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GROUP PURSUIT WITH PHASE CONSTRAINTS IN RECURRENT PONTRYAGIN'S EXAMPLE

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Abstract: We consider a generalized non-stationary Pontryagin's example under the same dynamic and inertial capabilities players and phase constraints on the state of the runaway. Boundary of phase constraints is not a "line of death" for the evader. Sufficient conditions for the capture of a group of pursuers one evader are obtained in this article.

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1. Introduction

Differential games of two players, first considered in the book of Isaacs [1], now present wide field of research [2]-[8]. Methods were developed for solving various classes of game problems: Isaaks' method, based on the analysis of a certain partial differential equation and its characteristics; Krasovskii's method of extremal guidans; Pontryagin's method and others. A natural generalization of differential games pursuit-evasion of two persons are games with a group of

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pursuers and one or several evaders [9]-[12]. These games are interesting from the theoretical point of view, they cannot be solved by theory for two-person games. One reason for this is, that the union sets of the reachability of all pursuers and the union of all target sets are sets, is non-convex and, furthermore, is not connected. On the other hand, there are some applications of these games to the problems of motion vehicles, collisions of avoidance for ships and others. In this case, the problem becomes much more complicated if the players on the state of the phase constraints are imposed. Among the large number of papers devoted to the problem of group pursuit with phase constraints, we mention the works [13]-[20]. In [13]-[17] there are the various tasks simple pursuit with phase constraints. The general problem of evasion with state constraints is considered in [18]. Sufficient mustache capture in stationary Pontryagin's example obtained in [19]. Terms of capture in an unsteady Pontryagin's example with phase constraints under other assumptions are presented in [20].

In this paper we consider the problem of persecution of a group of pursuers one evader at equal opportunities and inertial dynamic players. It is assumed that the evader in the game did not leave a convex polyhedral set, terminal sets - convex kompaty. Provided that some of the functions defined by the initial conditions and parameters of the game are recurrent, we obtain sufficient conditions for the solvability of persecution.

2. Statement of the Problem

In space $R^k (k \geq 2)$ we consider differential game $\Gamma(n, D)$ $n+1$ objects: n pursuers $P_1, \ldots, P_n$ and evader E.

The law of motion of each of the pursuers P_i has the form

$$x_i^{(l)} + a_1(t)x_i^{(l-1)} + a_2(t)x_i^{(l-2)} + \ldots + a_l(t)x_i = u_i, \quad u_i \in V, \tag{1}$$

The law of motion of evader E has the form

$$y^{(l)} + a_1(t)y^{(l-1)} + a_2 y^{(l-2)} + \ldots + a_l(t)y = v, \quad v \in V, \tag{2}$$

where $x_i, y_j, u_i, v_j \in R^k$, the functions $a_1(t), a_2(t), \ldots, a_l(t)$ are continuous $[t_0, \infty)$, V — convex compact. At $t = t_0$ the initial conditions are set

$$x_i^{(q)}(t_0) = x_i^q, \quad y(q)(t_0) = y^q, \tag{3}$$

where $x_i^0 - y^0 \notin M_i$ for all i, M_i is convex compact set. Here $i \in I = \{1, 2, \ldots, n\}$, $q = \{0, 1, \ldots, l-1\}$.

It is further assumed, that evader E in the course of the game does not leave a convex set

$$D = \{y : y \in R^k, (p_c, y) \leq \mu_c, c = 1, 2, \ldots, r\},$$

with non-empty interior, where (a, b) is the scalar product of vectors a and b, $p_1, \ldots, p_r$ are the unit vectors R^k, $\mu_1, \ldots, \mu_r$ are real numbers.

