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EXACT SOLUTIONS AND WRONSKIAN FORMULATION FOR A NEW FORM OF THE (3 + 1)-DIMENSIONAL BKP EQUATION

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Abstract: The Hirota's bilinear method is used to construct multiple-soliton solutions for a new form of the (3 + 1)-dimensional BKP equation. The resulting solutions involve generic phase shifts and wave frequencies containing some existing choices. By taking the long wave limit approach and extending the real parameters into complex parameters, a few classes of limit solutions and complexitons are generated respectively. Finally, a Wronskian formulation is presented.

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1. Introduction

The investigation of the exact solutions of nonlinear evolution equations (NLEEs) plays an important role in the study of nonlinear physical phenomena. Some methods have been developed to obtain the exact solutions, such as the Bäcklund transformation method [1], the Hirota's bilinear method [2], the Painlevé analysis [3] and the Wronskian technique [4, 5]. It is well known that the bilinear method first proposed by Hirota provides us with a comprehensive approach to construct exact solutions [2]. Once a NLEE is written in bilinear form, we are

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able to derive systematically particular solutions including multiple-soliton solutions. Based on the Hirota bilinear method, a variety of powerful methods have been developed. For example, Ablowitz and Satsuma studied a relationship between soliton and rational solutions of a certain class of NLEEs and developed a method to get rational solutions by taking a long wave limit on soliton solutions obtained by the Hirota bilinear method [6]. Another example, by applying the simplified Hirota bilinear method and extending the real parameters into complex parameters, Wazwaz et al. have obtained nonsingular complexiton solutions for two higher-dimensional nonlinear equations [7]. Recently, the multiple exp-function method [8] has been proposed as a generalization of Hirota's perturbation scheme [2]. Ma et al. have applied the multiple exp-function algorithm to construct multiple wave solutions to the (3+1)-dimensional generalized KP and BKP equation [9]. Additionally, according to the Hirota bilinear form, the Wronskian technique is an efficient method to establish multi-soliton solutions for NLEEs in Wronskian form [10, 11].

The aim of this paper is to study a new form of the (3 + 1)-dimensional BKP equation presented by

$$u_{xxxy} + \alpha(u_x u_y)_x + (u_x + u_y + u_z)_t - (u_{xx} + u_{yy} + u_{zz}) = 0, \tag{1.1}$$

where α is a non-zero parameter. First Wazwaz [12] has applied the simplified Hereman-Nuseir form to obtain one and two soliton solutions for Eq. (1.1), Second he has develop specific constraints that guarantee the existence of multiple soliton solutions for Eq. (1.1). In this paper, we would like to use the Hirota's bilinear method, the long wave limit approach and the Wronskian technique to shed light on diversity of exact solutions to Eq. (1.1).

The framework of this paper is as follows. In Section 2, we apply the Hirota's bilinear method to construct multiple soliton solutions to Eq. (1.1). In addition, by taking the long wave limit approach and extending the real parameters into complex parameters, rational solutions, complexitons, positons and negatons are generated. In Section 3, based on the Wronskian formulation of the KdV equation [13], a broad set of sufficient conditions consisting of systems of linear partial differential equations is presented which guarantees that the Wronskian determinant solves Eq. (1.1) in the bilinear form. Our conclusion and remarks are given in Section 4.

2. Soliton and Limit Solutions

Under the dependent variable transformation

$$u = \frac{6}{\alpha}(\ln f)_x, \tag{2.1}$$

equation (1.1) is mapped into a Hirota bilinear equation

$$(D_x^3 D_y + D_y D_t + D_x D_t + D_z D_t - D_x^2 - D_y^2 - D_z^2) f \cdot f = 0, \tag{2.2}$$

where D_x, D_y, D_z and D_t are Hirota bilinear differential operators [2]. Equivalently, we have

$$\begin{aligned}(f_{xxxy} + f_{yt} + f_{xt} + f_{zt} - f_{xx} - f_{yy} - f_{zz})f - f_{xxx}f_y - 3f_x f_{xxy} \\ +3f_{xx}f_{xy} - f_x f_t - f_y f_t - f_z f_t + f_x^2 + f_y^2 + f_z^2 = 0.\end{aligned} \tag{2.3}$$

