y \neq z$
4. ( $G-4$ )  $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ , symmetry in all three variables,
5. ( $G-5$ )  $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$  for all  $x, y, z, a \in X$

The function  $G$  is called a generalized metric or a  $G$ -metric on  $X$  and the pair  $(X, G)$  is called a  $G$ -metric space.

**Definition 2.2.** [12] Let  $(X, G)$  be a  $G$ -metric space, and let  $\{x_n\}$  be a sequence of points of  $X$ , a point  $x \in X$  is said to be the limit of the sequence  $\{x_n\}$ , if  $\lim_{n,m \rightarrow \infty} G(x, x_n, x_m) = 0$  and we say that the sequence  $\{x_n\}$  is  $G$ -convergent to  $x$  or  $\{x_n\}$   $G$ -convergent to  $x$ .

Thus,  $x_n \rightarrow x$  in a  $G$ -metric space  $(X, G)$  if for any  $\epsilon > 0$ , there exists  $k \in N$  such that  $G(x, x_n, x_m) < \epsilon$  for all  $m, n \geq k$ .

**Proposition 2.1.** [12] Let  $(X, G)$  be a  $G$ -metric space. Then the following are equivalent:

1.  $\{x_n\}$  is  $G$ -convergent to  $x$ ;
2.  $G(x_n, x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$ ;
3.  $G(x_n, x, x) \rightarrow 0$  as  $n \rightarrow \infty$ ;
4.  $G(x_n, x_m, x) \rightarrow 0$  as  $n, m \rightarrow \infty$ ;

**Definition 2.3.** [12] Let  $(X, G)$  be a  $G$ -metric space, a sequence  $\{x_n\}$  is called  $G$ -Cauchy if for every  $\epsilon > 0$ , there is  $k \in N$  such that  $G(x_n, x_m, x_l) < \epsilon$  for all  $n, m, l \geq k$ ; that is  $G(x_n, x_m, x_l) \rightarrow 0$  as  $n, m, l \rightarrow \infty$ .

**Proposition 2.2.** [12] Let  $(X, G)$  be a  $G$ -metric space. Then the following are equivalent:

1. The sequence  $\{x_n\}$  is  $G$ -cauchy;
2. For every  $\epsilon > 0$ , there is  $k \in N$ ,  $G(x_n, x_m, x_m) < \epsilon$  for all  $n, m \geq k$ ;

**Definition 2.4.** [12] Let  $(X, G)$  and  $(X', G')$  be  $G$ -metric space and let  $f : (X, G) \rightarrow (X', G')$  be a function. Then  $f$  is said to be  $G$ -continuous at a point  $a \in X$  if and only if, for every  $\epsilon > 0$ , there is  $\delta > 0$  such that  $x, y \in X$  and  $G(a, x, y) < \delta$  imply  $G'(f(a), f(x), f(y)) < \epsilon$ . A function  $f$  is  $G$ -continuous at  $X$  if and only if it is  $G$ -continuous at  $a \in X$ .

**Proposition 2.3.** [12] Let  $(X, G)$  be a  $G$ -metric space. Then the function  $G(x, y, z)$  is jointly continuous in all three of its variables.

**Definition 2.5.** [12] A  $G$  - metric space  $(X, G)$  is said to be  $G$ -complete if every  $G$ -Cauchy sequence in  $(X, G)$  is  $G$ -convergent in  $X$ .

**Definition 2.6.** [5] Two self mappings  $f$  and  $g$  of a  $G$ -metric space  $(X, G)$  are said to be weakly commuting if  $G(fgx, gfx, gfx) \leq G(fx, gx, gx)$  for all  $x \in X$ .

**Definition 2.7.** [5] Let  $f$  and  $g$  be two mappings from a  $G$ -metric space  $(X, G)$  into itself. Then the mappings  $f$  and  $g$  are said to be weakly compatible if  $G(fgx, gfx, gfx) = 0$  whenever  $G(fx, gx, gx) = 0$ .

**Proposition 2.4.** [12] Let  $(X, G)$  be a  $G$ -metric space. Then for all  $x, y, z, a \in X$ , it follows that:

1. If  $G(x, x, y) = 0$  then  $x = y = z$ ;
2.  $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$ ;
3.  $G(x, y, y) \leq 2G(y, x, x)$ ;
4.  $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$ ;
5.  $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$ ;
6.  $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$ ;

### 3. Main Results

**Theorem 3.1.** Let  $(X, G)$  be a complete  $G$ -metric space and let  $A, B, C, R, S$  and  $T$  be six mappings on  $X$  into itself satisfying the following conditions

$$(i) A(X) \subseteq T(X), B(X) \subseteq S(X), C(X) \subseteq R(X).$$

$$(ii) \forall x, y, z \in X.$$

$$\begin{aligned} G(Ax, By, Cz) \leq & \alpha\{G(Rx, Ty, Sz) + G(Rx, By, Cz) + G(Rx, Ax, Ax)\} \\ & + \beta\{\max\{G(Ty, By, Cz), G(By, Ty, Sz), G(Ty, Ax, Ax), \\ & G(Cz, Rx, Sz)\}\} \\ & + \gamma\{G(Rx, By, Ty) + G(Ty, Cz, Sz) + G(Sz, Ax, Ax)\} \end{aligned} \quad (3.1)$$

where  $\alpha, \beta, \gamma > 0$  and  $6\alpha + 4\beta + 5\gamma < \frac{1}{2}$

(iii) If one of the following condition is satisfied:

(a) Either  $A$  or  $R$  is  $G$ -continuous, the pair  $(A, R)$  is weakly commuting, the pair  $(B, T)$  and  $(C, S)$  are weakly compatible;

(b) Either  $B$  or  $T$  is  $G$ -continuous, the pair  $(B, T)$  is weakly commuting, the pair  $(A, R)$  and  $(C, S)$  are weakly compatible;

(c) Either  $C$  or  $S$  is  $G$ -continuous, the pair  $(C, S)$  is weakly commuting, the pair  $(A, R)$  and  $(B, T)$  are weakly compatible;

Then:

- (1) One of the pair  $(A, R)$ ,  $(B, T)$  or  $(C, S)$  has a coincidence point in  $X$ ;
- (2) The mappings  $A, B, C, S, T$  and  $R$  have a unique common fixed point in  $X$ .