Instead (1)–(3), we consider to solve the equation

$$z_i^{(l)} + a_1(t)z_i^{(l-1)} + a_2(t)z_i^{(l-2)} + \ldots + a_l(t)z_i = u_i - v, \tag{4}$$

with initial conditions

$$z_i^{(q)}(t_0) = z_i^q = x_i^q - y^q. \tag{5}$$

Let $\varphi_q(t, s)$ $(t \geq s \geq t_0)$ be the solution of the equation

$$\omega^{(l)} + a_1(t)\omega^{(l-1)} + a_2(t)\omega^{(l-2)} + \ldots + a_l(t)\omega = 0,$$

with the initial conditions

$$\omega(s) = 0, \ldots, \ \omega^{(q-1)}(s) = 0, \ \omega^{(q)}(s) = 1,$$

$$\omega^{(q+1)}(s) = 0, \ldots, \ \omega^{(l-1)}(s) = 0.$$

Suppose further

$$\xi_i(t) = \varphi_0(t, t_0)z_i^0 + \varphi_1(t, t_0)z_i^1 + \ldots + \varphi_{l-1}(t, t_0)z_i^{l-1},$$

$$\eta(t) = \varphi_0(t, t_0)y^0 + \varphi_1(t, t_0)y^1 + \ldots + \varphi_{l-1}(t, t_0)y^{l-1}.$$

Assume that $\xi_i(t) \notin M_i$ for all $i, t \geq t_0$, for if $\xi_i(\tau) \in M_i$ for some i, τ, then the pursuer P_i catches the evader E, supposing $u_i(t) = v(t)$.

Definition 1. We will say that a quasi-strategy U_i of pursuer P_i, is given if a mapping $U_i(t, z^0, v_t(\cdot))$ is defined that assigns to the initial state z^0, the point in time t, and an arbitrary past history of control $v_t(\cdot)$ of the evader E such that $y(t) \in D$ for all $t \geq t_0$, a measurable function $u_i(t)$ with values in V.

Definition 2. There is a capture in the game $\Gamma(n, D)$ if there is a time $T(z^0)$ and quasi-strategies $U_1(t, z^0, v_t(\cdot)), \ldots, U_n(t, z^0, v_t(\cdot))$ of pursuers $P_1, \ldots, P_n$ such that, for any measurable function $v(\cdot)$, $v(t) \in V$, $y(t) \in D, t \in [t_0, T(z^0)]$ there are numbers $s \in I$ and a time $\tau \in [t_0, T(z^0)]$, such that $z_s(\tau) \in M_s$.

Definition 3. (see [21]) The function $f : R^1 \to R^k$ is called recurrent if, for any $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ such that for any $a, t \in R^1$, there exists $\tau(t) \in [a, a + T(\varepsilon)]$ for which it holds that

$$\|f(t + \tau(t)) - f(t)\| < \varepsilon.$$

Definition 4. The function $f : [t_0, \infty) \to R^k$ is called recurrent on $[t_0, \infty)$ if there exists a recurrent function $F : R^1 \to R^k$ such that $f(t) = F(t)$ for all $t \in [t_0, \infty)$.

We define the functions

$$r(t,s) = \begin{cases} 1, \text{ if } \varphi_{l-1}(t,s) \geq 0, \\ -1, \text{ if } \varphi_{l-1}(t,s) < 0 \end{cases} \qquad (t_0 \leq s \leq t),$$

$$\lambda(v, \mu, b_i) = \sup\{\lambda \mid -\lambda\mu(b_i - M_i) \cap (V - v) \neq \emptyset\},$$

$$G_i(t, v(\cdot), b_i) = \int\limits_{t_0}^{t} |\varphi_{l-1}(t,s)| \lambda(v(s), r(t,s), b_i)) ds,$$

$$F(t) = \int\limits_{t_0}^{t} |\varphi_{l-1}(t,s) ds|.$$

3. Capture Conditions

Assumptions 1. *1) functions $\xi_i(t)$ are recurrent on $[t_0, \infty)$;*

2) function $\eta(t)$ is bounded on $[t_0, \infty)$;

3) $\lim\limits_{t \to \infty} F(t) = \infty$.