2.1. One-Soliton and Rational Solutions

Following the Hirota's bilinear method, the one-soliton solutions of Eq. (1.1) read

$$u = \frac{6}{\alpha}(\ln f)_x = \frac{6k_1 e^{k_1 x + l_1 y + m_1 z - w_1 t + \xi_1^0}}{\alpha(1 + e^{k_1 x + l_1 y + m_1 z - w_1 t + \xi_1^0})}, \tag{2.4}$$

with

$$f_1 = 1 + e^{\xi_1},$$

where $\xi_1 = k_1 x + l_1 y + m_1 z - w_1 t + \xi_1^0$, k_1, l_1, m_1 are constants, and the dispersion relation being satisfied

$$w_1 = \frac{k_1^3 l_1 - k_1^2 - l_1^2 - m_1^2}{k_1 + l_1 + m_1}.$$

The fact that one can recover rational solutions relies on our freedom of choosing the arbitrary phase constants ξ_1^0. For example, in (2.4), if we set $e^{\xi_1^0} = -1, l_1 = \gamma_1 k_1, m_1 = \gamma_2 k_1$, and take the limit of $k_1 \to 0$, then we find

$$f_1 = -k_1\theta_1 + o(k_1^2), \tag{2.5}$$

where $\theta_1 = x + \gamma_1 y + \gamma_2 z + \dfrac{1 + \gamma_1^2 + \gamma_2^2}{1 + \gamma_1 + \gamma_2} t$. Since u is given (2.1), f_1 is equivalent to θ_1. Thus, the first class of rational solutions to Eq. (1.1) is

$$u = \frac{6}{\alpha\theta_1}. \tag{2.6}$$

2.2. Two-Soliton, Rational and Complexiton Solutions

To determine the two-soliton solutions, we follow and use the auxiliary function

$$f_2 = 1 + e^{\xi_1} + e^{\xi_2} + a_{12}e^{\xi_1+\xi_2}, \tag{2.7}$$

where

$$\xi_i = k_i x + l_i y + m_i z - w_i t + \xi_i^0, i = 1, 2. \tag{2.8}$$

Applying Hirota's perturbation scheme leads to

$$a_{12} = -\frac{b_{12}}{c_{12}}, \tag{2.9}$$

where

$$\begin{aligned} b_{12} &= (k_1 - k_2)^3(l_1 - l_2) + (l_1 - l_2)(w_2 - w_1) + (k_1 - k_2)(w_2 - w_1) \\ &\quad +(m_1 - m_2)(w_2 - w_1) - (k_1 - k_2)^2 - (l_1 - l_2)^2 - (m_1 - m_2)^2, \\ c_{12} &= (k_1 + k_2)^3(l_1 + l_2) + (l_1 + l_2)(-w_2 - w_1) + (k_1 + k_2)(-w_2 - w_1) \\ &\quad +(m_1 + m_2)(-w_2 - w_1) - (k_1 + k_2)^2 - (l_1 + l_2)^2 - (m_1 + m_2)^2, \end{aligned}$$

and

$$w_i = \frac{k_i^3 l_i - k_i^2 - l_i^2 - m_i^2}{k_i + l_i + m_i}. \tag{2.10}$$

It is easy to find that the phase shift a_{12} depends on all coefficients k_i, l_i and m_i, $i = 1, 2$, of the spatial variables x, y and z, respectively. This means that two-soliton solutions are determined for k_i, l_i and m_i being free parameters. We will explore some interesting reductions of this kind of phase shifts later when discussing rational and complexiton solutions.

Firstly, if we take the choices

$$l_i = \beta_1 k_i,\ m_i = \beta_2 k_i,\ i = 1, 2, \tag{2.11}$$

where β_1 and β_2 are constants. This yields the wave frequencies

$$w_i = \frac{\beta_1 k_i^3 - (1 + {\beta_1}^2 + {\beta_2}^2)k_i}{1 + \beta_1 + \beta_2},\ i = 1, 2, \tag{2.12}$$

and the phase shift

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}. \tag{2.13}$$

In the case of (2.11), a special reduction

$$k_i = l_i = m_i,\ i = 1, 2,$$

presents the choice for the two-soliton solutions discussed in Ref. [12].