*Proof.* Suppose the mappings  $A, B, C, S, T$  and  $R$  satisfy the condition (3.1). Let  $x_0 \in X$  be an arbitrary point. Since  $A(X) \subseteq T(X)$ ,  $B(X) \subseteq S(X)$ ,  $C(X) \subseteq R(X)$  then there exist  $x_1, x_2, x_3 \in X$  such that

$$Ax_0 = Tx_1 = y_0, \quad Bx_1 = Sx_2 = y_1 \text{ and } Cx_2 = Rx_3 = y_2$$

Inductively we can construct a sequence  $\{y_n\}$  in  $X$  such that  $Ax_{3n} = Tx_{3n+1} = y_{3n}$ ,  $Bx_{3n+1} = Sx_{3n+2} = y_{3n+1}$  and  $Cx_{3n+2} = Rx_{3n+3} = y_{3n+2}$  for  $n = 0, 1, 2, 3, \dots$

If there exists  $n_0 \in N$  s.t.  $y_{n_0} = y_{n_0+1}$  then the conclusion (1) of Theorem(3.1) holds. In fact if there exists  $p \in N$  s.t.  $y_{3p+2} = y_{3p+3}$  then  $Au = Ru$  where  $u = x_{3p+3}$ . Thus the pair  $(A, R)$  has a coincidence point  $u \in X$ . if  $y_{3p} = y_{3p+1}$  then  $Bu = Tu$  where  $u = x_{3p+1}$ . Therefore the pair  $(B, T)$  has a coincidence point  $u \in X$ . if  $y_{3p+1} = y_{3p+2}$  then  $Cu = Su$  where  $u = x_{3p+2}$  and so the pair  $(C, S)$  has a coincidence point  $u \in X$ .

On the other hand, if there exists  $n_0 \in N$  s.t.  $y_{n_0} = y_{n_0+1} = y_{n_0+2}$  then  $y_n = y_{n_0}$  for any  $n \geq n_0$ . This implies  $\{y_n\}$  is a  $G$ -cauchy sequence. Actually, if there exists  $p \in N$  s.t.  $y_{3p} = y_{3p+1} = y_{3p+2}$  then applying the contractive condition (3.1) with  $x = y_{3p+3}$ ,  $y = y_{3p+1}$ ,  $z = y_{3p+2}$  we get,

$$\begin{aligned} &G(y_{3p+1}, y_{3p+2}, y_{3p+3}) = G(Ax_{3p+3}, Bx_{3p+1}, Cx_{3p+2}) \\ &\leq \alpha\{G(Rx_{3p+3}, Tx_{3p+1}, Sx_{3p+2}) + G(Rx_{3p+3}, Bx_{3p+1}, Cx_{3p+2}) \\ &\quad + G(Rx_{3p+3}, Ax_{3p+3}, Ax_{3p+3})\} + \beta\{\max\{G(Tx_{3p+1}, Bx_{3p+1}, Cx_{3p+2}), \\ &G(Bx_{3p+1}, Tx_{3p+1}, Sx_{3p+2}), G(Tx_{3p+1}, Ax_{3p+3}, Ax_{3p+3}), \\ &G(Cx_{3p+2}, Rx_{3p+3}, Sx_{3p+2})\}\} + \gamma\{G(Rx_{3p+3}, Bx_{3p+1}, Tx_{3p+1})+ \\ &G(Tx_{3p+1}, Cx_{3p+2}, Sx_{3p+2}) + G(Sx_{3p+2}, Ax_{3p+3}, Ax_{3p+3})\} \\ &\leq \alpha\{G(y_{3p+2}, y_{3p}, y_{3p+1}) + G(y_{3p+2}, y_{3p+1}, y_{3p+2}) + G(y_{3p+2}, y_{3p+3}, y_{3p+3})\}+ \\ &\beta\{\max\{G(y_{3p}, y_{3p+1}, y_{3p+2}), G(y_{3p+1}, y_{3p}, y_{3p+1}), G(y_{3p}, y_{3p+3}, y_{3p+3}) \\ &G(y_{3p+2}, y_{3p+2}, y_{3p+1})\}\} + \gamma\{G(y_{3p+2}, y_{3p+1}, y_{3p}) + G(y_{3p}, y_{3p+2}, y_{3p+1})+ \\ &G(y_{3p+1}, y_{3p+3}, y_{3p+3})\} \end{aligned}$$

$$\leq \alpha\{0 + 0 + G(y_{3p+2}, y_{3p+3}, y_{3p+3})\} + \beta\{\max\{0, 0, G(y_{3p}, y_{3p+3}, y_{3p+3}), 0\}\} \\ + \gamma\{0, 0, G(y_{3p+1}, y_{3p+3}, y_{3p+3})\}.$$

This implies that

$$G(y_{3p+1}, y_{3p+2}, y_{3p+3}) \leq (\alpha + \beta + \gamma)G(y_{3p+2}, y_{3p+3}, y_{3p+3})$$

If  $y_{3p+3} \neq y_{3p+1}$  then from  $G(3)$ ,  $G(4)$  and Proposition(2.4)(iii), we get

$$0 < G(y_{3p+1}, y_{3p+2}, y_{3p+3}) \leq 2(\alpha + \beta + \gamma)G(y_{3p+2}, y_{3p+2}, y_{3p+3}) \\ \leq 2(\alpha + \beta + \gamma)G(y_{3p+1}, y_{3p+2}, y_{3p+3})$$

This implies that  $\alpha + \beta + \gamma \geq \frac{1}{2}$  which is contradiction as  $6\alpha + 4\beta + 5\gamma < \frac{1}{2}$ .

So we find  $y_n = y_{3p}$  for any  $n \geq 3p$ .

This implies that  $\{y_n\}$  is a  $G$ -cauchy sequence.

Similarly we can show that  $\{y_n\}$  is a  $G$ -cauchy sequence for  $y_{3p+1} = y_{3p+2} = y_{3p+3}$  or  $y_{3p+2} = y_{3p+3} = y_{3p+4}$  for some  $p \in N$ .

For the rest of the paper assume that  $y_n \neq y_m$  for any  $n \neq m$ .