Assumptions 2. *There are times $\tau_i^0 \geq t_0$, are positive numbers ε, δ such that:*

1) for all i and for all $h_i \in S_\varepsilon(\xi_i(\tau_i^0))$ is satisfied $h_i \notin M_i$;

2) for all $h_i \in S_\varepsilon(\xi_i(\tau_i^0))$ inequality

$$\min_v \max\Big\{\max_i \lambda(v, +1, h_i), \max_j (p_j, v)\Big\} \geq \delta,$$

$$\min_v \max\bigl\{\max_i \lambda(v,-1,h_i), \max_j(-p_j,v)\bigr\}\geq \delta,$$

where $S_\varepsilon(a) = \{x \in R^k \mid \|x-a\| \leq \varepsilon\}$.

Denote

$$h = (h_1, h_2, \ldots, h_n), \quad S = S_\varepsilon(\xi_1(\tau_1^0)) \times S_\varepsilon(\xi_2(\tau_2^0)) \times \ldots \times S_\varepsilon(\xi_n(\tau_n^0)).$$

Lemma 1. *Suppose that assumptions 1, 2,* $r = 1$*. Then there time* $T \geq t_0$ *such that for any admissible control* $v(\cdot)$ *evader* E*, any* $h \in S$ *there exists a number* $m \in I$ *for which* $G_m(T, v(\cdot), h_m) \geq 1$.

Proof. Since control $v(t)$ of the evader E is admissible, then for all $t \geq t_0$

$$(p_1, y(t)) \leq \mu(t) = \mu_1 - (p_1, \eta(t)).$$

We define the set

$$\begin{aligned}
T^+(t) =& \{\tau : \tau \in [t_0, t], \varphi_{l-1}(t,\tau) \geq 0\},\\
T^-(t) =& \{\tau : \tau \in [t_0, t], \varphi_{l-1}(t,\tau) < 0\},\\
T_1^+(t) =& \{\tau : \tau \in T^+(t), (p_1, v(\tau)) \geq \delta\},\\
T_2^+(t) =& \{\tau : \tau \in T^+(t), (p_1, v(\tau)) < \delta\},\\
T_1^-(t) =& \{\tau : \tau \in T^-(t), (-p_1, v(\tau)) \geq \delta\},\\
T_2^-(t) =& \{\tau : \tau \in T^-(t), (-p_1, v(\tau)) < \delta\}.
\end{aligned}$$

Then

$$\begin{aligned}
\int_{t_0}^{t} \varphi_{l-1}(t,s)(p_1, v(s))\,ds =& \int_{T^+(t)} \varphi_{l-1}(t,s)(p_1, v(s))\,ds\\
&+ \int_{T^-(t)} (-\varphi_{l-1}(t,s))(-p_1, v(s))\,ds\\
=& \int_{T_1^+(t)} \varphi_{l-1}(t,s)(p_1, v(s))\,ds + \int_{T_2^+(t)} \varphi_{l-1}(t,s)(p_1, v(s))\,ds\\
&+ \int_{T_1^-(t)} (-\varphi_{l-1}(t,s))(-p_1, v(s))\,ds
\end{aligned}$$

$$
\begin{aligned}
&+ \int\limits_{T_2^-(t)} (-\varphi_{l-1}(t,s))(-p_1, v(s))\,ds \\
&\geq \delta \int\limits_{T_1^+(t)} \varphi_{l-1}(t,s)\,ds - \int\limits_{T_2^+(t)} \varphi_{l-1}(t,s)\,ds \\
&+ \delta \int\limits_{T_1^-(t)} (-\varphi_{l-1}(t,s))\,ds - \int\limits_{T_2^-(t)} (-\varphi_{l-1}(t,s))\,ds \\
&= \delta \int\limits_{T_1^+(t)\cup T_1^-(t)} |\varphi_{l-1}(t,s)|\,ds - \int\limits_{T_2^+(t)\cup T_2^-(t)} |\varphi_{l-1}(t,s)|\,ds.
\end{aligned}
$$

Deduce

$$
\delta \int\limits_{T_1^+(t)\cup T_1^-(t)} |\varphi_{l-1}(t,s)|\,ds - \int\limits_{T_2^+(t)\cup T_2^-(t)} |\varphi_{l-1}(t,s)|\,ds \leq \mu(t),
$$

$$
\int\limits_{T_1^+(t)\cup T_1^-(t)} |\varphi_{l-1}(t,s)|\,ds + \int\limits_{T_2^+(t)\cup T_2^-(t)} |\varphi_{l-1}(t,s)|\,ds = F(t).
$$

