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SECOND ORDER DIFFERENTIAL-DIFFERENCE EQUATION FOR DUNKL-CLASSICAL ORTHOGONAL POLYNOMIALS

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Abstract: In this paper, we show that Dunkl-classical polynomial sequences can be characterized by a differential-difference equation.

1. Introduction

Linear second order difference and differential equations with polynomial coefficients appear in many research domains such as atomic, molecular and nuclear physics, electrodynamics and acoustics. The links between them and classical orthogonal polynomials are extensively studied by many authors. For instance,

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we know since Bochner [2] that classical orthogonal polynomials of continuous variables are characterized by some second order differential equations. The literature on this topic is extremely vast. We quote for instance [8]-[11]. This characterization has been extended for q-classical orthogonal polynomials to a second order q-difference equation (see [7], [14]). Analogously, classical orthogonal polynomials of discrete variables were characterized by a linear second order difference equation [6].

A naturel question arises: Is there a similar characterization for Dunkl-classical polynomials?

Recall that a monic orthogonal polynomial sequence $\{P_n\}_{n\geq 0}$ is called Dunkl-classical polynomial sequence (the associated form is called Dunkl-classical form) if $\{T_\mu P_n\}_{n\geq 1}$ is an orthogonal polynomial sequence, where T_μ is the Dunkl operator [5] defined by $T_\mu = D + 2\mu H_{-1}$, whith $\mu > -\frac{1}{2}$ and D (resp. H_{-1}) denotes the derivative operator $D = \frac{d}{dx}$ (resp. the Hahn operator given by $(H_{-1}f)(x) = \frac{f(x)-f(-x)}{2x}$). [1]

Y. Ben Cheikh and his coworker [1], the first who introduced the notion of Dunkl-classical orthogonal polynomials, derived a differential-difference equation for Dunkl-classical symmetric polynomials in order to prove that the only Dunkl-classical symmetric orthogonal polynomials are the generalized Hermite polynomials and the generalized Gegenbauer polynomials. Later on, M. Sghaier [15] characterized Dunkl-classical orthogonal polynomials by certain linear second order differential-difference equation. In such characterization, only the symmetric polynomials are considered.

The aim of this paper is to extend this characterization to nonsymmetric Dunkl-classical polynomials. Namely, we prove the following result.

Theorem 1.1. *Let $\{P_n\}_{n\geq 0}$ a MOPS and let u_0 its associated form. The following assumptions are equivalent:*

(a) The sequence $\{P_n\}_{n\geq 0}$ is Dunkl-classical.

(b) There exist a complex number K, a sequence $(\lambda_n)_{n\geq 0}$ in $\mathbb{C}^*$ and three polynomials Φ (monic), $\widetilde{\Phi}$ and Ψ with $deg\widetilde{\Phi} = deg\Phi \leq 2$ and $deg\Psi = 1$ such that

$$x\Phi(x)u_0 = h_{-1}(x\widetilde{\Phi}(x)u_0) \tag{1}$$

and

$$\frac{K}{1-4\mu^2}\Big(\Phi(x) + 2\mu\widetilde{\Phi}(x)\Big)(T_\mu^2 P_{n+1})(x) - \Psi(x)(T_\mu P_{n+1})(x)$$

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$$-\frac{2\mu K}{1-2\mu}\Big(\Phi(x)+\widetilde{\Phi}(x)\Big)(H_{-1}T_\mu P_{n+1})(x)=\lambda_n P_{n+1}(x). \quad (2)$$

The structure of this paper is as follows: The second section is devoted to preliminary results and notations to be used in the sequel. In the third section, we prove the main theorem. In section 4, we illustrate results of the previous section by giving a differential-difference equation for a Dunkl-classical nonsymmetric polynomial sequence.

2. Preliminaries and Notations

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f \rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n = \langle u, x^n \rangle$, $n \geq 0$ the moments of u.

