THE GENERAL WEIERSTRASS $n$-SEMIGROUP OF A NON-CLASSICAL CURVE

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Abstract: We study the Weierstrass semigroup of $n$ general points of a non-classical curve of genus $g < p^2$, where $p$ is the characteristic of the base field.

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1. General $n$-Semigroup

Let $X$ be a smooth and connected projective curve of genus $g \geq 4$ defined over an algebraically closed field $\mathbb{K}$ with $\text{char}(\mathbb{K}) = p \geq 0$. We say that $X$ is non-classical, if the gap sequence of a general $P \in X$ is not $1, \ldots, g$. If either $p = 0$ or, say, $p > 2g - 2$ (the lower bound may be improved), then each genus $g$ curve is classical ([3, Theorem 11], [4, Theorem 15], [5]). For any $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ set $\|a\| = a_1 + \cdots + a_n$. Fix $P_1, \ldots, P_n \in X$ such that $P_i \neq P_j$ for all $i \neq j$. Let $H(P_1, \ldots, P_n) \subset \mathbb{N}^n$ be the set of all $n$-ples $(a_1, \ldots, a_n) \in \mathbb{N}^n$ such that $\mathcal{O}_X(a_1 P_1 + \cdots + a_n P_n)$ is spanned, i.e. such that there is a rational function on $X$ defined over $\mathbb{K}$ and with $a_1 P_1 + \cdots + a_n P_n$ as its divisor of poles. The set $H(P_1, \ldots, P_n)$ is a semigroup (called the Weierstrass semigroup of $P_1, \ldots, P_n$, for the componentwise addition $+: \mathbb{N}^n \times \mathbb{N}^n \to \mathbb{N}^n$ and $G(P_1, \ldots, P_n) := \mathbb{N}^n \setminus H(P_1, \ldots, P_n)$. We look at this semigroup when $(P_1, \ldots, P_n)$ is general in $X^n$. When either $p = 0$ or $p > 2g - 2$, then $H(P_1, \ldots, P_n) = \{a \in \mathbb{N}^n : \|a\| \geq g + 1\}$
(use [4, Theorem ] to all $|\omega_X(\sum b_i P_i)|$, $b_i \geq 0$). This is not true (for any $n \geq 1$) if $X$ is non-classical. We prove the following result.

**Theorem 1.** Assume $X$ non-classical of genus $g$ with $2g - 2 < p^2$. Fix an integer $n > 1$ and a general $(P_1, \ldots, P_n) \in X^n$. Fix $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ such that $a_1 \leq 2p - 1$ and $a_i < p$ for all $i \geq 2$. We have $a \in H(P_1, \ldots, P_n)$ if and only if one of the following conditions is satisfied:

1. $a_1 < p$ and $\|a\| \geq g + 1$;

2. $a_1 = p$ and $a_h = 0$ for all $h \geq 2$;

3. $a_1 \geq p$ and $\|a\| \geq g + 2$;

**Proof.** For any $i = 1, \ldots, n$ let $e_i$ be the i-th coordinate vector of $\mathbb{N}^n$. Fix $i \in \{1, \ldots, n\}$ and $b = (b_1, \ldots, b_n) \in \mathbb{N}^n$ with $b_i = 0$. Let $W$ be the linear system $|\omega_X(-\sum h b_i P_h)|$ and let $c$ be the dimension of the projective space $W$. Let $\{o_h\}_{h \geq 1}$ be the gap sequence of $W$ at a general $P \in X$. If $W$ is classical, then $o_h = h$ for $h = 1, \ldots, c$. If $X$ is not classical, then the first integer $t$ such that $\dim(W(-tP)) = \dim(W) - 2$ is a $p$-power ([2, 3.5]). Since $2g - 2 < p^2$, this $p$-power is exactly $p$. The linear system $W$ depends only on $\{P_1, \ldots, P_n\}\setminus\{P_i\}$. Since $P_i$ is general in $X$ (after fixing all $P_j$, $j \neq i$), the linear system $W$ has the same gap sequence at $P$ and at $P_i$. We get that the first non-zero non-gap of $W$ at $P_i$ is either $c + 1$ or it is a $p$-power. Since $p^2 > 2g - 2$, in the latter case the first non-zero gap of $P_i$ is $p$. By induction on $n$ we get $h^0(\omega_X(-\sum h a_h P_h)) = \max\{0, g - a_1 - \cdots - a_n\}$ (and hence $a = (a_1, \ldots, a_n) \notin H(P_1, \ldots, P_n)$ if $a_1 < p$. If $W$ is classical and $c = g - 1 - \|b\|$, then we get that the first integer $x$ with $x e_i + b \in H(P_1, \ldots, P_n)$ is the first integer $x \geq 0$ such that $x + \|b\| \geq g + 1$. If $a_1 < p$, then we get that $a \in H(P_1, \ldots, P_n)$ if and only if $\|a\| \geq g + 1$. Now assume $a_1 \geq p$. We have $h^0(\mathcal{O}_X(pP_i)) = 2$ and $\mathcal{O}_X(pP_1)$ is spanned. Riemann-Roch gives $h^1(\mathcal{O}_X(pP_1)) = g + 1 - p$. The integers between $p + 1$ and $2p - 1$ are non-classical gaps, by the $p$-adic criterion ([6, Corollary 1.9]). We get $h^0(\mathcal{O}_X(a_1 P_1)) = 2$ if $a_1 \leq g + 1$ and $h^0(\mathcal{O}_X(a_1 P_1)) = a_1 + 1 - g$ if $a_1 \geq g + 2$. Hence $ze_1 \in H(P_1)$.