The last two relations imply that

$$
\int\limits_{T_2^+(t)\cup T_2^-(t)} |\varphi_{l-1}(t,s)|\,ds \geq \frac{\delta F(t) - \mu(t)}{1+\delta}.
$$

Further, we have

$$
\begin{aligned}
\max_{i\in I} G_i(t, v(\cdot), h_i) &= \max_{i\in I} \int_{t_0}^{t} |\varphi_{l-1}(t,s)|\lambda(v(s), r(t,s), h_i)\,ds \\
&\geq \frac{1}{n} \int_{t_0}^{t} |\varphi_{l-1}(t,s)| \sum_{i\in I} \lambda(v(s), r(t,s), h_i)\,ds \\
&\geq \frac{\delta}{n} \int\limits_{T_2^+(t)\cup T_2^-(t)} |\varphi_{l-1}(t,s)|\,ds \\
&\geq \frac{\delta}{n}\Big(\frac{\delta F(t) - \mu(t)}{1+\delta}\Big).
\end{aligned}
$$

Since $F(t) \to \infty$ at $t \to \infty$ and $\mu(t)$ is bounded, then we obtain the desired result. The lemma is proved. □

Define the number T_0

$$T_0 = \min\{t \geq t_0 : \inf_{v(\cdot)} \min_{h \in S} \max_{i \in I} G_i(t, v(\cdot), h_i) \geq 1\}. \tag{6}$$

Assumptions 3. *There are times $\tau_i \geq T_0$ such that:*

1) $\xi_i(\tau_i) \in S_\varepsilon(\xi_i(\tau_i^0))$ for all i;

2) $\inf_{v(\cdot)} \max_i G_i(\tau_i, v(\cdot), \xi_i(\tau_i)) \geq 1$.

Remarks. a) the existence of τ_i in paragraph 1 of assumptions 3 guaranteed assumption that recurrence functions $\xi_i(t)$;

b) if the assumption 3 of all $\tau_i = \tau$, paragraph 2 of this assumption is made automatically by Lemma 1.

Theorem 1. *Suppose that assumptions 1, 2, 3, $r = 1$. Then in the game $\Gamma(n, D)$ capture occurs.*

Proof. By the Cauchy's formula, the solution of problem (4), (5) for all $t \geq t_0$ for any admissible control has the form

$$z_i(t) = \xi_i(t) + \int_{t_0}^{t} \varphi_{l-1}(t, s)(u_i(s) - v(s))\, ds.$$

Let τ_i — points satisfying assumption 3, $v(s), s \in [t_0, T_1]$ be an arbitrary admissible control of the evader E, where $T_1 = \max_i \tau_i$. Consider the function

$$H(t) = 1 - \max_i \int_{t_0}^{t} |\varphi_{l-1}(\tau_i, s)| \lambda(v(s), r(\tau_i, s), \xi_i(\tau_i)) ds.$$

We denote by $\tau_0 \geq t_0$ is the first root of this function. Note that, by the assumption 2, the time τ_0 exists , wherein $\tau_0 \leq \tau_i$ for at least one of i. Furthermore, there is an index m such that

$$1 - \int_{t_0}^{\tau_0} |\varphi_{l-1}(\tau_m, s)| \lambda(v(s), r(\tau_m, s), \xi_m(\tau_m)) ds = 0. \tag{7}$$

For $j \neq m$ is also denoted t_j — times for which the condition (7), if such times exist. By Filippov's lemma [22] for each i there are measurable functions $m_i(s)$, $u_i(s)$, $s \in [t_0, T_1]$, which for each fixed s, the solution of the equation

$$\lambda(v(s), r(\tau_j, s), \xi_i(\tau_j))(\xi_i(\tau_i) - m_i) = u_i - v(s).$$

