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THE LAPLACE TRANSFORM OF THE HURWITZ-LERCH ZETA-FUNCTION

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Abstract: Formula for Laplace transform of product of the Hurwitz and Lerch zeta-function is obtained.

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1. Introduction

Integral transforms (Laplace, Forier, Mellin) play an important role in the analytic number theory. In the works of Titchmarsh[7], Atkinson[1], Ivič[3][4], Jutila[5], Lukkarinen[6] it has been pointed out the connection between moments $I_k(I)$ of $|\zeta\left(\frac{1}{2}+it\right)|^{2k}$ on $[0,T]$ and Laplace transform $L_k\left(\frac{1}{2}\right)$ or modified Mellin transform $Z_k(\sigma+it)$, $k=1,2,\ldots$. Balčiunas and Laurincikas[2] investigated the Laplace transform of the Dirichlet L-function with principal character.

Our aim is to study the Laplace transform for product of the Hurwitz and Lerch functions.

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The classical Laplace transform $L_f(s)$, $s=\sigma+it$, of the function $f(x)$ is defined by

$$L_f(s)=\int\limits_0^\infty f(x)e^{-sx}dx$$

provided that the integral exists for $\sigma>\sigma_0$ with some σ_0.

Let $0<\alpha,\beta\le 1$. We define for $\Re s>1$:

$$\zeta(s;\alpha,0)=\sum_{n=0}^\infty\frac{1}{(n+\alpha)^s}$$

$$\zeta(s;0,\beta)=\sum_{n=1}^\infty\frac{e^{2\pi in\beta}}{(n)^s}.$$

The function $\zeta(s;\alpha,0)$ and $\zeta(s;0,\beta)$ call respectively the Hurwitz and Lerch zeta-functions. It is well-known that $\zeta(s;\alpha,0)$ has meromorphic continuation to the whole complex plane with simple pole $s=1$ with residue 1, and $\zeta(s;0,\beta)$ is an entire function if $\beta\neq 1$.

The Lerch zeta-function produces two functions

$$S(s;\beta):=\sum_{n=1}^\infty\frac{\sin(2\pi n\beta)}{n^s},\qquad C(s;\beta):=\sum_{n=1}^\infty\frac{\cos(2\pi n\beta)}{n^s},$$

which are the entire functions if $\beta\neq 1$.

These functions are connected by the Hurwitz relation

$$\begin{aligned}\zeta(s;\alpha,0)=&\frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\times\\ &\times\left(\sin\frac{\pi s}{2}\cdot C(a-1s;\alpha)+\cos\frac{2\pi}{2}\cdot S(1-s;\alpha)\right),\end{aligned}\tag{1}$$

$(\Re s<1)$.

Let us ℓ, q be positive integers, $\ell<q$, $(\ell,q)=1$, and let (ℓ_1,ℓ_2) runs complete system of solutions of the congruence

$$\ell_1\ell_2\equiv\ell\pmod q,\ 0<\ell_1,\ell_2<q.$$

Define the function

$$F(x)=\sum_{\substack{\ell_1,\ell_2=1\\ \ell_1\ell_2\equiv\ell\pmod q}}^{q}\zeta\left(\frac12+ix;\frac{\ell_1}{q},0\right)\zeta\left(\frac12-ix;0,\frac{\ell_2}{q}\right).\tag{2}$$

Consider the Laplace transform for $F(x)$

$$L_F(s) := \int_0^\infty F(x)e^{-sx}dx. \tag{3}$$

The main result of this paper is a proof the following statement.

