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PROJECTIVE CURVES SUCH THAT A GENERAL POINT OF THE AMBIENT PROJECTIVE IS CONTAINED IN A UNIQUE OSCULATING HYPERPLANE OF THE CURVE

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Abstract: We study the non-degenerate integral curves $X \subset \mathbb{P}^n$ such that a general point of $\mathbb{P}^n$ is contained in a unique osculating hyperplane of X (they are a generalization of the strange curves to the case $n > 2$).

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1. Introduction

Let $X \subset \mathbb{P}^n$, $n \geq 2$, be an integral and non-degenerate curve defined over an algebraically closed field K. We ask when the natural map from the variety of $(n-1)$-osculating spaces of X to $\mathbb{P}^n$ has separable degree 1, i.e. for a general $O \in \mathbb{P}^n$ there is a unique osculating hyperplane of X containing O. We call ♠ such a property. Set $p := \mathrm{char}(K)$. It is easy to check that X does not exists if $p = 0$ or if $p > \deg(X)$ (Proposition 4 and Remark 1). In positive characteristic we need to choose a definition of osculating variety and osculating space. We use [4], but essentially only use the existence of a non-empty open subset $U \subset X_{reg}$ and an integer $x \geq n$ such that for each $P \in U$ there is a unique hyperplane $O(X, P, n-1) \subset \mathbb{P}^n$ with order of contact x with U at P,

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while all other hyperplanes have order of contact $< x$ with X at P. This shows an ambiguity of the definition of ♠: unique hyperplane or unique $P \in U$ such that $O \in O(X, P, n-1)$? We take the second one as the definition of ♠, but usually it is easy to modify the proofs to adapt to the other interpretation (we call ♡ the alternative interpretation). We prove the following results.

Proposition 1. *X has property ♡ if and only if there is a codimension 2 linear subspace $M \subset \mathbb{P}^n$ such that $O(X, P, n-1) \supset M$ for a general $P \in U$. X has ♠ if and only if it has ♡ and for a general $P \in U$ the hyperplane $O(X, P, n-1)$ is not an osculating hyperplane of X at another point of U.*

Proposition 2. *Assume $n = 2$.*

(a) X has property ♡ if and only it is a strange curve.

(b) X has property ♠ if and only if it is a strange curve and the linear projection from its strange point has separable degree 1.

In particular by a theorem of Lluis ([5]) the case $p = 2$ and X a smooth conic is the only case with $n = 2$, ♡ and X smooth (it also has ♠).

We recall that there is a construction of all strange plane curves ([3] for $n = 2$, [1] when $n > 2$); in the case $n = 2$ it involves the multiplicity $\mu \geq 0$ of X at the strange point, o, the separable degree s of the the rational map τ induced on the normalization of X from the linear projection from o and the inseparable degree p^e of τ (we have $\deg(X) = \mu + sp^e$). We do not know a way to construct all curve with ♡ or ♠, but we have a way to construct two classes of such curves (see Examples 1 and 2).

2. The Proofs

Proposition 3. *If X has ♠, then it is rational, i.e. its normalization has genus 0.*

Proof. Let $f : C \to X$ be the normalization map. Write $\mathbb{P}^n = \mathbb{P}(V^\vee)$ with V an $(n+1)$-dimensional vector space. Let x be the degree of the intersection at a general $P \in X$ of the osculating hyperplane $O(X, P, n-1)$ and $\wp := P(X, x-1, f^*(\mathcal{O}_X(1)))$ the bundle of principal parts of order $x-1$ of the line bundle $f^*(\mathcal{O}_X(1))$. The composition $C \to X \hookrightarrow \mathbb{P}^n$ induces a map $W \otimes \mathcal{O}_C \to \wp$ whose image is a rank n vector bundle E on C and a surjective map $u' : \mathbb{P}(E) \to \mathbb{P}^n$ with separable degree one. The projection $\mathbb{P}(E)$ shows that C has genus 0. □

Proof of Proposition 1: First assume that M exists. For any $O \in \mathbb{P}^n$ there is a unique hyperplane containing both M and O. Since X is non-degenerate, we get that $\mathbb{P}^n$ is the image of the $(n-1)$-osculating variety of X. Hence X has $\heartsuit$. Now assume that X has $\heartsuit$. Fix a general $O \in \mathbb{P}^n$ and take any $P \in U$ with $O \in O(X, P, n-1)$. Let $S \subset X$ be the set of all $Q \in U$ with $O(X, Q, n-1) = O(X, P, n-1)$. S is a finite set. Fix any $e \in U \setminus S$ and set $M := O(X, P, n-1) \cap O(X, e, n-1)$. Since $e \notin S$, we have $O(X, e, n-1) \neq O(X, P, n-1)$ and M is a hyperplane of $O(X, P, n-1)$ not containing O. Since X has $\heartsuit$, then $O \notin O(X, Q, n-1)$ for any $Q \in U \in S$. Since O is general, we see that the union of all $O(X, P, n-1) \cap O(X, Q, n-1)$, $Q \in U \setminus S$ in not Zariski dense in $O(X, P, n-1)$. Therefore X has $\heartsuit$ with M as associated codimension two linear subspace. The statement concerning $\spadesuit$ follows from the one concerning $\heartsuit$. □

Proof of Proposition 2: Part (a) is the case $n = 2$ of Proposition 1. Part (b) follows from part (a). □

Proposition 4. *In characteristic zero there is no curve with* $\heartsuit$.