The Dirac mass at the point $c \in \mathbb{C}$ denoted by δ_c is the form defined by

$$\langle \delta_c, f \rangle = f(c), \ f \in \mathcal{P}.$$

For $u \in \mathcal{P}'$, we define the forms $h_a u$, gu and $Du = u'$ by:

$$\langle h_a u, f(x) \rangle = \langle u, (h_a f)(x) \rangle = \langle u, f(ax) \rangle, \ f \in \mathcal{P}, \ a \in \mathbb{C} \setminus \{0\},$$

$$\langle gu, f \rangle = \langle u, gf \rangle, \ \langle u', f \rangle = -\langle u, f' \rangle, \ f, g \in \mathcal{P}.$$

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials with $deg P_n = n$, $n \geq 0$ and let $\{u_n\}_{n\geq 0}$ be its dual sequence, i.e. $u_n \in \mathcal{P}'$ and defined by $<u_n, P_m> = \delta_{n,m}$, where $\delta_{n,m}$, n, $m \geq 0$ is the kronecker delta function.

Let us recall some result [12]:

Lemma 2.1. For any $u \in \mathcal{P}'$ and any integer $m \geq 1$, the following statements are equivalent:

(i) $<u, P_{m-1}> \neq 0, <u, P_n> = 0$, $n \geq m$

(ii) $\exists \lambda_\nu \in \mathbb{C}$, $0 \leq \nu \leq m-1$, $\lambda_{m-1} \neq 0$ such that $u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu$.

The form u is called regular if we can associate with it a PS $\{P_n\}_{n\geq 0}$ such that [4]:

$$<u, P_m P_n> = r_n \delta_{n,m}, \ n, m \geq 0; \ r_n \neq 0, \ n \geq 0. \quad (3)$$

The sequence $\{P_n\}_{n\geq 0}$ is then said orthogonal with respect to u. In this case, we have $u_n = r_n^{-1} P_n u_0$, $n \geq 0$ and reciprocally. According to the previous

lemma, we have $u = \lambda u_0$, where $(u)_0 = \lambda \neq 0$. In what follows all regular forms u will be taken normalized i.e, $(u)_0 = 1$.

According to Favard's theorem, a monic orthogonal polynomial sequence (MOPS, for shorter) $\{P_n\}_{n \geq 0}$ is characterized by the following three-term recurrence relation [4]:

$$P_0(x) = 1, \ P_1(x) = x - \beta_0,$$
$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \ n \geq 0. \tag{4}$$

A linear functional u is said symmetric if and only if $(u)_{2n+1} = 0$, $n \geq 0$. Or equivalently, in (4) $\beta_n = 0$, $n \geq 0$. Let us introduce the Dunkl's operator

$$(T_\mu f)(x) = f'(x) + 2\mu(H_{-1}f)(x), \ f \in \mathcal{P},$$

where

$$(H_{-1}f)(x) = \frac{f(x) - f(-x)}{2x}.$$

We define the operator T_μ from $\mathcal{P}'$ to $\mathcal{P}'$ as follows

$$< T_\mu u, f >= - < u, T_\mu f >, \ f \in \mathcal{P}, \ u \in \mathcal{P}'.$$

In particular, this yields to

$$(T_\mu u)_n = -\mu_n(u)_{n-1}, \ n \geq 0,$$

where

$$(u)_{-1} = 0, \ \mu_n = n + 2\mu[n], \ [n] = \frac{1 - (-1)^n}{2}, \ n \geq 0.$$

It is easy to see that

$$T_\mu u = Du + 2\mu H_{-1}u,$$

where

$$< H_{-1}u, f >= - < u, H_{-1}f > .$$

Now, consider a MOPS $\{P_n\}_{n \geq 0}$ as above and let

$$P_n^{[1]}(x, \mu) = \frac{1}{\mu_{n+1}}(T_\mu P_{n+1})(x), \ \mu \neq -n - \frac{3}{2}, \ n \geq 0.$$

Denoting by $\{u_n^{[1]}(\mu)\}_{n \geq 0}$ the dual sequence of $\{P_n^{[1]}(x, \mu)\}_{n \geq 0}$.