Theorem 1. *Let ℓ, q be the positive numbers, $(\ell, q) = 1$. Then for the Laplace transform of the function $F(x)$ defined by (2), we have*

$$L_F(s) = \pi i \left(e^{\frac{is}{2}} + e^{-\frac{is}{2}}\right) c_0(\ell, q) +$$
$$+ e^{-\frac{is}{2}} \sum_{m,n=1}^{\infty} K(m, -n\ell; q) \exp\left(-2\pi imn\ell^{-is}\right) + \lambda_\ell(s),$$

where $K(m, -n\ell; q)$ is the Kloosterman sum and $\lambda_\ell(s)$ is analytic in the strip $|\Re s| < \pi$, and, for $|\Re s| \le \pi - \varepsilon$, $\varepsilon > 0$ is arbitrary small, the estimate $\lambda_\ell(s) \ll (1 + |s|)^{-1}$ holds. Moreover,

$$c_0(\ell, q) = \frac{2}{\pi} \sum_{\ell_1\ell_2 \equiv \ell \pmod q} \left(\frac{1}{2} - \frac{\ell_1}{q}\right)\left(\frac{1}{2} - \frac{\ell_2}{q}\right).$$

2. Notation and Preliminary Results

We will use the following notations. Denote by $\mathbb{C}$ the set of complex numbers; for $z \in \mathbb{C}$ we write $\exp(z) = e^z$. $\sum\limits_{(\ell)}$ means the summation over all ℓ_1, ℓ_2 under condition $0 < \ell_1, \ell_2 < q$, $\ell_1\ell_1 \equiv \ell \pmod q$. Symbol $\int\limits_{(a)}$ always denotes integration an line $\Re z = a$, $-\infty < \Im z < +\infty$. The Vinogradov and Landau symbols (respectively, " $\ll$ " and "O") are equivalent. $\Gamma(z)$ signifies the gamma-function. $\tau(n)$ is a divisor function.

We will apply some auxiliary lemmas.

Lemma 1. *For $\frac{1}{2} \le \Re s \le 2$, $|\Im s| \ge 3$, we have*

$$\zeta\left(s; \frac{\ell}{q}, 0\right), \ \zeta\left(s; 0, \frac{\ell}{q}\right) \ll q^c\left(|\Im s| + 3\right), \ \ (0 < c < 2),$$

uniformly in ℓ, q, $|\Im s|$.

Lemma 2. *The functions e^{-y} and $\Gamma(s)$ form the Mellin pair, i.e. for $\Re s > 0$, $b > 0$,*

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx, \;\; e^{-y} = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \Gamma(s) y^{-s} ds.$$

The assertions of these lemmas are well-known (see, [7]).

Lemma 3. *On the line $\Re s = \frac{1}{2}$ we have*

$$\begin{aligned} &\sum_{(\ell)} \zeta\left(z; \frac{\ell_1}{q}, 0\right) \zeta\left(1 - z; 0, \frac{\ell_2}{q}\right) = \\ &= \sum_{(\ell)} \zeta\left(\overline{z}; \frac{\ell_1}{q}, 0\right) \zeta\left(1 - \overline{z}; 0, \frac{\ell_2}{q}\right). \end{aligned} \tag{4}$$

Indeed, applying the Hurwitz relation and taking into account symmetry over ℓ_1, ℓ_2 of the left-hand side of equality (4) we obtain the assertion of Lemma 3 immediately.

We stand the following notations

$$\begin{aligned} I_1(\ell, q) &:= \frac{e^{-\frac{is}{2}}}{2} \int_{(\frac{1}{2})} \sum_{(\ell)} \frac{\zeta\left(z; \frac{\ell_1}{q}, 0\right) C\left(1 - z; \frac{\ell_2}{q}\right)}{\sin \frac{\pi z}{2}} e^{-iz\left(\frac{\pi}{2} - s\right)} dz, \\ I_2(\ell, q) &:= \frac{e^{-\frac{is}{2}}}{2i} \int_{(\frac{1}{2})} \sum_{(\ell)} \frac{\zeta\left(z; \frac{\ell_1}{q}, 0\right) S\left(1 - z; \frac{\ell_2}{q}\right)}{\cos \frac{\pi z}{2}} e^{-iz\left(\frac{\pi}{2} - s\right)} dz, \end{aligned} \tag{5}$$

$$\begin{aligned} L_{F_1}(s) &:= \int_0^\infty \sum_{(\ell)} \zeta\left(\frac{1}{2} + ix; \frac{\ell_1}{q}, 0\right) C\left(\frac{1}{2} - ix; \frac{\ell_2}{q}\right) e^{-sx} dx, \\ L_{F_2}(s) &:= \int_0^\infty \sum_{(\ell)} \zeta\left(\frac{1}{2} + ix; \frac{\ell_1}{q}, 0\right) S\left(\frac{1}{2} - ix; \frac{\ell_2}{q}\right) e^{-sx} dx. \end{aligned} \tag{6}$$

