International Journal of Pure and Applied Mathematics
Volume 100 No. 4 2015, 455-459
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v100i4.3

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SPANNED VECTOR BUNDLES ON $C \times \mathbb{P}^k$

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Abstract: We introduce several notions associated to a rank r spanned vector bundles $\mathcal{E}$ on $C \times \mathbb{P}^k$, $k \geq 2$, C a smooth curve. We divide them according to the discrete invariants of the dependency locus of $r-1$ general sections of $\mathcal{E}$. In one case ($r = 2$, $\det(\mathcal{E}) = \mathcal{R} \boxtimes \mathcal{O}_{\mathbb{P}^{\|}}(\infty)$ with $h^0(R) = 2$) we get a complete classification. In many other cases our definitions should at least give strong necessary conditions for the numerical data associated to $\mathcal{E}$ and $\det(\mathcal{E})$.

AMS Subject Classification: 14J60, 14M20
Key Words: spanned vector bundle

1. Introduction

Let C be a smooth and connected projective curve of genus $q \geq 0$. Fix an integer $k \geq 2$ and set $X := C \times \mathbb{P}^k$. We use the case $q = 0$ and $k = 2$ done in [3]. There are many results that can be extended from the case $q = 0$ and $k = 2$ to the case $q = 0$ and $k > 2$. We chose to give only results in which the use of [2] and [3] is minimal. The case $q > 0$ is harder, at least for the tools of [2] and [3], because $h^1(\mathcal{O}_{\mathcal{X}}) = \text{II} > \prime$. Most of the tools in [2] and [3] only give necessary conditions for the existence of a spanned vector bundle with certain numerical invariants. We give one case (Theorem 1) in which we obtained a complete classification. Let $\pi_1 : X \to C$ and $\pi_2 : X \to \mathbb{P}^k$. For all line bundles L on C and R on $\mathbb{P}^k$ set $L \boxtimes R : \pi_1^*(L) \otimes \pi_2^*(R)$.

Received: December 7, 2014

Theorem 1. *Assume R spanned, $h^0(C,R) = 2$, $k \geq 2$, and take $\mathcal{L} := \mathcal{R}\boxtimes\mathcal{O}_{\mathbb{P}\|}(\infty)$. The vector bundle $\pi_1^*(R)\boxtimes\mathcal{O}_{\mathbb{P}\|}(\infty)$ is the only rank two spanned bundle on X with $\det(\mathcal{E}) \cong \mathcal{L}$.*

Let $\mathcal{E}$ be a rank $r \geq 2$ spanned vector bundle on X. Let $S \subset X$ be the dependency locus of $r-1$ general sections of $\mathcal{E}$ and call $S_1, \ldots, S_c$ the connected components of S. If $\pi_1(S_j)$ is a point, then we say that S_j is of *fiber type*. In this case there is a hypersurface $S'_j \subset \mathbb{P}^k$ such that $S_j = \{O\} \times S'_j$, where $\{O\} := \pi_1(S_j)$. In this case a necessary condition for the spannedness of $\mathcal{I}_\mathcal{S}\otimes\mathcal{L}$ is that $\deg(S'_j) \leq t$. If $\pi_1(S_j) = C$, then we say that S_j is of *dominant type*. Let S_j be a connected component of dominant type. We say that S_j is *fiber-connected* if a general fiber of $\pi_1|S_j$ is connected, i.e. if $\pi_{1*}(\mathcal{O}_{\mathcal{S}_|}) \cong \mathcal{O}_\mathcal{C}$, i.e. $\pi_1|S_j$ is its own Stein reduction. We prove the following result.

Proposition 1. *Take any C. If $k \geq 4$, then there is at most one component of dominant type, S_j, and any fiber of $\pi_1|S_j$ is connected.*

The second part of Proposition 1 says that each S_j of dominant type is fiber-connected if $k \geq 4$.

We work over an algebraically closed field with characteristic zero.

2. The Proofs

Let $\mathcal{E}$ be a rank $r \geq 2$ spanned vector bundle on X. Let $\mathcal{L} = \mathcal{R}\boxtimes\mathcal{O}_{\mathbb{P}\|}(\sqcup)$ be its determinant. R is a spanned line bundle on C and $t \geq 0$. Using the base change theorem ([6, page 13]) it is easy to check that $\mathcal{E} \cong \pi_\in^*(\mathcal{F})$ for some spanned bundle $\mathcal{F}$ on $\mathbb{P}^k$ with $\mathcal{O}_{\mathbb{P}\|}(\sqcup)$ as its determinant if and only $R \cong \mathcal{O}_\mathcal{C}$ and that $\mathcal{E} \cong \pi_\infty^*(\mathcal{G})$ for some spanned bundle $\mathcal{G}$ on C if and only if $t = 0$. Hence we assume $R \neq \mathcal{O}_\mathcal{C}$ (i.e. $h^0(C,R^\vee) = 0$) and $t > 0$. The Künneth formula gives $h^1(\mathcal{L}^\vee) = \prime$. Let $S \subset X$ be the dependency locus of $r-1$ general sections of $\mathcal{E}$. Since $\mathcal{E}$ is spanned, either $S = \emptyset$ or S has pure codimension 2 and its singular locus has dimension $\leq \dim(X) - 4 = k - 3$ ([5]). Since S has pure codimension 2, it is locally Cohen-Macaulay ([4, page 13]). Since S is locally Cohen-Macaulay and non-singular in codimension 2, it is normal. Hence each connected component of S is irreducible. We have an exact sequence

$$0 \to \mathcal{O}_\mathcal{X}^{\oplus(\nabla-\infty)} \to \mathcal{E} \to \mathcal{I}_\mathcal{S}\otimes\mathcal{L} \to \prime \tag{1}$$

Let $S_1, \ldots, S_c$, $c \geq 1$, be the connected components of S.