Lemma 2.2. *[15]*

$$T_\mu(u_n^{[1]}(\mu)) = -\mu_{n+1}u_{n+1}, \ n \geq 0. \tag{5}$$

3. Proof of the Main Theorem

For the proof, we need the following lemma:

Lemma 3.1. *[3] The following formula hold*

$$T_\mu \delta_0 = (1 + 2\mu)\delta_0',$$
(6)

$$H_{-1}\delta_0 = \delta_0',$$
(7)

$$H_{-1}(v - h_{-1}v) = 0, \ v \in \mathcal{P}',$$
(8)

$$T_\mu(fu) = fT_\mu u + (T_\mu f)u + 2\mu(H_{-1}f)(h_{-1}u - u), \ f \in \mathcal{P}, \ u \in \mathcal{P}',$$
(9)

Proof of Theorem 1.1. First of all, notice that for $\mu = 0$ we meet the D-classical orthogonal polynomial sequences, which satisfy a second order differential equation. Henceforth, we will suppose that $\mu \neq 0$.

$[(a) \Rightarrow (b)]$ From the assumption, we have

$$u_n = r_n^{-1} P_n u_0, \ n \geq 0$$
(10)

and

$$u_n^{[1]}(\mu) = (r_n^{[1]})^{-1} P_n^{[1]}(., \mu) u_0^{[1]}(\mu), \ n \geq 0.$$
(11)

Substitution of (10) and (11) in (5) gives

$$T_\mu(P_n^{[1]}(., \mu) u_0^{[1]}(\mu)) = -\chi_n P_{n+1} u_0, \ n \geq 0,$$
(12)

where

$$\chi_n = \mu_{n+1} \frac{r_n^{[1]}}{r_{n+1}}, \ n \geq 0.$$
(13)

Using formula (9), equation (12) becomes

$$P_n^{[1]}(., \mu) T_\mu(u_0^{[1]}(\mu)) + T_\mu(P_n^{[1]}(., \mu)) u_0^{[1]}(\mu) +$$
$$+ 2\mu H_{-1} P_n^{[1]}(., \mu) \left(h_{-1} u_0^{[1]}(\mu) - u_0^{[1]}(\mu) \right) = -\chi_n P_{n+1} u_0,$$
$$n \geq 0.$$
(14)

Putting $n = 0$ in (14) and taking into account (13), we get

$$T_\mu(u_0^{[1]}(\mu)) = -\Psi u_0,$$
(15)

where

$$\Psi = \frac{1 + 2\mu}{\gamma_1} P_1.$$
(16)

For $n = 1$, equation (14) becomes

$$P_1^{[1]}(.,\mu)T_\mu(u_0^{[1]}(\mu)) + u_0^{[1]}(\mu) + 2\mu h_{-1}u_0^{[1]}(\mu) = -2\frac{r_1^{[1]}}{r_2}P_2 u_0. \qquad (17)$$

Substitution of (15) in (17) gives

$$u_0^{[1]}(\mu) + 2\mu h_{-1}u_0^{[1]}(\mu) = K\Phi u_0, \qquad (18)$$

where

$$K\Phi = \frac{1+2\mu}{\gamma_1}P_1 P_1^{[1]}(.,\mu) - 2\frac{r_1^{[1]}}{r_2}P_2, \qquad (19)$$

(K is a constant to make Φ monic).

Applying the operator h_{-1} to (18), we get

$$2\mu u_0^{[1]}(\mu) + h_{-1}u_0^{[1]}(\mu) = Kh_{-1}(\Phi u_0). \qquad (20)$$

Multiplication of (20) by 2μ and substraction of the result from (18) gives

$$u_0^{[1]}(\mu) = \frac{K}{1-4\mu^2}(\Phi u_0 - 2\mu h_{-1}(\Phi u_0)). \qquad (21)$$

