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THE GENERAL WEIERSTRASS n-SEMIGROUP OF A NON-CLASSICAL CURVE

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Abstract: We study the Weierstrass semigroup of n general points of a non-classical curve of genus $g < p^2$, where p is the characteristic of the base field.

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1. General n-Semigroup

Let X be a smooth and connected projective curve of genus $g \geq 4$ defined over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = p \geq 0$. We say that X is non-classical, if the gap sequence of a general $P \in X$ is not $1, \dots, g$. If either $p = 0$ or, say, $p > 2g-2$ (the lower bound may be improved), then each genus g curve is classical ([3, Theorem 11], [4, Theorem 15], [5]). For any $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ set $\|a\| = a_1 + \cdots + a_n$. Fix $P_1, \dots, P_n \in X$ such that $P_i \neq P_j$ for all $i \neq j$. Let $H(P_1, \dots, P_n) \subset \mathbb{N}^n$ be the set of all n-ples $(a_1, \dots, a_n) \in \mathbb{N}^n$ such that $\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)$ is spanned, i.e. such that there is a rational function on X defined over $\mathbb{K}$ and with $a_1P_1 + \cdots + a_nP_n$ as its divisor of poles. The set $H(P_1, \dots, P_n)$ is a semigroup (called the Weierstrass semigroup of $P_1, \dots, P_n$) for the componentwise addition $+ : \mathbb{N}^n \times \mathbb{N}^n \to \mathbb{N}^n$ and $G(P_1, \dots, P_n) := \mathbb{N}^n \setminus H(P_1, \dots, P_n)$. We look at this semigroup when $(P_1, \dots, P_n)$ is general in X^n. When either $p = 0$ or $p > 2g-2$, then $H(P_1, \dots, P_n) = \{a \in \mathbb{N}^n : \|a\| \geq g+1\}$

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(use [4, Theorem] to all $|\omega_X(\sum b_iP_i)|$, $b_i \geq 0$). This is not true (for any $n \geq 1$) if X is non-classical. We prove the following result.

Theorem 1. *Assume X non-classical of genus g with $2g-2 < p^2$. Fix an integer $n > 1$ and a general $(P_1,\dots,P_n) \in X^n$. Fix $a = (a_1,\dots,a_n) \in \mathbb{N}^n$ such that $a_1 \leq 2p-1$ and $a_i < p$ for all $i \geq 2$. We have $a \in H(P_1,\dots,P_n)$ if and only if one of the following conditions is satisfied:*

1. *$a_1 < p$ and $\|a\| \geq g+1$;*
2. *$a_1 = p$ and $a_h = 0$ for all $h \geq 2$;*
3. *$a_1 \geq p$ and $\|a\| \geq g+2$;*

Proof. For any $i = 1,\dots,n$ let e_i be the i-th coordinate vector of $\mathbb{N}^n$. Fix $i \in \{1,\dots,n\}$ and $b = (b_1,\dots,b_n) \in \mathbb{N}^n$ with $b_i = 0$. Let W be the linear system $|\omega_X(-\sum_h b_hP_h)|$ and let c be the dimension of the projective space W. Let $\{o_h\}_{h\geq 1}$ be the gap sequence of W at a general $P \in X$. If W is classical, then $o_h = h$ for $h = 1,\dots,c$. If X is not classical, then the first integer t such that $\dim(W(-tP)) = \dim(W)-2$ is a p-power ([2, 3.5]). Since $2g-2 < p^2$, this p-power is exactly p. The linear system W depends only on $\{P_1,\dots,P_n\}\setminus\{P_i\}$. Since P_i is general in X (after fixing all P_j, $j \neq i$), the linear system W has the same gap sequence at P and at P_i. We get that the first non-zero non-gap of W at P_i is either $c+1$ or it is a p-power. Since $p^2 > 2g-2$, in the latter case the first non-zero gap of P_i is p. By induction on n we get $h^0(\omega_X(-\sum_h a_hP_h)) = \max\{0, g-a_1-\cdots-a_n\}$ (and hence $a = (a_1,\dots a_n) \notin H(P_1,\dots,P_n)$) if $a_1 < p$. If W is classical and $c = g-1-\|b\|$, then we get that the first integer x with $xe_i + b \in H(P_1,\dots,P_n)$ is the first integer $x \geq 0$ such that $x + \|b\| \geq g+1$. If $a_1 < p$, then we get that $a \in H(P_1,\dots,P_n)$ if and only if $\|a\| \geq g+1$. Now assume $a_1 \geq p$. We have $h^0(\mathcal{O}_X(pP_1)) = 2$ and $\mathcal{O}_X(pP_1)$ is spanned. Riemann-Roch gives $h^1(\mathcal{O}_X(pP_1)) = g+1-p$. The integers between $p+1$ and $2p-1$ are not non-classical gaps, by the p-adic criterion ([6, Corollary 1.9]). We get $h^0(\mathcal{O}_X(a_1P_1)) = 2$ if $a_1 \leq g+1$ and $h^0(\mathcal{O}_X(a_1P_1)) = a_1+1-g$ if $a_1 \geq g+2$. Hence $ze_1 \in H(P_1)$.