Since $t > 0$, $k \geq 2$ and $\dim(C) = 1$, the Künneth formula gives $h^1(\mathcal{L}^\vee) = \prime$. Since $h^1(\mathcal{L}^\vee) = \prime$, (1) gives that $\mathcal{E} \cong \mathcal{O}_\mathcal{X}^{\oplus(\nabla-\infty)} \oplus \mathcal{L}$. Hence from now on we assume $S \neq \emptyset$. Since $\mathcal{E}$ is spanned, $\mathcal{I}_{\mathcal{S}} \otimes \mathcal{L}$ is spanned. The converse is true if $h^1(\mathcal{O}_{\mathcal{X}}) = /$, i.e. if $q = 0$. If $q > 0$, we will only get necessary condition for the existence of a spanned $\mathcal{E}$. In each case we need to check the condition $h^2(\mathcal{L}^\vee) = /$. Since $\dim(C) = 1$, the Künneth formula gives $h^2(\mathcal{L}^\vee) = \langle'(\mathcal{C}, \mathcal{R}^\vee)\langle^{\in}(\mathcal{O}_{\mathbb{P}^{\parallel}}(-\sqcup)) + \langle^{\infty}(\mathcal{R}^\vee)\langle^{\infty}(\mathcal{O}_{\mathbb{P}^{\parallel}}(-\sqcup)) = \langle'(\mathcal{C}, \mathcal{R}^\vee)\langle^{\in}(\mathcal{O}_{\mathbb{P}^{\parallel}}(-\sqcup))$ (since $k \geq 2$). We always have $h^2(\mathcal{L}^\vee) = /$ if either $k \geq 3$ or if R is spanned, but not trivial. Recall that we always reducde to the case in which R is spanned, but not trivial. Hence we may always use that $h^2(\mathcal{L}^\vee) = /$. To apply [1, Theorem 1.1] we need to assume that S is locally a complete intersection; this condition is always satisfied if $\mathcal{E}$ has rank two.

Lemma 1. *Take S coming from a vector bundle $\mathcal{E}$ as in (1) and let S_j be a component of fiber type, say $S_j = \{O\} \times S'_j$. Then $\deg(S'_j) = t$ and $\omega_{S_{\rm j}} \otimes (\omega_X^\vee \otimes \mathcal{L}^\vee)|\mathcal{S}_{|} \cong \mathcal{O}_{\mathcal{S}_{|}}$. Every connected component of S is of fiber type.*

Proof. We know that $e := \deg(S'_j) \leq t$. Since $S_j \cong S'_j$, S_j is a locally complete intersection. Hence the conditions in [1, Theorem 1.1] restricted to S_j are necessary condition for the existence of some $\mathcal{E}$ with S_j as a connected component of its dependency locus. We have $\omega_X^\vee \otimes \mathcal{L}^\vee|\mathcal{S}_{|} \cong \mathcal{O}_{\mathcal{S}'_{|}}(\parallel + \infty - \sqcup)$ and $\omega_{S_{\rm j}} \cong \mathcal{O}_{\mathcal{S}_{\rangle}}(\rceil - \parallel - \infty)$ (up to the identification of S_j and S'_j and hence $\omega_{S_{\rm j}} \otimes (\omega_X^\vee \otimes \mathcal{L}^\vee)|\mathcal{S}_{|}$ has sections if and only if $e = t$. Assume the existence of a connected component S_i of S of dominant type. Since S_i is of dominant type, then $S_i \cap \{O\} \times \mathbb{P}^k \neq \emptyset$. Fix $Q = (O, P) \in S_i \cap \{O\} \times \mathbb{P}^k$. Since $S_i \cap S_j = \emptyset$, then $P \notin S'_j$. Since $\deg(S'_j) = t$, we get that $\{O\} \times \mathbb{P}^k$ is in the base locus of $\mathcal{I}_{\mathcal{S}} \otimes \mathcal{L}$, a contradiction. □

Proof of Proposition 1: Assume the existence of $i \neq j$ with S_i of dominant type, too. For any $O \in C$ $S_i \cap \{O\} \times \mathbb{P}^k$ and $S_j \cap \{O\} \times \mathbb{P}^k$ are disjoint schemes of pure dimension dimension $k-2$ of $\mathbb{P}^k$ by Bezout. Hence $k \leq 3$, a contradiction. In this case $S_j \cap \{O\} \times \mathbb{P}^k$ is connected, because it has pure dimension $k-2$. □

Remark 1. Set $d := \deg(R)$ and fix a $Z \in |R|$ and $M \in |\mathcal{O}_{\mathbb{P}^{\parallel}}(\sqcup)|$. The vector bundle $\mathcal{F} := \pi_{\infty}^*(\mathcal{R}) \oplus \pi_{\in}^*(\mathcal{O}_{\mathbb{P}^{\parallel}}(\sqcup))$ is spanned. If Z is reduced (e.g. if Z is a general in $|R|$), then $Z \times M$ is a dependency locus of $\mathcal{F}$ with d connected components of fiber type.