Now, putting $n = 2$ in (14), we obtain

$$P_2^{[1]}(.,\mu)T_\mu(u_0^{[1]}(\mu)) + T_\mu(P_2^{[1]}(.,\mu))u_0^{[1]}(\mu) +$$
$$+ 2\mu H_{-1}P_2^{[1]}(.,\mu)\left(h_{-1}u_0^{[1]}(\mu) - u_0^{[1]}(\mu)\right) = -\chi_2 P_3 u_0. \qquad (22)$$

Taking into account (15) and (21), we get

$$\frac{-2\mu K}{1-4\mu^2}\left(T_\mu(P_2^{[1]}(.,\mu)) - (1+2\mu)H_{-1}P_2^{[1]}(.,\mu)\right)h_{-1}(\Phi u_0) =$$
$$\left(\frac{1+2\mu}{\gamma_1}P_1 P_2^{[1]}(.,\mu) - \frac{K}{1-4\mu^2}\Phi T_\mu(P_2^{[1]}(.,\mu))\right.$$
$$\left. + \frac{2\mu K}{1-2\mu}\Phi H_{-1}P_2^{[1]}(.,\mu) - \chi_2 P_3\right)u_0.$$

Applying the operator h_{-1} to the last equation and taking into account the facts that

$$h_{-1}(xv) = -xh_{-1}v, \; v \in \mathcal{P}',$$
$$h_{-1}(h_{-1}v) = v, \; v \in \mathcal{P}'$$

and

$$T_\mu(P_2^{[1]}(.,\mu)) - (1 + 2\mu)H_{-1}P_2^{[1]}(.,\mu) = 2x,$$

we obtain

$$x\Phi(x)u_0 = h_{-1}(B(x)u_0), \tag{23}$$

where

$$B(x) = \frac{1 - 4\mu^2}{4\mu K}\left(\frac{1 + 2\mu}{\gamma_1}P_1 P_2^{[1]}(.,\mu) - \frac{K}{1 - 4\mu^2}\Phi T_\mu(P_2^{[1]}(.,\mu)) + \frac{2\mu K}{1 - 2\mu}\Phi H_{-1}P_2^{[1]}(.,\mu) - \chi_2 P_3\right). \tag{24}$$

To prove (1), we will show that in (23) polynomial B can be written as:

$$B(x) = x\widetilde{\Phi}(x),\ \widetilde{\Phi} \in \mathcal{P}. \tag{25}$$

Using the definition of the operator h_{-1}, we can easily see that

$$h_{-1}(f(x)v) = f(-x)h_{-1}v,\ f \in \mathcal{P},\ v \in \mathcal{P}'.$$

Then, (23) can be written as

$$x\Phi(x)u_0 = B(-x)h_{-1}u_0. \tag{26}$$

Application of h_{-1} to (26) gives

$$B(x)u_0 = -x\Phi(-x)h_{-1}u_0. \tag{27}$$

Multiplying (26) and (27) by $-x\Phi(-x)$ and $B(-x)$ respectively and subtracting the result, we obtain

$$-x^2\Phi(x)\Phi(-x)u_0 = B(x)B(-x)u_0.$$

By the regularity of u_0, we get

$$-x^2\Phi(x)\Phi(-x) = B(x)B(-x).$$

Hence,

$$B(0) = 0,\ \deg B = 1 + \deg \Phi.$$

Therefore, polynomial B can be written as in (25), with $\deg \widetilde{\Phi} = \deg \Phi$ and, then, (1) follows immediately from (23).

Now, multiplying (21) by x and taking into account (23), we get

$$xu_0^{[1]}(\mu) = \frac{K}{1-4\mu^2}(x\Phi(x) + 2\mu B(x))u_0. \tag{28}$$

Applying the operator h_{-1} to the last equation and using again (23), we obtain

$$xh_{-1}u_0^{[1]}(\mu) = -\frac{K}{1-4\mu^2}(B(x) + 2\mu x\Phi(x))u_0. \tag{29}$$