(a) First assume $a = a_1 e_1$. Since $a_1 \leq 2p - 1$, we saw that $a_1 \in H(P_1)$ if and only if either $a_1 = p$ or $a_1 \geq g + 2$.

(b) Now assume $a \neq a_1 e_1$. Since $a_h < p$ for all $h > 1$ and $P_2, \ldots, P_n$ are general, then either $h^0(\mathcal{O}_X(\sum h a_h P_h)) = h^0(\mathcal{O}_X(a_1 P_1))$ or $h^1(\mathcal{O}_X(\sum h a_h P_h)) = 0$. In the former case $a \notin H(P_1, \ldots, P_n)$. In the latter case we also get that no $a_h$, $h > 1$, is in the base locus of $\mathcal{O}_X(\sum h a_h P_h)$ if and only if either $h^1(\mathcal{O}_X(a_1 P_1)) = 0$ or $\|a\| - a_1 \geq h^1(\mathcal{O}_X(a_1 P_1)) + 1$.  

(b1) Assume $h^1(\mathcal{O}_X(a_1P_1)) > 0$ and $\|a\| - a_1 \geq h^1(\mathcal{O}_X(a_1P_1)) + 1$. If $a_1 \in H(P_1)$ (in step (b1) this is true if and only if $a_1 = p$), then $a \in H(P_1, \ldots, P_n)$. Now assume $a_1 \notin H(P_1)$. Obviously $a \in H(P_1, \ldots, P_n)$ if and only if $P_1$ is not in the base locus of $\mathcal{O}_X(\sum_i a_iP_i)$. By step (a) we get that $h^0(\mathcal{O}_X(a_1P_1)) = 2$, that $(a_1 - p)P_1$ is the base scheme of $\mathcal{O}_X(a_1P_1)$ and that $h^1(\mathcal{O}_X((a_1 - 1)P_1)) = h^1(\mathcal{O}_X(a_1P_1)) + 1$. Since $2g - 2 < p^2$ and $P_2, \ldots, P_n$ are general, we get $h^1(\mathcal{O}_X((\sum a_iP_i - P_1)) = \max\{0, h^1(\mathcal{O}_X(a_1P_1)) + 1 - \|a\| + a_1\}$. Hence $a \in H(P_1, \ldots, P_n)$ if and only if $\|a\| - a_1 \geq h^1(\mathcal{O}_X(a_1P_1)) + 1 = g + 2 - a_1$.

(b2) Assume $h^1(\mathcal{O}_X(a_1P_1)) = 0$. If $a_1 \in H(P_1)$ (i.e. if $a_1 \geq g + 2$), then $a \in H(P_1, \ldots, P_n)$. Now assume $a_1 \notin H(P_1)$. We saw that $h^0(\mathcal{O}_X(a_1P_1)) = 2$. Since $h^1(\mathcal{O}_X(a_1P_1)) = 0$, we get $a_1 = g + 1$ and $h^1(\mathcal{O}_X((a_1 - 1)P_1)) = 1$. We have $a \in H(P_1, \ldots, P_n)$ if and only if $\|a\| - a_1 \geq 2$, i.e. $\|a\| \geq g + 2$.

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References