Lemma 4. *There are exist the functions $\lambda_\ell^{(1)}(s)$, $\lambda_\ell^{(2)}(s)$ that are analytic* *in the strip* $|\Re s| < \pi$ *and* $\lambda_\ell^{(j)}(s) \ll (1+|s|)^{-1}$, $j = 1, 2$, *such that*

$$L_{F_1}(s) = \frac{e^{\frac{is}{2}}}{2} \int\limits_{(\frac{1}{2})} \sum_{(\ell)} \frac{\zeta\left(z; \frac{\ell_1}{q}, 0\right) C\left(1 - z; \frac{\ell_2}{q}\right)}{\sin \frac{\pi z}{2}} e^{-iz\left(\frac{\pi}{2} - s\right)} dz + \lambda_\ell^{(1)}(s), \tag{7}$$

$$L_{F_2}(s) = \frac{e^{\frac{is}{2}}}{2i} \int\limits_{(\frac{1}{2})} \sum_{(\ell)} \frac{\zeta\left(z; \frac{\ell_1}{q}, 0\right) S\left(1 - z; \frac{\ell_2}{q}\right)}{\cos \frac{\pi z}{2}} e^{-iz\left(\frac{\pi}{2} - s\right)} dz + \lambda_\ell^{(2)}(s). \tag{8}$$

Proof. By the identity

$$1 = \frac{\exp\left(\frac{\pi i}{4} - \frac{\pi x}{2}\right)}{\exp\left(\frac{\pi i}{4} - \frac{\pi x}{2}\right) - \exp\left(-\frac{\pi i}{4} + \frac{\pi x}{2}\right)} - \frac{\exp\left(-\frac{\pi i}{4} + \frac{\pi x}{2}\right)}{\exp\left(\frac{\pi i}{4} - \frac{\pi x}{2}\right) - \exp\left(-\frac{\pi i}{4} + \frac{\pi x}{2}\right)},$$

we have

$$L_{F_1}(s) = I_1(\ell, q) + I_{11}(s) - I_{12}(s), \tag{9}$$

where

$$I_{11} = I_{11}(s) = \int\limits_0^\infty \sum_{(\ell)} \frac{\zeta\left(\frac{1}{2} + ix; \frac{\ell_1}{q}, 0\right) C\left(\frac{1}{2} - ix; \frac{\ell_2}{q}\right)}{\exp\left(\frac{\pi i}{4} - \frac{\pi x}{2}\right) - \exp\left(-\frac{\pi i}{4} + \frac{\pi x}{2}\right)} e^{-xs} \times \exp\left(\frac{\pi i}{2} - \frac{\pi x}{2}\right) dx, \tag{10}$$

$$I_{12} = I_{12}(s) = \int\limits_0^\infty \sum_{(\ell)} \frac{\zeta\left(\frac{1}{2} + ix; \frac{\ell_1}{q}, 0\right) C\left(\frac{1}{2} - ix; \frac{\ell_2}{q}\right)}{\exp\left(\frac{\pi i}{4} + \frac{\pi x}{2}\right) - \exp\left(-\frac{\pi i}{4} - \frac{\pi x}{2}\right)} e^{xs} \times \exp\left(-\frac{\pi i}{2} - \frac{\pi x}{2}\right) dx. \tag{11}$$

The integrands in I_{11} and I_{12} involve two multipliers

$$\zeta\left(\frac{1}{2} + ix; \frac{\ell_1}{q}, 0\right) C\left(\frac{1}{2} - ix; \frac{\ell_2}{q}\right)$$

and

$$e^{\mp xs}\frac{\exp\left(\mp\frac{\pi i}{4}-\frac{\pi x}{2}\right)}{\exp\left(\frac{\pi i}{4}\mp\frac{\pi x}{2}\right)-\exp\left(-\frac{\pi i}{4}\pm\frac{\pi x}{2}\right)},$$

and moreover, the first multiplier increases as power of x (precisely $\ll x^{\frac{1}{3}}$) and the second multiplier decreases if $x\to\infty$ as exponential function (precisely $e^{-(\pi\pm s)x}$, $|\Re s|<\pi$).