Proof. If $n = 2$, just use Proposition 2 and that in characteristic zero no curve, except lines, have a strange point. Now assume $n > 2$ and take M as in Proposition 1. We have $x = n - 1$ by [4, Theorem 15]. Fix a general $Q \in M$ and call $\ell : \mathbb{P}^n \setminus \{Q\} \to \mathbb{P}^{n-1}$ and set $Y := \ell(X)$. Y is a non-degenerate curve whose general osculating hyperplane has order of contact $n+1$ with the curve at the contact point, contradicting [4, Theorem 15]. □

Remark 1. Assume $p > 0$. The proof of Proposition 4 and [4, Theorem 15] gives $\deg(X) \geq p$ for each curve X with $\heartsuit$.

Let $X \subset \mathbb{P}^n$ an integral non-degenerate curve. Fix an open subset $U \subseteq X_{reg}$ on which the osculating sequence in the sense of [4] is the general one for X. Fox each $P \in U$ and each integer $t \in \{1, \dots, n-1\}$ let $O(X, P, t)$ be the t-dimensional linear osculating subspace to X at P.

Example 1. We point out how to use [3] and [1] to construct all X with $\heartsuit$ (or $\spadesuit$) for which there are $n-1$ linearly independent points $P_1, \dots, P_t \in \mathbb{P}^n$ such that for each $t = 1, \dots, n-1$ a general t-osculating space of X contains the linear span of $\{P_1, \dots, P_t\}$; call $\clubsuit$ this property. Note that the construction will not depend, up to a projective equivalence, from the choice of the ordered $(n-1)$-ple of linearly independent points. If $n = 2$, then we use Proposition 2 and the construction of strange plane curves done in [3]. Now assume $n > 2$ and that the construction is done in $\mathbb{P}^{n-1}$. Let $\ell : \mathbb{P}^n \setminus \{P_1\} \to \mathbb{P}^{n-1}$ be the linear projection from P_n. Set $O_i := \ell(P_1)$, $2 \leq i \leq n-1$. The point $O_2, \dots, O_{n-1}$ are linearly independent. Take $Y \subset \mathbb{P}^{n-1}$ with $\clubsuit$ with respect to the points

$O_2, \dots, O_{n-1}$. Fix integer $m \geq 0$, $s \geq 1$ and $e > 0$. The construction in [1] gives all strange curve $X \subset \mathbb{P}^n$ with P_1 as their strange point, with Y as their image by the linear projection ℓ and such that $\ell|(X \setminus \{P_1\}$ has separable degree s and inseparable degree p^e. Among these curves the ones with ♠ are the one for which at each step the separable degree, s, is 1 (you need to start a strange plane curve with separable degree 1). The curve discovered by J. Rathmann ([6, Example 1.2]) has ♣ (see [2, Example 1] for the explicit equations of Y).

Example 2. If $n > 2$ there are non-strange curves with ♠. Here there is a way to construct them. Fix a codimension two linear subspace $M \subset \mathbb{P}^n$. Choose a system of homogeneous coordinates $x_0, \dots, x_n$ such that $M = \{x_0 = x_1 = 0\}$. Fix integers $e_i > 0$, $1 \leq i \leq n-1$, general $c_{i,j} \in K$, $1 \leq i \leq n-1$, $2 \leq j \leq n$, and general $u_i(x_0, x_1) \in K[x_0, x_1]$, $1 \leq i \leq n-1$, homogeneous of degree p^{e_i}. Set $f_i(x_0, \dots, x_n) := u_i(x_0, x_1) + \sum_{j=2}^{n} c_{i,j} x_j^{p^{e_i}}$, $X := \{f_1 = \cdots = f_{n-1}$ and $X_i := \{f_i\}$. For general $f_1, \dots, f_{n-1}$ the scheme X has dimension one and hence it is a locally complete intersection and $\deg(X) = p^{e_1 + \cdots + e_{n-1}}$. For general $f_1, \dots, f_{n-1}$ we have $X \cap M = \emptyset$ (e.g. if $e_i = e$ for all i it is sufficient that the $(n-1) \times (n-1)$ matrix with $c_{i,j}^{1/e}$ as entries is invertible). The linear projection $\ell : \mathbb{P}^n \setminus M \to \mathbb{P}^1$ from M induces a generically bijective $X_{red} \to \mathbb{P}^1$. Therefore to check that X is an integral curve it is sufficient to check that it is smooth at a general $P \in X$. Fix $P \in X_{reg}$ and let H be the hyperplane spanned by P and M. We have $(H \cap X)_{red} = \{P\}$, because $M \cap X = \emptyset$ and $\ell|X$ has separable degree one. Hence H is the osculating hyperplane to X at P.

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