Put  $x = y_{3n}$ ,  $y = y_{3n+1}$ ,  $z = y_{3n+2}$  in inequality(3.1) and using the condition  $G(3), G(4), G(5)$  and Proposition(2.4)(iv) we get

$$G(y_{3n}, y_{3n+1}, y_{3n+2}) = G(Ax_{3n}, Bx_{3n+1}, Cx_{3n+2}) \\ \leq \alpha\{G(Rx_{3n}, Tx_{3n+1}, Sx_{3n+2}) + G(Rx_{3n}, Bx_{3n+1}, Cx_{3n+2}) \\ + G(Rx_{3n}, Ax_{3n}, Ax_{3n})\} + \beta\{\max\{G(Tx_{3n+1}, Bx_{3n+1}, Cx_{3n+2}), \\ G(Bx_{3n+1}, Tx_{3n+1}, Sx_{3n+2}), G(Tx_{3n+1}, Ax_{3n}, Ax_{3n}), \\ G(Cx_{3n+2}, Rx_{3n}, Sx_{3n+2})\}\} + \gamma\{G(Rx_{3n}, Bx_{3n+1}, Tx_{3n+1}) + \\ G(Tx_{3n+1}, Cx_{3n+2}, Sx_{3n+2}) + G(Sx_{3n+2}, Ax_{3n}, Ax_{3n})\} \\ \leq \alpha\{G(y_{3n-1}, y_{3n}, y_{3n+1}) + G(y_{3n-1}, y_{3n+1}, y_{3n+2}) + G(y_{3n-1}, y_{3n}, y_{3n})\} + \\ \beta\{\max\{G(y_{3n}, y_{3n+1}, y_{3n+2}), G(y_{3n+1}, y_{3n}, y_{3n+1}), G(y_{3n}, y_{3n}, y_{3n}) \\ G(y_{3n+2}, y_{3n-1}, y_{3n+1})\}\} + \gamma\{G(y_{3n-1}, y_{3n+1}, y_{3n}) + G(y_{3n}, y_{3n+2}, y_{3n+1}) + \\ G(y_{3n+1}, y_{3n}, y_{3n})\} \\ \leq \alpha\{G(y_{3n-1}, y_{3n}, y_{3n+1}) + G(y_{3n-1}, y_{3n}, y_{3n+1}) + 2G(y_{3n}, y_{3n+1}, y_{3n+2}) + \\ G(y_{3n-1}, y_{3n}, y_{3n+1})\} + \beta\{\max\{G(y_{3n}, y_{3n+1}, y_{3n+2}), G(y_{3n-1}, y_{3n}, y_{3n+1}), \\ 0, G(y_{3n-1}, y_{3n}, y_{3n+1}) + 2G(y_{3n}, y_{3n+1}, y_{3n+2})\}\} + \gamma\{G(y_{3n-1}, y_{3n}, y_{3n+1}) + \\ G(y_{3n}, y_{3n+1}, y_{3n+2}) + G(y_{3n-1}, y_{3n}, y_{3n+1})\} \\ \leq \alpha\{3G(y_{3n-1}, y_{3n}, y_{3n+1}) + 2G(y_{3n}, y_{3n+1}, y_{3n+2})\} + \\ \beta\{G(y_{3n-1}, y_{3n}, y_{3n+1}) + 2G(y_{3n}, y_{3n+1}, y_{3n+2})\} + \\ \gamma\{2G(y_{3n-1}, y_{3n}, y_{3n+1}) + G(y_{3n}, y_{3n+1}, y_{3n+2})\}$$

Now we show that  $G(y_{3n}, y_{3n+1}, y_{3n+2}) \leq G(y_{3n-1}, y_{3n}, y_{3n+1})$  for every  $n \in N$ . If  $G(y_{3n}, y_{3n+1}, y_{3n+2}) > G(y_{3n-1}, y_{3n}, y_{3n+1})$  for some  $n \in N$  then by above we have

$$\begin{aligned} G(y_{3n}, y_{3n+1}, y_{3n+2}) &< 5\alpha\{G(y_{3n}, y_{3n+1}, y_{3n+2})\} + 3\beta\{G(y_{3n}, y_{3n+1}, y_{3n+2})\} + \\ &3\gamma\{G(y_{3n}, y_{3n+1}, y_{3n+2})\} \\ &< (5\alpha + 3\beta + 3\gamma)\{G(y_{3n}, y_{3n+1}, y_{3n+2})\} \end{aligned}$$

This implies that  $5\alpha + 3\beta + 3\gamma > 1 > \frac{1}{2}$  which is contradiction as  $6\alpha + 4\beta + 5\gamma < \frac{1}{2}$ . Hence we get  $G(y_{3n}, y_{3n+1}, y_{3n+2}) \leq G(y_{3n-1}, y_{3n}, y_{3n+1})$  for every  $n \in N$ .

This implies that

$$\begin{aligned} G(y_{3n}, y_{3n+1}, y_{3n+2}) &\leq 5\alpha\{G(y_{3n-1}, y_{3n}, y_{3n+1})\} + 3\beta\{G(y_{3n-1}, y_{3n}, y_{3n+1})\} + \\ &3\gamma\{G(y_{3n-1}, y_{3n}, y_{3n+1})\} \\ &\leq (5\alpha + 3\beta + 3\gamma)\{G(y_{3n-1}, y_{3n}, y_{3n+1})\} \\ &< (6\alpha + 4\beta + 5\gamma)\{G(y_{3n-1}, y_{3n}, y_{3n+1})\} \\ &\Rightarrow G(y_{3n}, y_{3n+1}, y_{3n+2}) \leq q \cdot \{G(y_{3n-1}, y_{3n}, y_{3n+1})\} \end{aligned} \quad (3.2)$$

where  $q = 6\alpha + 4\beta + 5\gamma$

Similarly we can show that

$$G(y_{3n+1}, y_{3n+2}, y_{3n+3}) < q \cdot \{G(y_{3n}, y_{3n+1}, y_{3n+2})\} \quad (3.3)$$

Thus it follows from inequality (3.2) and (3.3) that for all  $n \in N$

$$\begin{aligned} G(y_n, y_{n+1}, y_{n+2}) &< q\{G(y_{n-1}, y_n, y_{n+1})\} < q^2 \cdot \{G(y_{n-2}, y_{n-1}, y_n)\} \\ &< \dots < q^n \{G(y_0, y_1, y_2)\} \end{aligned}$$

Therefore for all  $n, m \in N$ ,  $n < m$ , by  $G(3)$  and  $G(5)$  we have

$$\begin{aligned} G(y_n, y_m, y_m) &\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + \\ &\dots + G(y_{m-1}, y_m, y_m) \\ &< q^n G(y_0, y_1, y_2) + q^{n+1} G(y_0, y_1, y_2) + \dots + q^{m-1} G(y_0, y_1, y_2) \\ &< (q^n + q^{n+1} + q^{n+2} + \dots + q^{m-1}) G(y_0, y_1, y_2) \\ &< \frac{q^n}{1 - q} G(y_0, y_1, y_2) \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . Hence  $\{y_n\}$  is a  $G$ -cauchy sequence in  $X$ .

Since  $X$  is complete  $G$ -metric space, then there exists a point  $u \in X$  s.t.  $y_n \rightarrow u$  as  $n \rightarrow \infty$ .