(a) First assume $a = a_1e_1$. Since $a_1 \leq 2p-1$, we saw that $a_1 \in H(P_1)$ if and only if either $a_1 = p$ or $a_1 \geq g+2$.

(b) Now assume $a \neq a_1e_1$. Since $a_h < p$ for all $h > 1$ and $P_2,\dots,P_n$ are general, then either $h^0(\mathcal{O}_X(\sum_h a_hP_h)) = h^0(\mathcal{O}_X(a_1P_1))$ or $h^1(\mathcal{O}_X(\sum_h a_hP_h)) = 0$. In the former case $a \notin H(P_1,\dots,P_n)$. In the latter case we also get that no a_h, $h > 1$, is in the base locus of $\mathcal{O}_X(\sum_h a_hP_h)$ if and only if either $h^1(\mathcal{O}_X(a_1P_1)) = 0$ or $\|a\| - a_1 \geq h^1(\mathcal{O}_X(a_1P_1)) + 1$.

(b1) Assume $h^1(\mathcal{O}_X(a_1P_1)) > 0$ and $\|a\|-a_1 \ge h^1(\mathcal{O}_X(a_1P_1))+1$. If $a_1 \in H(P_1)$ (in step (b1) this is true if and only if $a_1 = p$), then $a \in H(P_1,\dots,P_n)$. Now assume $a_1 \notin H(P_1)$. Obviously $a \in H(P_1,\dots,P_n)$ if and only if P_1 is not in the base locus of $\mathcal{O}_X(\sum_i a_iP_i)$. By step (a) we get that $h^0(\mathcal{O}_X(a_1P_1)) = 2$, that $(a_1-p)P_1$ is the base scheme of $\mathcal{O}_X(a_1P_1)$ and that $h^1(\mathcal{O}_X((a_1-1)P_1)) = h^1(\mathcal{O}_X(a_1P_1))+1$. Since $2g-2 < p^2$ and $P_2,\dots,P_n$ are general, we get $h^1(\mathcal{O}_X((\sum a_iP_i - P_1)) = \max\{0, h^1(\mathcal{O}_X(a_1P_1))+1-\|a\|+a_1\}$. Hence $a \in H(P_1,\dots,P_n)$ if and only if $\|a\|-a_1 \ge h^1(\mathcal{O}_X(a_1P_1))+1 = g+2-a_1$.

(b2) Assume $h^1(\mathcal{O}_X(a_1P_1)) = 0$. If $a_1 \in H(P_1)$ (i.e. if $a_1 \ge g+2$), then $a \in H(P_1,\dots,P_n)$. Now assume $a_1 \notin H(P_1)$. We saw that $h^0(\mathcal{O}_X(a_1P)) = 2$. Since $h^1(\mathcal{O}_X(a_1P_1)) = 0$, we get $a_1 = g+1$ and $h^1(\mathcal{O}_X((a_1-1)P_1)) = 1$. We have $a \in H(P_1,\dots,P_n)$ if and only if $\|a\|-a_1 \ge 2$, i.e. $\|a\| \ge g+2$. □

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