The long wave limit approach can be applied to f_2. Under the choices (2.11), choosing $e^{\xi_1^0} = -e^{\xi_2^0} = \dfrac{k_1 + k_2}{k_1 - k_2}$ and taking the limit $k_i \to 0\,(i = 1, 2)$ in (2.7), we get

$$f_2 = -\frac{1}{6}k_1 k_2(k_1 + k_2)\theta_2 + o(k^4), \tag{2.14}$$

where

$$\theta_2 = \frac{12\beta_1}{1+\beta_1+\beta_2}t + \left(x + \beta_1 y + \beta_2 z + \frac{1+{\beta_1}^2+{\beta_2}^2}{1+\beta_1+\beta_2}t\right)^3.$$

The constant $-\dfrac{1}{6}k_1 k_2(k_1 + k_2)$ can be neglected in (2.14) as before. Thus, the second class of rational solutions is

$$u = \frac{18}{\alpha\theta_2}\left(x + \beta_1 y + \beta_2 z + \frac{1+{\beta_1}^2+{\beta_2}^2}{1+\beta_1+\beta_2}t\right)^2. \tag{2.15}$$

By virtue of the two-soliton solutions with the phase shift given in (2.13), we next discuss and illustrate complexiton and positon solutions of Eq. (1.1).

Case 1 Assuming

$$k_1 = a + \mathrm{i}b,\ k_2 = a - \mathrm{i}b,\ (a, b \in R), \tag{2.16}$$

and noticing expressions (2.8),(2.11),(2.12) and (2.13), we have

$$\xi_1 = \Omega_1 + \mathrm{i}\psi_1 + \xi_1^0,\ \xi_2 = \Omega_1 - \mathrm{i}\psi_1 + \xi_2^0,\ a_{12} = -\frac{b^2}{a^2}, \tag{2.17}$$

where

$$\Omega_1 = ax + \beta_1 ay + \beta_2 az - \frac{\beta_1(a^3 - 3ab^2)t - (1+{\beta_1}^2+{\beta_2}^2)at}{1+\beta_1+\beta_2},$$

$$\psi_1 = bx + \beta_1 by + \beta_2 bz + \frac{\beta_1(b^3 - 3a^2b)t + (1+{\beta_1}^2+{\beta_2}^2)bt}{1+\beta_1+\beta_2}.$$

Substituting (2.17) into (2.7) and using the choice $e^{\xi_1^0} = e^{\xi_2^0} = \dfrac{a}{b}$, we can now compute

$$f_2 = 1 + \frac{2a}{b}e^{\Omega_1}\cos\psi_1 - e^{2\Omega_1}. \tag{2.18}$$

Consequently, we obtain a class of explicit solution of Eq. (1.1) as follows:

$$u = \frac{12(a^2 e^{\Omega_1}\cos\psi_1 - abe^{\Omega_1}\sin\psi_1 - abe^{2\Omega_1})}{\alpha(b + 2ae^{\Omega_1}\cos\psi_1 - be^{2\Omega_1})}, \tag{2.19}$$

the parameters a and b are suitably chosen.

When $ab \neq 0$, expression (2.19) is a class of complexiton solutions of Eq. (1.1), i.e., solution involving two kinds of transcendental functions—exponential functions and trigonometric functions.

Furthermore, in (2.18), f_2 is also written as

$$f_2 = \frac{2a}{b}\big(e^{\Omega_1}\cos\psi_1 + \frac{b}{2a}(1 - e^{2\Omega_1})\big), \tag{2.20}$$

where the multiplicative factor $\frac{2a}{b}$ as a constant does not affect the solution (2.1) and can be neglected. Setting the limit $a \to 0$ and employing the transformation (2.1), a class of limit solutions of Eq. (1.1) is given by

$$u = \frac{6}{\alpha}\big(\ln(\cos\widehat{\psi}_1 - b\widehat{\Omega}_1)\big)_x = \frac{-6b(\sin\widehat{\psi}_1 + 1)}{\alpha(\cos\widehat{\psi}_1 - b\widehat{\Omega}_1)}, \tag{2.21}$$

where

$$\widehat{\Omega}_1 = x + \beta_1 y + \beta_2 z + \frac{3\beta_1 b^2 t + (1 + {\beta_1}^2 + {\beta_2}^2)t}{1 + \beta_1 + \beta_2},$$

$$\widehat{\psi}_1 = bx + \beta_1 by + \beta_2 bz + \frac{\beta_1 b^3 t + (1 + {\beta_1}^2 + {\beta_2}^2)bt}{1 + \beta_1 + \beta_2}.$$

We easily see that (2.21) involves only one kind of transcendental functions—trigonometric functions, and so it gives a class of positon solutions.