Since the sequences  $\{Ax_{3n}\} = \{Tx_{3n+1}\}$ ,  $\{Bx_{3n+1}\} = \{Sx_{3n+2}\}$  and  $\{Cx_{3n-1}\} = \{Rx_{3n}\}$  are also the subsequences of  $\{y_n\}$ , then they all converges to  $u$ .

Hence as  $n \rightarrow \infty$

$$\begin{aligned}y_{3n} &= Ax_{3n} = Tx_{3n+1} \rightarrow u \\y_{3n+1} &= Bx_{3n+1} = Sx_{3n+2} \rightarrow u \\y_{3n-1} &= Cx_{3n-1} = Rx_{3n} \rightarrow u\end{aligned}\tag{3.4}$$

Now we prove that  $u$  is the common fixed point of  $A, B, C, R, S$  and  $T$  under condition (a).

**Case I.** First suppose that  $R$  is continuous, the pair  $(A, R)$  is weakly commuting, the pairs  $(B, T)$  and  $(C, S)$  is weakly compatible. First we will prove that  $u = Au = Ru$ .

By equation(3.4) and weakly commuting of the pair  $(A, R)$ , we have as  $n \rightarrow \infty$

$$G(ARx_{3n}, RAx_{3n}, RAx_{3n}) \leq G(Ax_{3n}, Rx_{3n}, Rx_{3n}) \rightarrow 0\tag{3.5}$$

Since  $R$  is continuous then by inequality(3.5)  $R^2x_{3n} \rightarrow Ru$ ,  $RAx_{3n} \rightarrow Ru$  as  $n \rightarrow \infty$ .

We know that  $ARx_{3n} \rightarrow Ru$  then from inequality (3.1) we have

$$\begin{aligned}G(ARx_{3n}, Bx_{3n+1}, Cx_{3n+2}) &\leq \alpha\{G(R^2x_{3n}, Tx_{3n+1}, Sx_{3n+2}) \\&+ G(R^2x_{3n}, Bx_{3n+1}, Cx_{3n+2})+ \\G(R^2x_{3n}, ARx_{3n}, ARx_{3n})\} &+ \beta\{max\{G(Tx_{3n+1}, Bx_{3n+1}, Cx_{3n+2}), \\G(Bx_{3n+1}, Tx_{3n+1}, Sx_{3n+2}), &G(Tx_{3n+1}, ARx_{3n}, ARx_{3n}), \\G(Cx_{3n+2}, R^2x_{3n}, Sx_{3n+2})\}\} &+ \gamma\{G(R^2x_{3n}, Bx_{3n+1}, Tx_{3n+1})+ \\G(Tx_{3n+1}, Cx_{3n+2}, Sx_{3n+2}) &+ G(Sx_{3n+2}, ARx_{3n}, ARx_{3n})\}\end{aligned}$$

Taking  $n \rightarrow \infty$

$$\begin{aligned}G(Ru, u, u) &\leq \alpha\{G(Ru, u, u) + G(Ru, u, u) + G(Ru, Ru, Ru)\} \\&+ \beta\{max\{G(u, u, u), G(u, u, u), \\G(u, Ru, Ru), G(u, Ru, u)\}\} &+ \gamma\{G(Ru, u, u) + G(u, u, u) + G(u, Ru, Ru)\} \\&\leq \alpha\{2G(Ru, u, u) + 0\} + \beta\{max\{0, 0, G(u, Ru, Ru), G(u, Ru, u)\}\}+ \\&\gamma\{G(Ru, u, u) + 0 + G(u, Ru, Ru)\} \\&\leq 2\alpha G(Ru, u, u) + 2\beta G(Ru, u, u) + 3\gamma G(Ru, u, u)\end{aligned}$$

$$\Rightarrow G(Ru, u, u) \leq (2\alpha + 2\beta + 3\gamma)G(Ru, u, u)$$

which implies that  $G(Ru, u, u) = 0$ , so that  $Ru = u$  as  $6\alpha + 4\beta + 5\gamma < \frac{1}{2}$ .

Again using inequality(3.1) we have

$$\begin{aligned} G(Au, Bx_{3n+1}, Cx_{3n+2}) &\leq \alpha\{G(Ru, Tx_{3n+1}, Sx_{3n+2}) + G(Ru, Bx_{3n+1}, Cx_{3n+2}) + \\ &G(Ru, Au, Au)\} + \beta\{\max\{G(Tx_{3n+1}, Bx_{3n+1}, Cx_{3n+2}), \\ &G(Bx_{3n+1}, Tx_{3n+1}, Sx_{3n+2}), G(Tx_{3n+1}, Au, Au), \\ &G(Cx_{3n+2}, Ru, Sx_{3n+2})\}\} + \gamma\{G(Ru, Bx_{3n+1}, Tx_{3n+1}) + \\ &G(Tx_{3n+1}, Cx_{3n+2}, Sx_{3n+2}) + G(Sx_{3n+2}, Au, Au)\} \end{aligned}$$

Taking  $n \rightarrow \infty$  and  $u = Ru$ , we have

$$\begin{aligned} G(Au, u, u) &\leq \alpha\{G(u, u, u) + G(u, u, u) + G(u, Au, Au)\} \\ &+ \beta\{\max\{G(u, u, u), G(u, u, u), \\ &G(u, Au, Au), G(u, u, u)\}\} + \gamma\{G(u, u, u) + G(u, u, u) + G(u, Au, Au)\} \\ &\leq \alpha\{2G(Au, u, u)\} + \beta\{2G(Au, u, u)\} + \gamma\{2G(Au, u, u)\} \\ &\Rightarrow G(Au, u, u) \leq (2\alpha + 2\beta + 2\gamma)G(Au, u, u) \end{aligned}$$

which implies that  $G(Au, u, u) = 0$ , so that  $Au = u$ .

Thus we have  $u = Au = Ru$ . Hence  $u$  is the common fixed point of  $A$  and  $R$ .

Again we show that  $u$  is the common fixed point of  $B$  and  $T$ .

Since  $A(X) \subseteq T(X)$  and  $u = Au \in A(X)$ , there exists a point  $v \in X$  s.t.  $u = Au = Tv$ .