Multiplying (14) by x and taking into account (15), (28) and (29), we get

$$\frac{K}{1-4\mu^2}(x\Phi(x) + 2\mu B(x))T_\mu P_n^{[1]}(.,\mu) - x\Psi(x)P_n^{[1]}(.,\mu)+$$
$$-\frac{2\mu K}{1-2\mu}(x\Phi(x) + B(x))H_{-1}P_n^{[1]}(.,\mu) = -\chi_n x P_{n+1}(x),$$
$$\tag{30}$$

or, equivalently,

$$\frac{K}{1-4\mu^2}(x\Phi(x) + 2\mu B(x))T_\mu^2 P_{n+1} - x\Psi(x)T_\mu P_{n+1}+$$
$$-\frac{2\mu K}{1-2\mu}(x\Phi(x) + B(x))H_{-1}T_\mu P_{n+1} = \lambda_n x P_{n+1}(x),$$
$$\tag{31}$$

where

$$\lambda_n = -\chi_n \mu_{n+1}.$$

Substitution of (25) in (31) and simplification by x give (2).

$[(b) \Rightarrow (a)]$ Assume that equation (2) is fulfilled then using the definition of $\{P_n^{[1]}\}_{n\geq 0}$, we get

$$\frac{K}{1-4\mu^2}\left(\Phi(x) + 2\mu\widetilde{\Phi}(x)\right)(T_\mu P_n^{[1]})(x) - \Psi(x)P_n^{[1]}(x)+$$
$$-\frac{2\mu K}{1-2\mu}\left(\Phi(x) + \widetilde{\Phi}(x)\right)(H_{-1}P_n^{[1]})(x) = \frac{\lambda_n}{\mu_{n+1}}P_{n+1}(x).$$
$$\tag{32}$$

Therefore,

$$0 = \frac{\lambda_n}{\mu_{n+1}} < u_0, P_{n+1} > \; = \; < u_0, \frac{K}{1-4\mu^2}\left(\Phi(x) + 2\mu\widetilde{\Phi}(x)\right)T_\mu P_n^{[1]}$$
$$-\Psi(x)P_n^{[1]} - \frac{2\mu K}{1-2\mu}\left(\Phi(x) + \widetilde{\Phi}(x)\right)H_{-1}P_n^{[1]} >$$
$$= \; - < \frac{K}{1-4\mu^2}T_\mu\left((\Phi(x) + 2\mu\widetilde{\Phi}(x))u_0\right) + \Psi(x)u_0$$
$$-\frac{2\mu K}{1-2\mu}H_{-1}\left((\Phi(x) + \widetilde{\Phi}(x))\right)u_0, P_n^{[1]} >, n \geq 0.$$

Taking into account the fact that $\{P_n^{[1]}(x)\}_{n\geq 0}$ is a basis sequence, we obtain

$$\frac{K}{1-4\mu^2}T_\mu\Big((\Phi(x)+2\mu\widetilde{\Phi}(x))u_0\Big) + \Psi(x)u_0$$

$$-\frac{2\mu K}{1-2\mu}H_{-1}\Big((\Phi(x)+\widetilde{\Phi}(x))u_0\Big) = 0. \quad (33)$$

On the other hand, multiplying (23) by x^{-1} and applying the formula

$$x^{-1}(xv) = v - (v)_0\delta_0, \ v \in \mathcal{P}', \quad (34)$$

we get

$$\Phi u_0 = -h_{-1}(\widetilde{\Phi}u_0) + <u_0, \Phi(x)+\widetilde{\Phi}(x)>\delta_0,$$

or, equivalently,

$$\widetilde{\Phi}u_0 = -h_{-1}(\Phi u_0) + <u_0, \Phi(x)+\widetilde{\Phi}(x)>\delta_0. \quad (35)$$

Substituting (35) in (33) and taking account of (6), (7) and (8) we obtain

$$\frac{K}{1-4\mu^2}T_\mu\Big(\Phi u_0 - 2\mu h_{-1}(\Phi u_0)\Big) + \Psi u_0 = 0. \quad (36)$$

Therefore

$$0 = <\frac{K}{1-4\mu^2}T_\mu\Big(\Phi u_0 - 2\mu h_{-1}(\Phi u_0)\Big) + \Psi u_0, 1> = <u_0, \Psi>.$$