Case 2 The same limiting procedure can be applied to the case

$$k_1 = a + \mathrm{i}b,\ k_2 = -a + \mathrm{i}b,\ (a, b \in R). \tag{2.22}$$

Substituting (2.22) into (2.13) and taking $e^{\xi_1^0} = \frac{a\mathrm{i}}{b}$, $e^{\xi_2^0} = -\frac{a\mathrm{i}}{b}$ in (2.7), we find

$$f_2 = \frac{2b\mathrm{i}}{a}\big(e^{\mathrm{i}\psi_1}\sinh\Omega_1 + \frac{a}{2b\mathrm{i}}(1 - e^{2\mathrm{i}\psi_1})\big), \tag{2.23}$$

where Ω_1, ψ_1 are defined by (2.17). Taking the limit $b \to 0$ and using the transformation (2.1), the resulting negaton solutions read

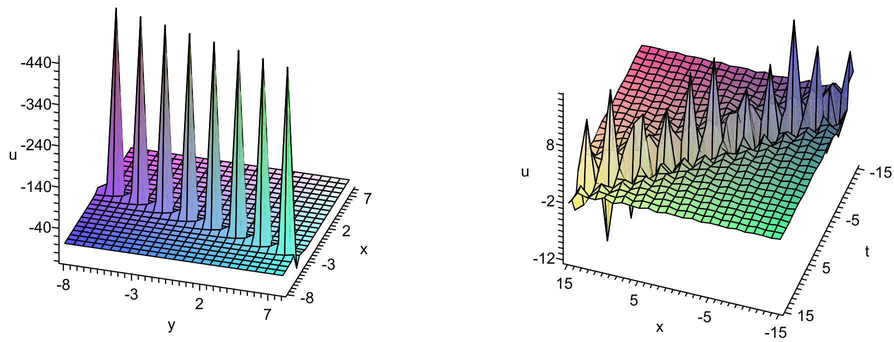
$$u = \frac{6}{\alpha}\big(\ln(\sinh\widetilde{\Omega}_1 - a\widetilde{\psi}_1)\big)_x = \frac{6a(\cosh\widetilde{\Omega}_1 - 1)}{\alpha(\sinh\widetilde{\Omega}_1 - a\widetilde{\psi}_1)}. \tag{2.24}$$

where

$$\widetilde{\Omega}_1 = ax + \beta_1 ay + \beta_2 az - \frac{\beta_1 a^3 t - (1+{\beta_1}^2+{\beta_2}^2)at}{1+\beta_1+\beta_2},$$

$$\widetilde{\psi}_1 = x + \beta_1 y + \beta_2 z - \frac{3\beta_1 a^2 t - (1+{\beta_1}^2+{\beta_2}^2)t}{1+\beta_1+\beta_2}.$$

The graphs of complexiton and positon solutions with specific values being chosen for the parameters are presented in Figure 1, which show some singularities of the solutions.



(a) (b)

Figure 1: (a) The plot of the complexiton solution given by (2.19) with parameters: $a = 2$, $b = 1$, $\beta_1 = \frac{1}{3}$, $\beta_2 = 1$, $\alpha = 1$, $z = 0$, $t = 2$. (b) The plot of positon solution given by (2.21) with parameters: $b = -2$, $\beta_1 = -\frac{1}{3}$, $\beta_2 = 1$, $\alpha = 2$, $z = 0$, $y = 2$.

Secondly, let us take the choices

$$l_1 = \beta_1 k_1,\ l_2 = -\beta_1 k_2,\ m_1 = \beta_2 k_1,\ m_2 = \delta k_2, \tag{2.25}$$

where β_1, β_2 and δ are constants. This leads to the wave frequencies

$$w_1 = \frac{\beta_1 k_1^3 - (1+{\beta_1}^2+{\beta_2}^2)k_1}{1+\beta_1+\beta_2},\ w_2 = \frac{-\beta_1 k_2^3 - (1+{\beta_1}^2+\delta^2)k_2}{1-\beta_1+\delta}, \tag{2.26}$$

and the phase shift $a_{12} = 1$.