Proof of Theorem 1. Set $z := \deg(R)$. Since $k = 2$, S is a smooth curve. Fix $P \in \mathbb{P}^2$ and set $D := C \times \{P\}$. Since $\mathcal{I}_{\mathcal{S} \cap \mathcal{D}, \mathcal{C}} \otimes \mathcal{R}$ is spanned and R is a spanned line bundle of degree z with $h^0(C, R) = 2$, either $S \cap D = \emptyset$ or $D \subseteq S$ or $D \cap S \in |R|$. First assume $D \subseteq S$. Since S is smooth, D is a connected component of S isomorphic to C. We have $\omega_D \otimes (\omega_C^\vee \otimes R^\vee \boxtimes \mathcal{O}_{\mathcal{X}}(\in)|\mathcal{D} \cong \mathcal{R}^\vee$

(up to the identification of D with C given by $\pi_1|D$. Hence no S with D as a connected component is associated to a bundle using the Serre correspondence. We get that $\pi_1|S : S \to \pi_2(S)$ is a degree z morphism. Assume for the moment the existence of a component S_1 of fiber type, say $S_1 = \{O\} \times L$ with $L \subset \mathbb{P}^2$ a plane curve and $O \in C$. Since $\{O\} \times \mathbb{P}^2$ is not in the base locus of $\mathcal{I_S} \otimes \mathcal{L}$, L is a line. Since $h^0(\mathcal{I}_{\{\mathcal{O}\}\times\mathcal{L}} \otimes \mathcal{L}) = \langle'(\mathcal{L}) - \in = \langle'(\mathcal{I}_{\mathcal{Z}\times\mathcal{L}} \otimes \mathcal{L})$, we get $Z \times L \subseteq S$. Hence Z is reduced. By Lemma 1 every connected component of S is of fiber type. To get that $S = Z \times L$ it is sufficient to check that $c = z$. Assume $c > z$. We get the existence of z other components of fiber type whose union is $Z_1 \times L_1$ with $L_1 \subset \mathbb{P}^2$ a line, Z_1 a reduced element of $|R|$ and $(Z_1, L_1) \neq (Z, L)$. First assume $Z_1 \neq Z$. Since $|R|$ is a base point free pencil, we have $Z \cap Z_1 = \emptyset$. Since L, L_1 are lines of $\mathbb{P}^2$, there is $Q \in L \cap L_1$. Since $\deg(\pi_2^{-1}(Q \cap S) \geq 2z$, we get a contradiction. Now assume $Z = Z_1$, since $L \neq L_1$, we get that $Z \times \mathbb{P}^2$ is in the base locus of $\mathcal{I_S} \otimes \mathcal{L}$, a contradiction. Since $h^i(\mathcal{L}^*) = \prime$, $i = 1, 2$, the Serre correspondence gives a unique rank 2 bundle with $Z \times L$ as the zero-locus of one of its sections ([1, Theorem 1.1]). Hence $\mathcal{E} \cong \mathcal{R} \boxtimes \mathcal{O}_{\mathbb{P}^\in}(\infty)$.

Now assume the existence of a component S_1 of S of fiber type. By Lemma 1 all component of S are of fiber type. For each connected component S_i of S set $e[i]_1 := \deg(S_i \cap \{O\} \times \mathbb{P}^2)$, $O \in C$, and $e[i]_2 := \deg(S_i \cap C \times L)$, L a general line of $\mathbb{P}^2$. Since $r = 2$, then $\omega_S \cong \pi^*((\omega_C^\vee) \otimes R^\vee) \boxtimes \mathcal{O}_{\mathbb{P}^\in}(-\in))|\mathcal{S}$. Since $p_a(C) > 0$ and $\deg(R) > 0$, $(\omega_C \otimes R) \boxtimes \mathcal{O}_{\mathbb{P}^\in}(\in))$ is ample. Since $\deg(\omega_{C_i}) = -2$, we get $(2p_a(C) - 2 + z)e[i]_1 + 2e[i]_2 = 2$, a contradiction.

Now assume $k > 2$. By induction on k we may assume that $\mathcal{E}|\mathcal{C} \times \mathcal{V} \cong \mathcal{R} \boxtimes \mathcal{O}_{\mathcal{V}}(\infty)$ for every hyperplane V of $\mathbb{P}^k$. We first get that $c = z$ and that S has exactly z components of fiber type, then that $S = Z \times H$ with Z a reduced element of $|R|$ and then we conclude using again □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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