Now by inequality(3.1) we have

$$\begin{aligned} G(Au, Bv, Cx_{3n+2}) &\leq \alpha\{G(Ru, Tv, Sx_{3n+2}) + G(Ru, Bv, Cx_{3n+2}) \\ &+ G(Ru, Au, Au)\} + \\ &\beta\{\max\{G(Tv, Bv, Cx_{3n+2}), G(Bv, Tv, Sx_{3n+2}), G(Tv, Au, Au), \\ &G(Cx_{3n+2}, Ru, Sx_{3n+2})\}\} + \gamma\{G(Ru, Bv, Tv) + \\ &G(Tv, Cx_{3n+2}, Sx_{3n+2}) + G(Sx_{3n+2}, Au, Au)\} \end{aligned}$$

Taking  $n \rightarrow \infty$  and using  $u = Au = Ru = Tv$ , we get

$$\begin{aligned} G(u, Bv, u) &\leq \alpha\{G(u, u, u) + G(u, Bv, u) + G(u, u, u)\} \\ &+ \beta\{\max\{G(u, Bv, u), G(Bv, u, u), \\ &G(u, u, u), G(u, u, u)\}\} + \gamma\{G(u, Bv, u) + G(u, u, u) + G(u, u, u)\} \\ &\leq (\alpha + \beta + \gamma)G(u, Bv, u) \end{aligned}$$

This implies that  $G(u, Bv, u) = 0 \Rightarrow u = Bv$ .

Also  $Tv = u \Rightarrow Tv = u = Bv$ .

Since the pair  $(B, T)$  is weakly compatible, we have  $Bu = BTv = TBv = Tu$ .

Again using inequality(3.1) we have

$$\begin{aligned} G(Au, Bu, Cx_{3n+2}) &\leq \alpha\{G(Ru, Tu, Sx_{3n+2}) + G(Ru, Bu, Cx_{3n+2}) \\ &+ G(Ru, Au, Au)\} + \\ &\beta\{\max\{G(Tu, Bu, Cx_{3n+2}), G(Bu, Tu, Sx_{3n+2}), G(Tu, Au, Au), \\ &G(Cx_{3n+2}, Ru, Sx_{3n+2})\}\} + \gamma\{G(Ru, Bu, Tu) + \\ &G(Tu, Cx_{3n+2}, Sx_{3n+2}) + G(Sx_{3n+2}, Au, Au)\} \end{aligned}$$

Taking  $n \rightarrow \infty$ , using  $u = Au = Ru$  and  $Bu = Tu$ , we get

$$\begin{aligned} G(u, Bu, u) &\leq \alpha\{G(u, Bu, u) + G(u, Bu, u) + G(u, u, u)\} \\ &+ \beta\{\max\{G(Bu, Bu, u), \\ &G(Bu, Bu, u), G(Bu, u, u), G(u, u, u)\}\} + \gamma\{G(u, Bu, Bu) + \\ &G(Bu, u, u) + G(u, u, u)\} \\ &\leq (2\alpha + 2\beta + 2\gamma)G(u, Bu, u) \end{aligned}$$

This implies that  $G(u, Bu, u) = 0 \Rightarrow u = Bu$ .

Also  $Tu = Bu \Rightarrow Tu = u = Bu$ .

This shows that  $u$  is a common fixed point of  $B$  and  $T$ .

Now we show that  $u$  is a common fixed point of  $C$  and  $S$ .

Since  $B(X) \subseteq S(X)$  and  $u = Bu \in B(X)$ , therefore there exists a point  $w \in X$  s.t.  $u = Bu = Sw$ .

Using the inequality(3.1) we have

$$\begin{aligned} G(Au, Bu, Cw) &\leq \alpha\{G(Ru, Tu, Sw) + G(Ru, Bu, Cw) + G(Ru, Au, Au)\} + \\ &\beta\{\max\{G(Tu, Bu, Cw), G(Bu, Tu, Sw), G(Tu, Au, Au), G(Cw, Ru, Sw)\}\} + \\ &\gamma\{G(Ru, Bu, Tu) + G(Tu, Cw, Sw) + G(Sw, Au, Au)\} \end{aligned}$$

using  $u = Au = Ru = Bu = Tu = Sw$ , we get

$$\begin{aligned} G(u, u, Cw) &\leq \alpha\{G(u, u, u) + G(u, u, Cw) + G(u, u, u)\} + \\ &\beta\{\max\{G(u, u, Cw), G(u, u, u), G(u, u, u), G(Cw, u, u)\}\} + \\ &\gamma\{G(u, u, u) + G(u, Cw, u) + G(u, u, u)\} \\ &\leq (\alpha + \beta + \gamma)G(u, u, Cw) \end{aligned}$$

This implies that  $G(u, u, Cu) = 0 \Rightarrow u = Cu$ .

Also  $u = Sw \Rightarrow Sw = u = Cu$ .

Since the pair  $(C, S)$  is weakly compatible, we have  $Cu = CSw = SCw = Su$ .

Again using inequality(3.1) we have

$$G(Au, Bu, Cu) \leq \alpha\{G(Ru, Tu, Su) + G(Ru, Bu, Cu) + G(Ru, Au, Au)\} + \beta\{\max\{G(Tu, Bu, Cu), G(Bu, Tu, Su), G(Tu, Au, Au), G(Cu, Ru, Su)\}\} + \gamma\{G(Ru, Bu, Tu) + G(Tu, Cu, Su) + G(Su, Au, Au)\}$$

using  $u = Au = Ru = Bu = Tu$  and  $Cu = Su$ , we get

$$G(u, u, Cu) \leq \alpha\{G(u, u, Cu) + G(u, u, Cu) + G(u, u, u)\} + \beta\{\max\{G(u, u, Cu), G(u, u, Cu), G(u, u, u), G(Cu, u, Cu)\}\} + \gamma\{G(u, u, u) + G(u, Cu, Cu) + G(Cu, u, u)\} \leq (2\alpha + 2\beta + 3\gamma)G(u, u, Cu)$$

This implies that  $G(u, u, Cu) = 0 \Rightarrow u = Cu$ .

Also  $Su = Cu$ , therefore we get  $Cu = u = Su$ .

This shows that  $u$  is a common fixed point of  $C$  and  $S$ .

Hence  $u$  is a common fixed point of  $A, B, C, R, S$  and  $T$ .

**Case II.** Suppose that  $A$  is continuous, the pair  $(A, R)$  is weakly commuting, the pairs  $(B, T)$  and  $(C, S)$  are weakly compatible.

First we prove that  $u = Au$ .

Since the pair  $(A, R)$  is weakly commuting, then by inequality(3.4) we have as  $n \rightarrow \infty$

$$G(ARx_{3n}, RAx_{3n}, RAx_{3n}) \leq G(Ax_{3n}, Rx_{3n}, Rx_{3n}) \rightarrow 0.$$

Since  $A$  is continuous then  $A^2x_{3n} \rightarrow Au$ , as  $n \rightarrow \infty$ .

By inequality(3.5) we have  $RAx_{3n} \rightarrow Au$ , as  $n \rightarrow \infty$ .