We specify the control of pursuers P_i, in the following way:

$$u_i(t) = v(t) - \lambda(v(t), r(\tau_i, t), \xi_i(\tau_i))(\xi_i(\tau_i) - m_i(t)), \quad t \in [t_0, \min\{t_i, T_1\}],$$
$$u_i(t) = v(t), \ t \in (\min\{t_i, T_1\}, T_1].$$

Then

$$\begin{aligned} z_i(\tau_i) =& \xi_i(\tau_i) + \int_{t_0}^{\tau_i} \varphi(\tau_i, s)(u_i(s) - v(s))ds \\ =& \xi_i(\tau_i) - \int_{t_0}^{\tau_i} |\varphi_{l-1}(\tau_i, s)|\lambda(v(s), r(\tau_i, s), \xi_i(\tau_i))(\xi_i(\tau_i) - m_i(s))ds \\ =& \xi_i(\tau_i) \left(1 - \int_{t_0}^{t_i} |\varphi_{l-1}(\tau_i, s)|\lambda(v(s), r(\tau_i, s), \xi_i(\tau_i))ds\right) \\ &+ \int_{t_0}^{t_i} |\varphi_{l-1}(\tau_i, s)|\lambda(v(s), r(\tau_i, s), \xi_i(\tau_i))m_i(s)ds. \end{aligned}$$

(7) it follows that

$$z_m(\tau_m) = \int_{t_0}^{t_m} |\varphi_{l-1}(\tau_m, s)|\lambda(v(s), r(\tau_m, s), \xi_m(\tau_m))m_m(s)ds \in M_m$$

The theorem is proved. □

4. Capture Conditions in the Case of $V = S_1(0)$

We denote Int$X$, co$X$ respectively the interior and the convex hull of the set X,

$$\lambda(M_i, v) = \sup\{\lambda \geq 0 \ \lambda M_i \cap (V - v) \neq \emptyset\}.$$

Lemma 2. *Let $V = S_1(0)$, $Q_i, i \in I$ — convex compact sets R^k, $0 \notin Q_i$ for all i. Then*

$$\delta^+ = \min_v \max\big\{\max_i, \lambda(Q_i, v), \max_j (p_j, v)\big\}> 0$$

if and only if

$$0 \in \operatorname{Intco}\{Q_1, \ldots, Q_n, p_1, \ldots, p_r\}.$$

Proof. Note that (see [9], p. 46)

$$\lambda(Q_i, v) = \max_{q_i \in Q_i} \frac{(q_i, v) + \sqrt{(q_i, v)^2 + \|q_i\|^2(1 - \|v\|^2)}}{\|q_i\|^2}.$$

Suppose that $\delta^+ = 0$. Then there exists an $v_0, \|v_0\| = 1$ such that

$$\lambda(Q_i, v_0) = 0 \text{ for all } i, \ (p_j, v_0) \leq 0 \text{ for all } j.$$

Therefore, the following inequalities

$$(q_i, v_0) \leq 0 \text{ for all } i, q_i \in Q_i, \ (p_j, v_0) \leq 0 \text{ for all } j. \tag{8}$$

Therefore, 0 and $\operatorname{co}\{Q_1, \ldots, Q_n, p_1, \ldots, p_r\}$ separable. Hence $0 \notin \operatorname{Intco}\{Q_1, \ldots, Q_n, p_1, \ldots, p_r\}$.

We now assume that $0 \notin \operatorname{Intco}\{Q_1, \ldots, Q_n, p_1, \ldots, p_r\}$. Then 0 and

$$\operatorname{co}\{Q_1, \ldots, Q_n, p_1, \ldots, p_r\}$$

separable. Consequently, there exists $v_0, \|v_0\| = 1$ such that the inequalities (8). Hence $\lambda(Q_i, v_0) = 0$ for all i. Therefore $\delta^+ = 0$. The lemma is proved. □