Furthermore, choosing $e^{\xi_1^0} = e^{\xi_2^0} = -1$ and taking the limit $k_i \to 0\,(i = 1, 2)$ in (2.7), we have

$$f_2 = \theta_3\theta_4, \tag{2.27}$$

where

$$\theta_3 = \left(x + \beta_1 y + \beta_2 z + \frac{(1 + {\beta_1}^2 + {\beta_2}^2)t}{1 + \beta_1 + \beta_2}\right), \tag{2.28}$$

$$\theta_4 = \left(x - \beta_1 y + \delta z + \frac{(1 + {\beta_1}^2 + \delta^2)t}{1 - \beta_1 + \delta}\right).$$

Then the third class of rational solutions is determined by

$$u = \frac{6}{\alpha\theta_3\theta_4}\left(2x + (\beta_2 + \delta)z + \frac{(1 + {\beta_1}^2 + {\beta_2}^2)t}{1 + \beta_1 + \beta_2} + \frac{(1 + {\beta_1}^2 + \delta^2)t}{1 - \beta_1 + \delta}\right). \tag{2.29}$$

2.3. Three-Soliton and Rational Solutions

We now consider three-soliton solutions

$$u = \frac{6}{\alpha}(\ln f_3)_x,$$

with f_3 being defined by

$$f_3 = 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + a_{12}e^{\xi_1+\xi_2} + a_{23}e^{\xi_2+\xi_3} + a_{13}e^{\xi_1+\xi_3} + a_{123}e^{\xi_1+\xi_2+\xi_3}, \tag{2.30}$$

$$a_{123} = a_{12}a_{13}a_{23},$$

where

$$\xi_i = k_i x + l_i y + m_i z - w_i t + \xi_i^0, 1 \le i \le 3. \tag{2.31}$$

We would like to search for three-soliton solutions with the selection of

$$w_i = \frac{k_i^3 l_i - k_i^2 - l_i^2 - m_i^2}{k_i + l_i + m_i}, 1 \le i \le 3 \tag{2.32}$$

and

$$a_{ij} = -\frac{b_{ij}}{c_{ij}}, 1 \le i < j \le 3, \tag{2.33}$$

where

$$b_{ij} = (k_i - k_j)^3(l_i - l_j) + (l_i - l_j)(w_j - w_i) + (k_i - k_j)(w_j - w_i)$$
$$+(m_i - m_j)(w_j - w_i) - (k_i - k_j)^2 - (l_i - l_j)^2 - (m_i - m_j)^2,$$
$$c_{ij} = (k_i + k_j)^3(l_i + l_j) + (l_i + l_j)(-w_j - w_i) + (k_i + k_j)(-w_j - w_i)$$
$$+(m_i + m_j)(-w_j - w_i) - (k_i + k_j)^2 - (l_i + l_j)^2 - (m_i + m_j)^2.$$

Since the Eq. (1.1) is not completely integrable [12], we need to determine conditions to generated three-soliton solutions.

The first class of three-soliton solutions is associated with the choices

$$l_i = \beta_1 k_i,\ m_i = \beta_2 k_i,\ 1 \le i \le 3, \tag{2.34}$$

where β_1 and β_2 are constants. This yields the wave frequencies

$$w_i = \frac{\beta_1 k_i^3 - (1 + {\beta_1}^2 + {\beta_2}^2)k_i}{1 + \beta_1 + \beta_2},\ 1 \le i \le 3, \tag{2.35}$$

and the phase shifts

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2},\ 1 \le i < j \le 3, \tag{2.36}$$

$$a_{123} = \frac{(k_1 - k_2)^2 (k_2 - k_3)^2 (k_1 - k_3)^2}{(k_1 + k_2)^2 (k_2 + k_3)^2 (k_1 + k_3)^2}.$$

Taking

$$e^{\xi_1^0} = \frac{(k_1 + k_2)(k_1 + k_3)}{(k_1 - k_2)(k_1 - k_3)},\ e^{\xi_2^0} = \frac{(k_2 + k_1)(k_2 + k_3)}{(k_2 - k_1)(k_2 - k_3)},$$

$$e^{\xi_3^0} = \frac{(k_3 + k_1)(k_3 + k_2)}{(k_3 - k_1)(k_3 - k_2)},$$

and passing to the limit $k_i \to 0\,(1 \le i \le 3\ )$ in (2.30), we can obtain a class of limit solutions

$$f_3 = \theta_3^6 + \frac{60\beta_1 t}{1 + \beta_1 + \beta_2}\theta_3^3 - 720\left(\frac{\beta_1 t}{1 + \beta_1 + \beta_2}\right)^2, \tag{2.37}$$

where θ_3 is defined by (2.28). Thus, we also obtain a class of rational solutions of Eq. (1.1) by the transformation (2.1), which is different from the above three types of rational solutions.