From inequality(3.1) we have

$$G(A^2x_{3n}, Bx_{3n+1}, Cx_{3n+2}) \leq \alpha\{G(RAx_{3n}, Tx_{3n+1}, Sx_{3n+2}) + G(RAx_{3n}, Bx_{3n+1}, Cx_{3n+2}) + G(RAx_{3n}, A^2x_{3n}, A^2x_{3n})\} + \beta\{\max\{G(Tx_{3n+1}, Bx_{3n+1}, Cx_{3n+2}), G(Bx_{3n+1}, Tx_{3n+1}, Sx_{3n+2}), G(Tx_{3n+1}, A^2x_{3n}, A^2x_{3n}), G(Cx_{3n+2}, RAx_{3n}, Sx_{3n+2})\}\} + \gamma\{G(RAx_{3n}, Bx_{3n+1}, Tx_{3n+1}) + G(Tx_{3n+1}, Cx_{3n+2}, Sx_{3n+2}) + G(Sx_{3n+2}, A^2x_{3n}, A^2x_{3n})\}$$

Taking  $n \rightarrow \infty$

$$\begin{aligned} G(Au, u, u) &\leq \alpha\{G(Au, u, u) + G(Au, u, u) + G(Au, Au, Au)\} \\ &\quad + \beta\{\max\{G(u, u, u), G(u, u, u) \\ &\quad , G(u, Au, Au), G(u, Au, u)\}\} + \gamma\{G(Au, u, u) \\ &\quad + G(u, u, u) + G(u, Au, Au)\} \\ &\leq 2\alpha G(Au, u, u) + 2\beta G(Au, u, u) + 3\gamma G(Au, u, u) \\ &\leq (2\alpha + 2\beta + 3\gamma)G(Au, u, u) \end{aligned}$$

which implies that  $G(Au, u, u) = 0$ , so that  $Au = u$ .

Now we prove that  $u$  is the common fixed point of  $B$  and  $T$ .

Since  $A(X) \subseteq T(X)$  and  $u = Au \in A(X)$ , there exists a point  $v \in X$  s.t.  $u = Au = Tv$ .

Now by inequality(3.1) we have

$$\begin{aligned} G(A^2x_{3n}, Bv, Cx_{3n+2}) &\leq \alpha\{G(RAx_{3n}, Tv, Sx_{3n+2}) + G(RAx_{3n}, Bv, Cx_{3n+2}) + \\ &\quad G(RAx_{3n}, A^2x_{3n}, A^2x_{3n})\} + \beta\{\max\{G(Tv, Bv, Cx_{3n+2}), \\ &\quad G(Bv, Tv, Sx_{3n+2}), G(Tv, A^2x_{3n}, A^2x_{3n}), \\ &\quad G(Cx_{3n+2}, RAx_{3n}, Sx_{3n+2})\}\} + \gamma\{G(RAx_{3n}, Bv, Tv) + \\ &\quad G(Tv, Cx_{3n+2}, Sx_{3n+2}) + G(Sx_{3n+2}, A^2x_{3n}, A^2x_{3n})\} \end{aligned}$$

Taking  $n \rightarrow \infty$  and using  $u = Au = Tv$ , we get

$$\begin{aligned} G(u, Bv, u) &\leq \alpha\{G(u, u, u) + G(u, Bv, u) + G(u, u, u)\} \\ &\quad + \beta\{\max\{G(u, Bv, u), G(Bv, u, u), \\ &\quad G(u, u, u), G(u, u, u)\}\} + \gamma\{G(u, Bv, u) + G(u, u, u) + G(u, u, u)\} \\ &\leq (\alpha + \beta + \gamma)G(u, Bv, u) \end{aligned}$$

This implies that  $G(u, Bv, u) = 0 \Rightarrow u = Bv$ .

Also  $Tv = u \Rightarrow Tv = u = Bv$ .

Since the pair  $(B, T)$  is weakly compatible, we have  $Bu = BTv = TBv = Tu$ .

Again using inequality(3.1) we have

$$\begin{aligned} G(Ax_{3n}, Bu, Cx_{3n+2}) &\leq \alpha\{G(Rx_{3n}, Tu, Sx_{3n+2}) + G(Rx_{3n}, Bu, Cx_{3n+2}) + \\ &\quad G(Rx_{3n}, Ax_{3n}, Ax_{3n})\} + \beta\{\max\{G(Tu, Bu, Cx_{3n+2}), \\ &\quad G(Bu, Tu, Sx_{3n+2}), G(Tu, Ax_{3n}, Ax_{3n}), \\ &\quad G(Cx_{3n+2}, Rx_{3n}, Sx_{3n+2})\}\} + \gamma\{G(Rx_{3n}, Bu, Tu) + \\ &\quad G(Tu, Cx_{3n+2}, Sx_{3n+2}) + G(Sx_{3n+2}, Ax_{3n}, Ax_{3n})\} \end{aligned}$$

Taking  $n \rightarrow \infty$ , using  $u = Au = Ru$  and  $Bu = Tu$ , we get

$$\begin{aligned} G(u, Bu, u) &\leq \alpha\{G(u, Bu, u) + G(u, Bu, u) + G(u, u, u)\} \\ &\quad + \beta\{\max\{G(Bu, Bu, u), G(Bu, Bu, u), \\ &\quad G(Bu, u, u), G(u, u, u)\}\} + \gamma\{G(u, Bu, Bu) \\ &\quad + G(Bu, u, u) + G(u, u, u)\} \\ &\leq (2\alpha + 2\beta + 3\gamma)G(u, Bu, u) \end{aligned}$$

This implies that  $G(u, Bu, u) = 0 \Rightarrow u = Bu$ .

Also  $Tu = Bu \Rightarrow Tu = u = Bu$ .

Thus  $u$  is a common fixed point of  $B$  and  $T$ .

Now we show that  $u$  is a common fixed point of  $C$  and  $S$ .

Since  $B(X) \subseteq S(X)$  and  $u = Bu \in B(X)$ , therefore there exists a point  $w \in X$  s.t.  $u = Bu = Sw$ .