So, the integrals I_{11} and I_{12} are convergent in the strip $|\Re s|\le\pi-\varepsilon$ for any positive $\varepsilon\le\frac{1}{2}$. Hence, we accomplish that $I_{11}-I_{12}$ defines analytic function in the strip $|\Re s|<\pi$.

Furthermore,

$$|I_{1j}|\le\left|\int_0^1\right|+\left|\int_1^\infty\right|\ll 1+\left|\int_1^\infty G_j(x)e^{-x(\pi\pm s)}dx\right|,\ (j=1,2),\tag{12}$$

where

$$G_1(x)=\sum_{(\ell)}\zeta\left(\frac{1}{2}+ix;\frac{\ell_1}{q},0\right)C\left(\frac{1}{2}-ix;\frac{\ell_2}{q}\right)\cdot\left(1-ie^{-\pi x}\right)^{-1},$$
$$G_2(x)=\sum_{(\ell)}\zeta\left(\frac{1}{2}+ix;\frac{\ell_1}{q},0\right)C\left(\frac{1}{2}-ix;\frac{\ell_2}{q}\right)\cdot\left(1+ie^{-\pi x}\right)^{-1}.$$

Since, the function $C\left(z;\frac{\ell_2}{q}\right)$ in the region $\frac{1}{2}\le|\Re z|\le 2$, $|\Im z|\ge 3$, has the same estimate as $\zeta\left(z;0,\frac{\ell_2}{q}\right)$, then the integral Cauchy formula gives that

$$\left(\zeta\left(z;\frac{\ell_1}{q},0\right)\right)',\ \left(C\left(z;\frac{\ell_2}{q}\right)\right)'\ll q^c|\Im z|.\tag{13}$$

And hence, computing the integrals $\int_1^\infty G_j(x)e^{-x(\pi\mp s)}dx$, $j=1,2$ by integrating by parts, we derive:

$$\begin{aligned}\int_1^\infty G_j(x)e^{-x(\pi\mp s)}dx&=\frac{e^{-x(\pi\mp s)}}{-\pi\pm s}G_j(x)\Bigg|_1^\infty-\\&-\int_1^\infty\frac{e^{-x(\pi\mp s)}}{-\pi\pm s}dG_j(x)\ll(1+|s|)^{-1}.\end{aligned}\tag{14}$$

Hence, we have that

$$\lambda_\ell^{(1)} = I_{11}(s) - I_{12}(s)$$

is analytic in the strip $|\Re(s)| < \pi$, and

$$\lambda_\ell^{(1)} \ll (1 + |s|)^{-1}$$

From (9)-(13) the relation (7) and properties of $\lambda_\ell^{(1)}$ follow. In the same way we can prove the equality (8).

This completes the proof of Lemma 4. □

Corollary. *In the strip $|\Re s| < \pi$ the following relation*

$$L_F(s) = I_1(\ell, q) + iI_2(\ell, q) + \lambda_\ell(s)$$

where $\lambda_\ell(s)$ is analytic function, $\lambda_\ell(s) \ll (1 + |s|)^{-1}$, holds.

3. Main Result

In this section we obtain the main result of our paper.

We start with calculation of the integrals $I_j(\ell, q)$, $j = 1, 2$.

Note that the subintegral functions in $I_1(\ell, q)$ and $I_2(\ell, q)$ have simple poles at the points $s = 0$ and $s = -1$, respectively.