Taking into account that Ψ is a polynomial of degree one, we get

$$\Psi(x) = \lambda P_1(x), \ \lambda \neq 0.$$

So, (36) becomes

$$T_\mu\Big(\frac{K}{1-4\mu^2}\big(\Phi u_0 - 2\mu h_{-1}(\Phi u_0)\big)\Big) = -\lambda P_1(x)u_0.$$

But, according to Lemma 2.2,

$$T_\mu u_0^{[1]} = -\frac{1+2\mu}{\gamma_1}P_1(x)u_0.$$

Therefore,

$$T_\mu u_0^{[1]} = T_\mu\Big(\frac{\alpha K}{1-4\mu^2}\big(\Phi u_0 - 2\mu h_{-1}(\Phi u_0)\big)\Big),$$

where
$$\alpha = \frac{1 + 2\mu}{\lambda \gamma_1}.$$

Hence,
$$u_0^{[1]} = \frac{\alpha K}{1 - 4\mu^2} \big(\Phi u_0 - 2\mu h_{-1}(\Phi u_0) \big). \tag{37}$$

Taking into account (23), we get
$$x u_0^{[1]} = \frac{\alpha K}{1 - 4\mu^2} (x\Phi(x) + 2\mu B(x)) u_0, \tag{38}$$

$$x h_{-1} u_0^{[1]}(\mu) = -\frac{\alpha K}{1 - 4\mu^2} (B(x) + 2\mu x\Phi(x)) u_0. \tag{39}$$

Multiplying (32) by $\alpha x u_0$ and taking into account (37), (38), (39) and (23), we get
$$x T_\mu(P_n^{[1]} u_0^{[1]}) = \alpha \frac{\lambda_n}{\mu_{n+1}} x P_{n+1} u_0, \ n \geq 0. \tag{40}$$

Using (34), we obtain
$$T_\mu(P_n^{[1]} u_0^{[1]}) = \alpha \frac{\lambda_n}{\mu_{n+1}} P_{n+1} u_0, \ n \geq 0. \tag{41}$$

The regularity of $u_0^{[1]}$ comes from (3), (41) and the definition of $\{P_n^{[1]}\}_{n \geq 0}$: For $n \geq m$ we have
$$\begin{aligned}
< u_0^{[1]}, P_m^{[1]} P_n^{[1]} > &= -\frac{1}{\mu_{m+1}} < T_\mu(P_n^{[1]} u_0^{[1]}), P_{m+1} > \\
&= -\frac{\alpha \lambda_n}{\mu_{n+1} \mu_{m+1}} < P_{n+1} u_0, P_{m+1} > \\
&= -\frac{\alpha \lambda_n}{\mu_{n+1} \mu_{m+1}} < u_0, P_{m+1} P_{n+1} > \\
&= -\frac{\alpha \lambda_n}{\mu_{n+1}^2} r_n \delta_{n,m}.
\end{aligned}$$

Therefore $\{P_n\}_{n \geq 0}$ is a Dunkl-classical orthogonal polynomial sequence. $\quad\square$

Remark. The first author [3] showed that Dunkl-classical forms are characterized by (23) and (36) with the following condition:
$$\Psi'(0) + \frac{1}{2} \frac{K\Phi''(0)}{1 - 4\mu^2} (4\mu^2[n] - n) + \frac{1}{3} \frac{K B'''(0)}{(1 - 4\mu^2)} \mu([n] - n) \neq 0. \tag{42}$$

In this paper, following the previous proof, we can easily see that from (23) and (36) only (without condition (42)) we prove the Dunkl-classical character.