Using the inequality(3.1) we have

$$\begin{aligned} G(Ax_{3n}, Bu, Cw) &\leq \alpha\{G(Rx_{3n}, Tu, Sw) + G(Rx_{3n}, Bu, Cw) \\ &\quad + G(Rx_{3n} + Ax_{3n}, Ax_{3n})\} + \\ &\quad \beta\{\max\{G(Tu, Bu, Cw), G(Bu, Tu, Sw), G(Tu, Ax_{3n}, Ax_{3n}), \\ &\quad G(Cw, Rx_{3n}, Sw)\}\} + \gamma\{G(Rx_{3n}, Bu, Tu) \\ &\quad + G(Tu, Cw, Sw) + \\ &\quad G(Sw, Ax_{3n}, Ax_{3n})\} \end{aligned}$$

Taking  $n \rightarrow \infty$  and using  $u = Bu = Tu = Sw$ , we get

$$\begin{aligned} G(u, u, Cw) &\leq \alpha\{G(u, u, u) + G(u, u, Cw) + G(u, u, u)\} + \\ &\quad \beta\{\max\{G(u, u, Cw), G(u, u, u), G(u, u, u), G(Cw, u, u)\}\} + \\ &\quad \gamma\{G(u, u, u) + G(u, Cw, u) + G(u, u, u)\} \\ &\leq (\alpha + \beta + \gamma)G(u, u, Cw) \end{aligned}$$

This implies that  $G(u, u, Cw) = 0 \Rightarrow u = Cw$ .

Also  $u = Sw \Rightarrow Sw = u = Cw$ .

Since the pair  $(C, S)$  is weakly compatible, we have  $Cu = CSw = SCw = Su$ .

Again using inequality(3.1) we have

$$\begin{aligned} G(Ax_{3n}, Bu, Cu) &\leq \alpha\{G(Rx_{3n}, Tu, Su) + G(Rx_{3n}, Bu, Cu) \\ &\quad + G(Rx_{3n}, Ax_{3n}, Ax_{3n})\} + \\ &\quad \beta\{\max\{G(Tu, Bu, Cu), G(Bu, Tu, Su), G(Tu, Ax_{3n}, Ax_{3n}), \\ &\quad G(Cu, Rx_{3n}, Su)\}\} + \gamma\{G(Rx_{3n}, Bu, Tu) + G(Tu, Cu, Su) + \\ &\quad G(Su, Ax_{3n}, Ax_{3n})\} \end{aligned}$$

Taking  $n \rightarrow \infty$ , using  $u = Bu = Tu$  and  $Cu = Su$ , we get

$$\begin{aligned} G(u, u, Cu) &\leq \alpha\{G(u, u, Cu) + G(u, u, Cu) + G(u, u, u)\} + \\ &\quad \beta\{\max\{G(u, u, Cu), G(u, u, Cu), G(u, u, u), G(Cu, u, Cu)\}\} + \\ &\quad \gamma\{G(u, u, u) + G(u, Cu, Cu) + G(Cu, u, u)\} \\ &\leq (2\alpha + 2\beta + 3\gamma)G(u, u, Cu) \end{aligned}$$

This implies that  $G(u, u, Cu) = 0 \Rightarrow u = Cu$ .

Also  $Su = Cu$ , therefore we get  $Cu = u = Su$ .

This shows that  $u$  is a common fixed point of  $C$  and  $S$ .

Finally we will show that  $u = Ru$ .

Since  $C(X) \subseteq R(X)$  and  $u = Cu \in C(X)$ , therefore there exists a point  $t \in X$  s.t.  $u = u = Rt$ .

Using the inequality(3.1) we have

$$\begin{aligned} G(At, Bu, Cu) &\leq \alpha\{G(Rt, Tu, Su) + G(Rt, Bu, Cu) + G(Rt, At, At)\} + \\ &\quad \beta\{\max\{G(Tu, Bu, Cu), G(Bu, Tu, Su), \\ &\quad G(Tu, At, At), G(Cu, Rt, Su)\}\} + \\ &\quad \gamma\{G(Rt, Bu, Tu) + G(Tu, Cu, Su) + G(Su, At, At)\} \\ &\leq \alpha\{G(u, u, u) + G(u, u, u) + G(u, At, At)\} + \beta\{\max\{G(u, u, u), \\ &\quad G(u, u, u), G(u, At, At), G(u, u, u)\}\} + \gamma\{G(u, u, u) \\ &\quad + G(u, u, u) + G(u, At, At)\} \\ &\leq (2\alpha + 2\beta + 2\gamma)G(At, u, u) \end{aligned}$$

This implies that  $G(At, u, u) = 0 \Rightarrow u = At$ .

Since the pair  $(A, R)$  is weakly commuting, we have  $Au = ARt = RAt = Ru = u$ .

This shows that  $u$  is a common fixed point of  $A$  and  $R$ .

Hence  $u$  is a common fixed point of  $A, B, C, R, S$  and  $T$  when  $A$  is continuous and the pair  $(A, R)$  is weakly commuting, the pairs  $(B, T)$  and  $(C, S)$  is weakly compatible.

Similarly we can prove that the  $u$  is a common fixed point of  $A, B, C, R, S$  and  $T$  when the condition (b) and (c) are hold.

**Uniqueness.** Let  $u$  and  $z$  be two common fixed point of  $A, B, C, R, S$  and  $T$ . Using inequality(3.1) we have

$$\begin{aligned} G(z, u, u) &\leq \alpha\{G(Rz, Tu, Su) + G(Rz, Bu, Cu) + G(Rz, Az, Az)\} + \\ &\quad \beta\{\max\{G(Tu, Bu, Cu), G(Bu, Tu, Su), \\ &\quad G(Tu, Az, Az), G(Cu, Rz, Su)\}\} + \\ &\quad \gamma\{G(Rz, Bu, Tu) + G(Tu, Cu, Su) + G(Su, Az, Az)\} \\ &\leq \alpha\{G(z, u, u) + G(z, u, u) + G(z, z, z)\} + \beta\{\max\{G(u, u, u), G(u, u, u), \\ &\quad G(u, z, z), G(u, z, u)\}\} + \gamma\{G(z, u, u) + G(u, u, u) + G(u, z, z)\} \\ &\leq (2\alpha + 2\beta + 3\gamma)G(z, u, u) \end{aligned}$$

This implies that  $G(z, u, u) = 0 \Rightarrow z = u$ .

Thus the common fixed point is unique.

This completes the proof.  $\square$

In Theorem(3.1), if we take  $R = S = T = I$  then following Corollary is obtained.

**Corollary 3.1.** *Let  $(X, G)$  be a complete  $G$ -metric space and let  $A, B$  and  $C$  be three mappings of  $X$  into itself satisfying the following conditions,*

$$\begin{aligned} G(Ax, By, Cz) &\leq \alpha\{G(x, y, z) + G(x, By, Cz) + G(x, Ax, Ax)\} + \\ &\quad \beta\{\max\{G(y, By, Cz), G(By, y, z), G(y, Ax, Ax), G(Cz, x, z)\}\} + \\ &\quad \gamma\{G(x, By, y) + G(y, Cz, z) + G(z, Ax, Ax)\} \end{aligned} \tag{3.6}$$

$\forall x, y, z \in X$ , where  $\alpha, \beta, \gamma > 0$  and  $6\alpha + 4\beta + 5\gamma < \frac{1}{2}$ . Then mappings  $A, B$  and  $C$  have a unique common fixed point in  $X$ .