Thus, denoting by $H_1(z, s)$ and $H_2(z, s)$ the subintegral functions in $I_1(\ell, q)$ and $I_2(\ell, q)$ (see, (5)), we obtain

$$I_1(\ell, q) = \frac{e^{-\frac{is}{2}}}{2} \left\{ \int\limits_{(-\frac{3}{2})} H_1(z, s)dz + 2\pi i \operatorname*{res}_{z=0} H_1(z, s) \right\}. \tag{15}$$

But we have

$$\operatorname*{res}_{z=0} H_1(z, s) = \frac{2}{\pi} \sum_{(\ell)} \zeta\left(0; \frac{\ell_1}{q}, 0\right) \cdot \sum_{m=1}^{\infty} \frac{\cos\left(2\pi n \frac{\ell_2}{q}\right)}{n} =$$

$$= \frac{2}{\pi} \sum_{(\ell)} \left(\frac{1}{2} - \frac{\ell_1}{q}\right)\left(\frac{1}{2} - \frac{\ell_2}{q}\right) = c_0(\ell_q),$$

because the Hurwitz relation gives

$$\zeta(0;u,0)=\frac{1}{2}-u,\qquad \zeta(1;0,u)=\frac{1}{2}-u,\ \ if u\neq 1.$$

Hence

$$I_1(\ell,q)=e^{-\frac{is}{2}}\pi i c_0(\ell,q)+\frac{e^{-\frac{is}{2}}}{2}\int\limits_{(-\frac{3}{2})} H_1(z,s)dz. \tag{16}$$

Further, by the Hurwitz relation, we deduce

$$\int\limits_{(-\frac{3}{2})} H_1(z,s)dz=\sum_{(\ell)}\sum_{m,n=1}^{\infty}[\mathbf{L}+\mathbf{R}]=I_1^{(1)}(s)+I_1^{(2)}(s), \tag{17}$$

where

$$\mathbf{L}=\cos\left(2\pi n\frac{\ell_1}{q}\right)\cos\left(2\pi m\frac{\ell_2}{q}\right)\int\limits_{(-\frac{3}{2})}\frac{2\Gamma(1-z)}{(2\pi mn)^{1-z}}e^{-iz(\frac{\pi}{2}-s)}dz,$$

$$\mathbf{R}=\sin\left(2\pi\frac{n\ell_1}{q}\right)\cos\left(2\pi\frac{m\ell_2}{q}\right)\int\limits_{(-\frac{3}{2})}\frac{2\Gamma(1-z)}{(2\pi mn)^{1-z}}\cdot\operatorname{ctg}\frac{\pi z}{2}\cdot e^{-iz(\frac{\pi}{2}-s)}dz,$$

say.

Putting $1-z=w$ and applying Lemma 2, we find

$$I_1^{(1)}(s)=\sum_{m,n=1}^{\infty}\sum_{(\ell)}\cos\left(2\pi n\frac{\ell_1}{q}\right)\cos 2\pi m\frac{\ell_2}{q}\exp\left(-2\pi mne^{-is}\right). \tag{18}$$

For $I_1^{(2)}(s)$ we take into account that with every pairs (ℓ_1,ℓ_2) involving in the sum $\sum\limits_{(\ell)}$ as well as pair $(-\ell_1,-\ell_2)$ involves. Hence,

$$\sum_{\ell_1\ell_2\equiv 1 \pmod q}\sin\left(2\pi n\frac{\ell_1}{q}\right)\cos\left(2\pi m\frac{\ell_2}{q}\right)=0,$$

and, consequently, $I_1^{(2)}=0$.

So, we obtain

$$\begin{aligned} I_1(\ell,q) = e^{-\frac{is}{2}} \sum_{m,n=1}^{\infty} \sum_{(\ell)} \cos\left(2\pi m \frac{\ell_1}{q}\right) \times \\ \times \cos\left(2n\frac{\ell_2}{q}\right) \exp\left(-2\pi i m n e^{-is}\right) + \\ + \pi i e^{-\frac{is}{2}} c_0(\ell,q). \end{aligned} \tag{19}$$

Similarly as above, we obtain

$$I_2(\ell,q) = \frac{e^{-\frac{is}{2}}}{2i} \left\{ 2\pi i \operatorname*{res}_{z=-1} H_2(z,s) + \int\limits_{(-\frac{3}{2})} H_2(z,s)dz \right\}. \tag{20}$$