The second class of three-soliton solutions is associated with the choices

$$l_1 = \beta_1 k_1,\ l_2 = -\beta_1 k_2,\ l_3 = \beta_1 k_3,\ m_1 = \beta_2 k_1,\ m_2 = \delta k_2,\ m_3 = \beta_2 k_3, \tag{2.38}$$

where β_1, β_2 and δ are constants, which leads to the wave frequencies

$$w_i = \frac{\beta_1 k_i^3 - (1 + {\beta_1}^2 + {\beta_2}^2)k_i}{1 + \beta_1 + \beta_2},\ i = 1, 3,$$

$$w_2 = \frac{-\beta_1 k_2^3 - (1+{\beta_1}^2+\delta^2)k_2}{1-\beta_1+\delta}, \tag{2.39}$$

and the phase shifts

$$a_{12}=1,\, a_{13}=\frac{(k_1-k_3)^2}{(k_1+k_3)^2},\, a_{23}=1. \tag{2.40}$$

Taking

$$e^{\xi_1^0}=\frac{(k_1+k_3)}{(k_1-k_3)},\, e^{\xi_2^0}=-1,\, e^{\xi_3^0}=\frac{(k_3+k_1)}{(k_3-k_1)},$$

and passing to the limit $k_i \to 0\,(1 \le i \le 3\ )$ in (2.30), we have

$$f_3 = \left(x-\beta_1 y+\delta z+\frac{1+{\beta_1}^2+\delta^2}{1-\beta_1+\delta}t\right)\left(\frac{\beta_1 t}{1+\beta_1+\beta_2}\right). \tag{2.41}$$

The corresponding class of rational solutions is similiar to the above rational solutions (2.6).

3. Wronskian Formulation of Eq. (1.1)

Next, applying the Wronskian technique, we will present the Wronskian formulation to Eq. (1.1). To use this technique, we adopt the compact notation introduced by Freeman and Nimmo [4, 5], they set

$$W(\phi_1,\phi_2,\cdots,\phi_N)=\begin{vmatrix}\phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)}\\ \phi_2 & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)}\\ \vdots & \vdots & \vdots & \vdots\\ \phi_N & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)}\end{vmatrix}$$

$$=|0,1,\cdots,N-1|=|\widehat{N-1}|, \tag{3.1}$$

where $\phi_i^{(j)}=\dfrac{\partial^j\phi_i}{\partial x^j}$.

First we state useful results about the KdV equation

$$u_t-6uu_x+u_{xxx}=0, \tag{3.2}$$

which has the bilinear form

$$(D_xD_t+D_x^4)f\cdot f=0, \tag{3.3}$$

through the transformation $u = -2(\ln f)_{xx}$. Equivalently, we have

$$f f_{xt} - f_x f_t + f f_{xxxx} - 4 f_x f_{xxx} + 3 f_{xx}^2 = 0. \tag{3.4}$$

In Ref. [13], the following Theorem 1 has been proved:

Theorem 1. *Assume that a group of functions $\phi_i = \phi_i(x,t)$, $(1 \le i \le N)$ satisfies the two sets of conditions*

$$-\phi_{i,xx} = \sum_{j=1}^{N} \lambda_{ij}(t)\phi_j,\ 1 \le i \le N, \tag{3.5a}$$

$$\phi_{i,t} = -4\phi_{i,xxx} + \zeta(t)\phi_i,\ 1 \le i \le N, \tag{3.5b}$$

simultaneously, where $\lambda_{ij}(t)$ are arbitrary differentiable real functions of t and $\zeta(t)$ is an arbitrary continuous real function of t. Then $f = |\widehat{N-1}|$ defined by (3.1) solves the bilinear KdV equation (3.3).

Using the Theorem 1, we would like to present a broad set of sufficient conditions which make the Wronskian determinant a solution to the bilinear BKP equation (2.2).