**Remark 3.1.** Theorem(3.1) and Corollary(3.1) improve and generalize the corresponding results of Abbas and Rhoades[6], Mustafa et al.[13], Chugh, Kadian, Rani and Rhoades[7].

**Theorem 3.2.** *Let  $(X, G)$  be a complete  $G$ -metric space and let  $A, B, C, R, S$  and  $T$  be six mappings on  $X$  into itself satisfying the following conditions:*

(i)  $A(X) \subseteq T(X)$ ,  $B(X) \subseteq S(X)$ ,  $C(X) \subseteq R(X)$ .

(ii) The pairs  $(A, R)$ ,  $(B, T)$  and  $(C, S)$  are commuting mapping.

(iii)  $\forall x, y, x \in X$

$$\begin{aligned}
 G(A^m x, B^m y, C^m z) \leq & \alpha \{G(Rx, Ty, Sz) + G(Rx, B^m y, C^m z) \\
 & + G(Rx, A^m x, A^m x)\} + \\
 & \beta \{ \max \{G(Ty, B^m y, C^m z), G(B^m y, Ty, Sz), \\
 & G(Ty, A^m x, A^m x), \\
 & G(C^m z, Rx, Sz)\} \} + \gamma \{G(Rx, B^m y, Ty) + G(Ty, C^m z, Sz) \\
 & + G(Sz, A^m x, A^m x)\}
 \end{aligned} \tag{3.7}$$

where  $\alpha, \beta, \gamma > 0$  and  $6\alpha + 4\beta + 5\gamma < \frac{1}{2}$ ,  $m \in N$  then  $A, B, C, R, S$  and  $T$  have a unique fixed point in  $X$ .

*Proof.* Suppose the mappings  $A, B, C, R, S$  and  $T$  satisfy inequality(3.7).

Since  $A^m(X) \subset A^{m-1}(X) \subset A^{m-2}(X) \subset \dots \subset A(X) \subseteq T(X)$ . Similarly we can show that  $B^m(X) \subset S(X)$  and  $C^m(X) \subset R(X)$ . From Theorem(3.1), we can show that  $A^m, B^m, C^m, R, S$  and  $T$  have a unique fixed point  $u$  in  $X$ .

Now using inequality(3.7), we have

$$\begin{aligned}
 G(A^m Au, B^m u, C^m u) \leq & \alpha \{G(RAu, Tu, Su) + G(RAu, B^m u, C^m u) \\
 & + G(RAu, A^m Au, A^m Au)\} + \\
 & \beta \{ \max \{G(Tu, B^m u, C^m u), G(B^m u, Tu, Su), G(Tu, A^m Au, A^m Au), \\
 & G(C^m u, RAu, Su)\} \} + \gamma \{G(RAu, B^m u, Tu) + G(Tu, C^m u, Su) + \\
 & G(Su, A^m Au, A^m Au)\}
 \end{aligned}$$

By condition(ii) we have  $RAu = ARu = Au$  and  $Au = A(A^m u) = A^{m+1}u = A^m Au$  then we get

$$\begin{aligned}
 G(Au, u, u) \leq & \alpha \{G(Au, u, u) + G(Au, u, u) + G(Au, Au, Au)\} + \beta \{ \max \{G(u, u, u), \\
 & G(u, u, u), G(u, Au, Au), G(u, Au, u)\} \} + \gamma \{G(Au, u, u) + \\
 & G(u, u, u) + G(u, Au, Au)\} \\
 \leq & 2\alpha G(Au, u, u) + 2\beta G(Au, u, u) + 3\gamma G(Au, u, u) \\
 \leq & (2\alpha + 2\beta + 3\gamma)G(Au, u, u)
 \end{aligned}$$

This implies that  $G(Au, u, u) = 0, \Rightarrow Au = u$ .

Similarly we can show that  $u = Bu$  and  $u = Cu$ .

Thus we have  $u = Au = Bu = Cu = Ru = Su = Tu$ . So that  $A, B, C, R, S$  and  $T$  have a common fixed point in  $X$ .

Let  $v$  be another common fixed point of  $A, B, C, R, S$  and  $T$  then using the inequality(3.6), we have

$$\begin{aligned} G(v, u, u) &\leq \alpha\{G(Rv, Tu, Su) + G(Rv, B^m u, C^m u) + G(Rv, A^m v, A^m v)\} + \\ &\quad \beta\{\max\{G(Tu, B^m u, C^m u), G(B^m u, Tu, Su), G(Tu, A^m v, A^m v), \\ &\quad G(C^m u, Rv, Su)\}\} + \gamma\{G(Rv, B^m u, Tu) + G(Tu, C^m u, Su) \\ &\quad + G(Su, A^m v, A^m v)\} \\ &\leq \alpha\{G(v, u, u) + G(v, u, u) + G(v, v, v)\} \\ &\quad + \beta\{\max\{G(u, u, u), G(u, u, u), \\ &\quad G(u, v, v), G(u, v, u)\}\} + \gamma\{G(v, u, u) + G(u, u, u) + G(u, v, v)\} \\ &\leq (2\alpha + 2\beta + 3\gamma)G(v, u, u) \end{aligned}$$

This implies that  $G(v, u, u) = 0 \Rightarrow v = u$ .

Thus  $u$  is the unique common fixed point of  $A, B, C, R, S$  and  $T$ .

**Corollary 3.2.** *Let  $(X, G)$  be a complete  $G$ -metric space and let  $A$  be a mapping of  $X$  into itself satisfying the following conditions*

$$\begin{aligned} G(A^m x, A^m y, A^m z) &\leq \alpha\{G(x, y, z) + G(x, A^m y, A^m z) + G(x, A^m x, A^m x)\} + \\ &\quad \beta\{\max\{G(y, A^m y, A^m z), G(A^m y, y, z), G(y, A^m x, A^m x), \\ &\quad G(A^m z, x, z)\}\} + \gamma\{G(x, A^m y, y) + G(y, A^m z, z) \\ &\quad + G(Sz, A^m x, A^m x)\} \end{aligned} \tag{3.8}$$

$\forall x, y, z \in X$ , where  $\alpha, \beta, \gamma > 0$  and  $6\alpha + 4\beta + 5\gamma < \frac{1}{2}$ ,  $m \in N$  then  $A$  has a unique fixed point in  $X$ .

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