Corollary 1. *Let $V = S_1(0)$, $Q_i, i \in I$ — convex compact sets R^k, $0 \notin Q_i$ for all i. Then*

$$\delta^- = \min_v \max\big\{\max_i, \lambda(-Q_i, v), \max_j (-p_j, v)\big\}> 0$$

if and only if

$$0 \in \operatorname{Intco}\{Q_1, \ldots, Q_n, p_1, \ldots, p_r\}.$$

Assumptions 4. *There $\tau_i^0 \geq t_0$ such that*

$$0 \in \operatorname{Intco}\big\{\xi_i(\tau_i^0) - M_i, i \in I, p_1, \ldots, p_r\big\}$$

Lemma 3. *Suppose that assumptions 1, 4. Then there are positive numbers $\varepsilon, T(\varepsilon)$ for which the following propositions:*

1) for all $h_i \in S_\varepsilon(\xi_i(\tau_i^0))$ the following inclusion

$$h_i \notin M_i,\ 0 \in \operatorname{Intco}\big\{h_i - M_i, i \in I, p_1, \ldots, p_r\big\};$$

2) for each $t \geq t_0$ there are moments $\tau_i \in [t, t+T(\varepsilon)]$, that $\xi_i(\tau_i) \in S_\varepsilon(\xi_i(\tau_i^0))$.

The first proposition follows from the properties of the strict separation of convex sets, and the second assertion follows from the properties of recurrent functions.

We fix $\varepsilon > 0$ and $T(\varepsilon) > 0$, so that there have been propositions of Lemma 4.

Lemma 4. *Let $V = S_1(0)$ and assumption 4 is fulfiled. Then, for any $h \in S$ we have the inequalities*

$$\delta^+(h) = \min_v \max\{\max_i \lambda(h_i, 1, v), \max_j (p_j, v)\} > 0,$$
$$\delta^-(h) = \min_v \max\{\max_i \lambda(h_i, -1, v), \max_j (-p_j, v)\} > 0,$$
$$\delta = \min_{h \in S} \min\{\delta^+(h), \delta^-(h)\} > 0.$$

Proof. Let $h \in S$. Then are fulfiled for the sets $Q_i = h_i - M_i$ the conditions of Lemmas 3 and 4. Therefore, $\delta^+(h) > 0$. By Lemma 1.3.13 [9, p.30] function $\delta^+(h)$ is continuous on S. Therefore, by the Weierstrass's theorem $\min_{h \in S} \delta^+(h) > 0$. Similarly $\min_{h \in S} \delta^-(h) > 0$. Consequently, $\delta > 0$. The lemma is proved. □

Definition 5. (see [23]) The vectors $a_1, \ldots, a_s$ form a positive basis of R^k, if, for any $x \in R^k$, there exist positive real numbers $\gamma_1, \ldots, \gamma_s$ such that

$$x = \gamma_1 a_1 + \cdots + \gamma_s a_s.$$

Theorem 2. (see [23]) *Vectors $a_1, \ldots, a_s$ a positive basis to form R^k if and only if*

$$0 \in \operatorname{Intco}\{a_1, \ldots, a_s\}.$$

Lemma 5. *Let $Q_i, i \in I$ be convex compact sets R^k, $0 \notin Q_i$ for all i and the following conditions:*

1) $0 \in \mathrm{Intco}\{Q_1, \ldots, Q_n, p_1, \ldots p_r\}$;

2) the number of elements $\bigcup_{i=1}^{n} Q_i$ *at least to* k;

3) in the set $\bigcup_{i=1}^{n} Q_i$ *exist* k *linearly independent vectors.*

Then there exist $p \in R^k, \mu \in R^1$ *such that:*

1) $D \subset D_1 = \{z \mid z \in R^k, (p, z) \leq \mu\}$;

2) $0 \in \mathrm{Intco}\{Q_1, \ldots, Q_n, p\}$.