Obviously, that

$$\operatorname*{res}_{z=-1} H_2(z,s) = -c_0(\ell,q) e^{-i\left(\frac{\pi}{2}-s\right)} = i e^{is} c_0(\ell,q). \tag{21}$$

Moreover

$$\begin{aligned} \int\limits_{(-\frac{3}{2})} H_2(z,s)dz = \sum_{m,n=1}^{\infty} \sum_{(\ell)} \Bigg(\sin\left(2\pi n \frac{\ell_1}{q}\right) \sin\left(2\pi m \frac{\ell_2}{q}\right) \times \\ \times \int\limits_{(-\frac{3}{2})} \frac{2\Gamma(1-z)}{(2\pi m n)^{1-z}} e^{-iz\left(\frac{\pi}{2}-s\right)} dz + \\ + \cos\left(2\pi n \frac{\ell_1}{q}\right) \sin\left(2\pi m \frac{\ell_2}{q}\right) \times \\ \times \int\limits_{(-\frac{3}{2})} \frac{2\Gamma(1-z)}{(2\pi m n)^{1-z}} e^{-iz\left(\frac{\pi}{2}-s\right)} dz \Bigg) = \\ = \sum_{m,n=1}^{\infty} \sum_{(\ell)} \sin\left(2\pi m \frac{\ell_1}{q}\right) \sin\left(2\pi n \frac{\ell_2}{q}\right) \exp\left(-2\pi i m n e^{-is}\right). \end{aligned} \tag{22}$$

Thus, by combining (20)-(22), we have established the following representation

$$\begin{aligned} I_2(\ell,q) = \pi e^{\frac{is}{2}} c_0(\ell,q) + \\ + \frac{e^{-\frac{is}{2}}}{2i} \sum_{m,n=1}^{\infty} \sum_{(\ell)} \sin\left(2\pi m \frac{\ell_1}{q}\right) \sin\left(2\pi n \frac{\ell_2}{q}\right) \exp\left(-2\pi i m n e^{-is}\right). \end{aligned} \tag{23}$$

Nowm from (19), (23) we have

$$L_F(s) = \pi i \left(e^{-\frac{is}{2}} + e^{\frac{is}{2}}\right) c_0(\ell, q) + \\ + e^{-\frac{is}{2}} \sum_{m,n=1}^{\infty} \sum_{(\ell)} \cos\left(2\pi \frac{m\ell_1 - n\ell_2}{q}\right) \exp\left(-2\pi i m n e^{-is}\right) + \lambda_\ell(s). \tag{24}$$

Since, $|\exp(-2\pi i m n e^{-is})| = \exp(-2\pi m n \sin(\pi - \Re s))$, we conclude that the series

$$\sum_{m,n=1}^{\infty} \sum_{(\ell)} \cos\left(2\pi \frac{m\ell_1 - n\ell_2}{q}\right) \exp\left(-2\pi i m n e^{-is}\right)$$

is an absolutely convergent series in $|\Re(s)| \leq \tau - \varepsilon$, and consequently $L_F(s)$ is an analytical function in this strip.

Further, we have

$$\sum_{(\ell)} \cos\left(2\pi \frac{m\ell_1 - n\ell_2}{q}\right) = \\ = \frac{1}{2} \sum_{(\ell)} \left(e^{2\pi i \frac{m\ell_1 - n\ell_2}{2}} + e^{2\pi i \frac{n\ell_2 - m\ell_1}{2}}\right) = \\ = \sum_{\substack{\ell_1, \ell_2 \pmod q \\ \ell_1 \ell_2 \equiv \ell \pmod q}} e^{2\pi i \frac{m\ell_1 - n\ell_2}{q}} = K(m, -n\ell; q), \tag{25}$$

where $K(a, b; q)$ is the classical Kloosterman sum, for which the bound of Weil[9]

$$K(a, b; q) \ll (gcd(a, b, q))^{\frac{1}{2}} q^{\frac{1}{2}} \tau(q)$$

is well-known.

Now, by the Corollary from Lemma 4 it follows that we completed the proof of main theorem.

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