Theorem 2. *Assume that a group of functions $\phi_i = \phi_i(x,y,z,t)$, $(1 \le i \le N)$ satisfies the following conditions:*

$$\phi_{i,xx} = \sum_{j=1}^{N} \lambda_{ij}(t)\phi_j, \tag{3.6a}$$

$$\phi_{i,y} = \beta_1 \phi_{i,x}, \tag{3.6b}$$

$$\phi_{i,z} = \beta_2 \phi_{i,x}, \tag{3.6c}$$

$$\phi_{i,t} = \beta_3 \phi_{i,x} + \beta_4 \phi_{i,xxx} + \zeta(t)\phi_i, \tag{3.6d}$$

with

$$\beta_3 = \frac{1+\beta_1^2+\beta_2^2}{1+\beta_1+\beta_2}\ , \beta_4 = \frac{-4\beta_1}{1+\beta_1+\beta_2}, \tag{3.6e}$$

where β_1, β_2 are two real constants not to be zero, $1+\beta_1+\beta_2 \ne 0$, $\lambda_{ij}(t)$ are arbitrary differentiable real functions of t, and $\zeta(t)$ is an arbitrary continuous real function of t. Then the Wronskian determinant $f = |\widehat{N-1}|$ defined by (3.1) solves the bilinear Eq.(2.2).

Proof. By using the conditions (3.6), we can now compute that

$$f_y = \beta_1 f_x,\ f_{xy} = \beta_1 f_{xx},\ f_{xxy} = \beta_1 f_{xxx},\ f_{xxxy} = \beta_1 f_{xxxx}, f_{yy} = {\beta_1}^2 f_{xx},$$
$$f_z = \beta_2 f_x,\ f_{zz} = {\beta_2}^2 f_{xx},\ f_t = \beta_3 f_x + \beta_4 f_{(\widetilde{3x})} + N\zeta f,$$
$$f_{xt} = \beta_3 f_{xx} + \beta_4 f_{(\widetilde{3x})x} + N\zeta f_x,\ f_{yt} = \beta_1\beta_3 f_{xx} + \beta_1\beta_4 f_{(\widetilde{3x})x} + \beta_1 N\zeta f_x,$$
$$f_{zt} = \beta_2\beta_3 f_{xx} + \beta_2\beta_4 f_{(\widetilde{3x})x} + \beta_2 N\zeta f_x,$$

where

$$f_{(\widetilde{3x})} = |\widehat{N-4}, N-2, N-1, N| - |\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|,$$

$$f_{(\widetilde{3x})x} = |\widehat{N-5}, N-3, N-2, N-1, N| - |\widehat{N-3}, N, N+1| + |\widehat{N-2}, N+3|.$$

Substituting these derivatives into (2.3) and using the condition (3.6e), we have

$$\begin{aligned}&(f_{xxxy} + f_{yt} + f_{xt} + f_{zt} - f_{xx} - f_{yy} - f_{zz})f - f_{xxx}f_y - 3f_x f_{xxy}\\ &+3f_{xx}f_{xy} - f_x f_t - f_y f_t - f_z f_t + f_x^2 + f_y^2 + f_z^2\\ =\ &\beta_1(-4ff_{(\widetilde{3x})x} + 4f_x f_{(\widetilde{3x})} + ff_{xxxx} - 4f_x f_{xxx} + 3f_{xx}^2).\end{aligned} \tag{3.7}$$

By virtue of Theorem 1, expression (3.7) is equal to zero. Therefore, we have shown that $f = |\widehat{N-1}|$ solves the bilinear Eq.(2.2). □

4. Conclusion and Remarks

In summary, by using the Hirota's bilinear method and the long wave limit approach, we have presented multi-soliton, rational, positon, negaton and complexiton solutions. Although we need to make additional constraint for the parameters to guarantee the existence of multi-soliton solutions, the newly presented generic phase shifts and wave frequencies in this paper lead to the richness and diversity of exact solutions for the considered Eq. (1.1).

On the other hand, based on the Wronskian formulation of the KdV equation, we have established a Wronskian formulation for Eq. (1.1), with all generating functions for matrix entries satisfying a linear system of partial differential equations involving free parameters. Of course, the obtained Wronskian solution formulas of Eq. (1.1) allow us to construct rational solutions. However, our rational solutions obtained by taking the long wave limit approach contain the rational solutions generated from Wronskian formulation.

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