Proof. According to the conditions of Lemma there are $q_1, \ldots, q_s \in \bigcup_{i=1}^{n} Q_i$ such that

$$0 \in \mathrm{Intco}\{q_1, \ldots, q_s, p_1, \ldots, p_r\}.$$

We can assume that $s \geq k$ and the vectors $q_1, \ldots, q_k$ linearly independent. By theorem 2, the vectors $q_1, \ldots, q_s, p_1, \ldots, p_r$ form a positive basis. Therefore, there exist positive integers $\alpha_1, \ldots, \alpha_s$, $\beta_1, \ldots, \beta_r$ such that

$$0 = \alpha_1 q_1 + \cdots + \alpha_s q_s + \beta_1 p_1 + \cdots + \beta_r p_r. \tag{9}$$

Consider the vector $p = \beta_1 p_1 + \cdots + \beta_r p_r$. We show that the set of$q_1, \ldots, q_s, p$ forms a positive basis.

Let $x \in R^k$. Since $q_1, \ldots q_k$ is a basis of R^k, then there exist numbers $\gamma_1, \ldots, \gamma_k$ such that

$$x = \gamma_1 q_1 + \cdots + \gamma_k q_k.$$

By (9) we obtain that for any $d \in R^1$ equality

$$x = \gamma_1 q_1 + \cdots + \gamma_k q_k + d\big(\alpha_1 q_1 + \cdots + \alpha_s q_s + \beta_1 p_1 + \cdots + \beta_r p_r\big).$$

Taking $d > 0$ such that the inequality $\gamma_c + d\alpha_c > 0$ for all $c = 1, \ldots, k$ obtain that

$$x = \gamma_1^0 q_1 + \gamma_s^0 q_s + dp$$

and wherein all the coefficients are positive. Consequently, $q_1, \ldots, q_s$, p constitute a positive basis. So $0 \in \mathrm{Intco}\{q_1, \ldots, q_s, p\}$ and therefore

$$0 \in \mathrm{Intco}\{Q_1, \ldots, Q_s, p\}.$$

Consider the set

$$D_1 = \{x \mid x \in R^k, (p, x) \leq \mu\},$$

where $\mu = \beta_1\mu_1 + \cdots + \beta_r\mu_r$. Then $D \subset D_1$. Note that if $p = 0$, then $D_1 = R^k$. The lemma is proved. □

Assumptions 5. *For any $h \in S$ in the set $\bigcup_{i=1}^{n}(h_i - M_i)$ there is a k linear independent vectors.*

Corollary 2. *Suppose that assumptions 1, 4, 5. Then, for any $h \in S$ exist a vector $p(h) \in R^k$ and the number of $\mu(h) \in R^1$ such that:*

1) $0 \in \operatorname{Intco}\{h_i - M_i, i \in I, p(h)\}$;

2) $D \subset D_1 = \{z \mid z \in R^k, (p(h), z) \leq \mu(h)\}$.

Lemma 6. *Let $V = S_1(0)$ and assumptions 1, 4, 5 be fulfiled. Then there is the time $T \geq t_0$ such that for any admissible control $v(\cdot)$ of the evader E in the game $\Gamma(n, D_1)$, any $h \in S$ there exists a number $m \in I$ such that*

$$G_m(T, v(\cdot), h_m) \geq 1,$$

where D_1 is defined in corollary fact 2.

Proof. Suppose $h \in S$. By Lemma 5, we have $\delta > 0$. Therefore, the assumption that the conditions 2. Therefore, we can apply Lemma 1, which implies the assertion. The lemma is proved. □

Theorem 3. *Let the assumptions of 1,4,5, and there are $\tau_i \geq T_0$ such that:*

1) $\xi_i(\tau_i) \in S_\varepsilon(\xi_i(\tau_i^0))$;

2) $\inf_{v(\cdot)} \max_i G_i(\tau_i, v(\cdot), \xi_i(\tau_i)) \geq 1$ *in the game $\Gamma(n, D_1)$.*

Then in the game $\Gamma(n, D_1)$ capture occurs.

The validity of this assertion follows from theorem 1.