Corollary 3.2. *A symmetric MOPS $\{P_n\}_{n\geq 0}$ is Dunkl-classical if and only if there exist a complex number K, a sequence $(\lambda_n)_{n\geq 0}$ in $\mathbb{C}^*$ and two polynomials Φ (monic), and Ψ with $deg\Phi \leq 2$ and $deg\Psi = 1$ such that*

$$\frac{K}{1-4\mu^2}\Big(\Phi(x)-2\mu\Phi(-x)\Big)(T_\mu^2 P_{n+1})(x)-\Psi(x)(T_\mu P_{n+1})(x)+$$

$$-\frac{2\mu K}{1-2\mu}\Big(\Phi(x)-\Phi(-x)\Big)(H_{-1}T_\mu P_{n+1})(x)=\lambda_n P_{n+1}(x). \quad (43)$$

Proof. The sequence $\{P_n\}_{n\geq 0}$ is symmetric. Then, its corresponding form u_0 satisfies

$$u_0 = h_{-1}u_0.$$

Multiplication of the above equation by $x\Phi(x)$ gives (1) with

$$\widetilde{\Phi}(x) = -\Phi(-x). \quad (44)$$

Therefore, according to Theorem 1.1, $\{P_n\}_{n\geq 0}$ is a Dunkl-classical sequence if and only if it satisfies (2). Using (44), we can easily see that equation (2) is equivalent to (43). This completes the proof. $\square$

4. Example

In order to illustrate Theorem 1.1, we will consider the Dunkl-classical form $\widehat{\mathcal{G}}$ defined by:

$$\widehat{\mathcal{G}} = (x-1)\mathcal{G}(\alpha, \mu - \frac{1}{2}), \quad (45)$$

where $\mathcal{G}(\alpha, \mu - \frac{1}{2})$ is the generalized Gegenbauer form.

This form is regular for $\alpha \neq 0$, $\mu + \frac{1}{2} \neq -n$, $n \geq 1$ and satisfies [15]

$$T_\mu\Big((x^2-1)\widehat{\mathcal{G}}\Big) + \Big((-2\alpha - 2\mu - 3)x - (1+2\mu)\Big)\widehat{\mathcal{G}} = 0. \quad (46)$$

The last equation is equivalent to (36), with

$$\Phi(x) = (x+1)(x - \frac{1+2\mu}{1-2\mu}),$$

$$\Psi(x) = \frac{(2\alpha + 2\mu + 3)^2}{2\alpha + 2}(x + \frac{1+2\mu}{2\alpha + 2\mu + 3}),$$

$$K = -\frac{(1-2\mu)(2\alpha + 2\mu + 3)}{2\alpha + 2}.$$

Furthermore, using (45) and the fact that $\mathcal{G}(\alpha, \mu - \frac{1}{2})$ is symmetric, we can easily see that $\widehat{\mathcal{G}}$ satisfies (1) with

$$\widetilde{\Phi}(x) = (x+1)(x + \frac{1+2\mu}{1-2\mu}).$$

Therefore, according to Theorem 1.1 and its proof, the MOPS corresponding to $\widehat{\mathcal{G}}$ satisfies

$$\frac{K}{1-2\mu}(x^2-1)(T_\mu^2 P_{n+1})(x) - \Psi(x)(T_\mu P_{n+1})(x)$$

$$-\frac{4\mu K}{1-2\mu}x(x+1)(H_{-1}T_\mu P_{n+1})(x) = \lambda_n P_{n+1}(x), \quad (47)$$

where λ_n is a non-zero complex number.

Comparing the coefficients of higher degree of (47), we get

$$\lambda_n = \frac{2\alpha + 2\mu + 3}{2\alpha + 2}\Big(\mu_n - 2n - 2\alpha - 2\mu - 3\Big)\mu_{n+1}.$$

Notes that u_0 is a nonsymmetric form. Since the sequence $\{P_n\}_{n \geq 0}$ satisfies (4) with [15]:

$$\beta_n = a_{n+1} - a_n, \ n \geq 0,$$

where

$$a_{2n} = \frac{2n + 2\alpha + 2\mu + 1}{4n + 2\alpha + 2\mu + 1}, \quad a_{2n+1} = \frac{2n + 2\alpha + 2}{4n + 2\alpha + 2\mu + 3}, \qquad n \geq 0.$$

Acknowledgments

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