Corollary 3. *Suppose that all the conditions of theorem 3. Then in the game $\Gamma(n, D)$ capture occurs.*

5. Examples

Example 1. Suppose that $k = 2$, $n = 1$, $M_1 = \{0\}$, $p_1 = (1, 0)$, $\mu_1 = 2$, $x^0 = (0, 0)$, $y^0 = (1, 0)$, $t_0 = 0$, the system (3),(4) has the form

$$\dot{z} = u - v, \ u, v \in V = \{(v_1, v_2) \bigm| v_1, v_2 \in [-1, 1]\}.$$

Proposition 1. *In the game* $\Gamma(1, D)$ *capture occurs.*

Proof. In this case fulfiled $\varphi_0(t) = 1$, $\xi(t) = z^0 = (-1, 0)$, $F(t) = t$ and therefore assumption 1 . Furthermore assumption 2, as $\delta = 0, 5$. Assumption 3 is automatically satisfied. Theorem 1 is applicable. □

Example 2. Suppose that the system (3),(4) has the form

$$\dot{z}_i = u_i - v, \ z_i(0) = z_i^0.$$

For this example, assumption 1.

Proposition 2. *Let* $V = S_1(0)$, *in the set* $(\bigcup_{i=1}^{n}(z_i^0 - M_i)$ *there is a* k *linearly independent vectors and*

$$0 \in \mathrm{Intco}\{z_i^0 - M_i, i \in I, p_1, \ldots, p_r\}.$$

In the game $\Gamma(n, D)$ *capture occurs.*

Proposition 3. (see [13]) *Let* $V = S_1(0)$, D *be polyhedron,* $M_i = \{0\}$ *for all* i *and* $n \geq k$.

Then in the game $\Gamma(n, D)$ *capture occurs.*

Example 3. Let $V = S_1(0)$, in the system (3),(4) $l = 1$, $t_0 = 0$, the function $a_1(t)$ has the form

$$a_1(t) = \begin{cases} 0, \ \text{if } t \in [0, 2\pi], \\ \sin t, \ \text{if } t > 2\pi \end{cases}$$

Then the function $\varphi_0(t)$ has the form

$$\varphi_0(t) = \begin{cases} 1, \ \text{if } t \in [0, 2\pi], \\ e^{1 - \cos t}, \ \text{if } t > 2\pi. \end{cases}$$

Function $\varphi_0(t)$ is recurrent, but is not almost periodic (see [21]). Assumption 1 is satisfied.

Proposition 4. *Let $D = R^k$, the initial position z_i^0 such that*

$$0 \in \operatorname{Intco}\{z_i^0 - M_i, i \in I\}.$$

Then in the game $\Gamma(n, D)$ capture occurs.

Example 4. Let $V = S_1(0)$, the system (3),(4) has the form

$$\ddot{z}_i + \frac{2}{3t}\dot{z}_i + \frac{1}{9t^{2/3}}z_i = u_i - v,$$

and $t_0 = 8\pi^2$. Then

$$\begin{aligned}\varphi_0(t,s) =& \cos(\sqrt[3]{t} - \sqrt[3]{s}),\\ \varphi_1(t,s) =& 3s^{2/3}\sin(\sqrt[3]{t} - \sqrt[3]{s}),\\ \xi_i(t) =& z_i^0\cos(\sqrt[3]{t}) + 12\pi^2 z_i^1 \sin(\sqrt[3]{t}).\end{aligned}$$

Recurrence functions $\xi_i(t)$ from the results of (see [21]).

Proposition 5. *Assume that there exists time $\tau \in [t_0, \infty)$ such that the set $\bigcup\limits_{i=1}^{n}(\xi_i(\tau) - M_i)$ there is a k linearly independent vectors and*

$$0 \in \operatorname{Intco}\{\xi_i(\tau) - M_i,\ i \in I, p_1, \ldots, p_r\}.$$

Then in the game $\Gamma(n, D)$ capture occurs.

Taking as $\tau_i^0 = t_0 = 8\pi^2$, fairly

Proposition 6. *Let the initial position z_i^0 such that the set $\bigcup\limits_{i=1}^{n}(z_i^0 - M_i)$ there is a k linearly independent vectors and*

$$0 \in \operatorname{Intco}\{z_i^0 - M_i, i \in I, p_1, \ldots, p_r\}.$$

Then in the game $\Gamma(n, D)$ capture occurs.

Note that for this example are not fulfilled the conditions of [20].

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