International Journal of Pure and Applied Mathematics
Volume 100 No. 4 2015, 469-473
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v100i4.5

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DYNAMICS OF PARABOLIC WEIGHTED COMPOSITION OPERATORS

Bahmann Yousefi[1], Fariba Ershad[2 §]

[1,2]Department of Mathematics
Payame Noor University
P.O. Box 19395-3697, Tehran IRAN

Abstract: In the present paper we investigate conditions under which a parabolic self-map of the open unit disk induces a hypercyclic weighted composition operator in some Banach function spaces.

AMS Subject Classification: 47B33, 47B38
Key Words: weighted composition operator, hypercyclic operator, Denjoy-Wolff point, parabolic map

1. Introduction

For $z = (z_1, ..., z_N)$ and $w = (w_1, ..., w_N)$ in $\mathbb{C}^N$, write $< z, w >$ for the Euclidean inner product $\sum_{j=1}^{N} z_j \bar{w}_j$ and let $|z| =< z, z >^{1/2}$. With this notation, the unit ball in $\mathbb{C}^N$ is the set $B_N = \{z \in \mathbb{C}^N : \ |z| < 1\}$ and the unit sphere in $\mathbb{C}^N$ is the set $S_N = \{z \in \mathbb{C}^N : \ |z| = 1\}$, analogously to the unit disc and circle for $N = 1$. The space $H(B_N)$, is the set of all holomorphic fuctions on B_N, can be made into a F-space by a complete metric for which a sequence $\{f_n\}$ in $H(B_N)$ converges to $f \in H(B_N)$ if and only if $f_n \longrightarrow f$ uniformly on every compact subset of B_N. Each $\varphi \in H(B_N)$ and holomorphic self-map ψ of

Received: December 24, 2014

§Correspondence author

B_N induces a linear weighted composition operator $C_{\varphi,\psi} : H(B_N) \longrightarrow H(B_N)$ by $C_{\varphi,\psi}(f)(z) = \varphi(z)f(\psi(z))$ for every $f \in H(B_N)$ and $z \in B_N$. Indeed, $C_{\varphi,\psi} = M_\varphi C_\psi$ where M_φ denotes the operator of multiplication by φ and C_ψ is a composition operator by means of the definition $C_\psi(f) = f \circ \psi$ for every $f \in H(B_N)$.

A bounded linear operator T on a F-space X is said to be hypercyclic if there exists a vector $x \in X$ for which the orbit $Orb(T, x) = \{T^n x : n \in \mathbb{N}$ is dense in X and in this case we refer to x as a hypercyclic vector for T.

The holomorphic self maps of B_N are divided into classes of elliptic and non-elliptic. The elliptic type is an automorphism and has a fixed point in B_N. It is well known that this map is conjugate to a rotation.

For simplicity, throughout this paper we use the notation " $\stackrel{k}{\rightarrow}$ " for indicating uniform convergence on compact subsets of B_N. Also, by ψ_n we denote the nth iterate of ψ.

Suppose that $\mathcal{X}$ is a separable Banach space of analytic functions on the open unit ball B_N. The functional of evaluation at λ, $e_\lambda : \mathcal{X} \to \mathcal{C}$ is defined by $e_\lambda(f) = f(\lambda)$ for all $f \in \mathcal{X}$. A complex valued function φ on B_N for which $\varphi\mathcal{X} \subseteq \mathcal{X}$ is called a multiplier of $\mathcal{X}$. The set of all multipliers of $\mathcal{X}$ is denoted by $M(\mathcal{X})$ and it is well-known that $M(\mathcal{X}) \subseteq H^\infty(B_N)$.

For the algebra $\mathcal{B}(\mathcal{X})$ of all bounded linear operators on a Banach space $\mathcal{X}$, the weak operator topology (WOT) is the one in which a net A_α converges to A if $A_\alpha x \to Ax$ weakly, $x \in \mathcal{X}$. Also, the strong operator topology (SOT) is the one in which a net A_α converges to A if $A_\alpha x \to Ax$, $x \in \mathcal{X}$.

The next section of the present paper shows that weighted composition operators with non-constant weight function and parabolic compositional symbol can be hypercyclic on a Banach space $\mathcal{X} \subset H(B_N)$. For simplicity, we call a weighted composition operator $C_{\varphi,\psi}$, a parabolic weighted composition operator whenever the compositional symbol ψ is parabolic. For some sources see [1–9].

2. Main Result

In this section ψ will denote a holomorphic self-map of B_N and φ is a nonzero holomorphic map on B_N.

Theorem 2.1. *Suppose that $\mathcal{X} \subset \mathcal{H}(\mathcal{B}_\mathcal{N})$ is a separable Banach space such that $\mathcal{X}$ contains constants, the multiplication operator by the variable z is polynomially bounded on $\mathcal{X}$, and for all $\lambda \in B_N$ the functional of evaluation at*

λ is bounded on $\mathcal{X}$. Let ψ be a parabolic automorphism with Denjoy-Wolff fixed point w such that for some $0 < \alpha < 1$, the series $\sum_{n=0}^{\infty}(1-|\psi_n(z)|)^{\alpha/2}$ converges uniformly on compact subsets of B_N. Let $\varphi \in M(\mathcal{H})$ satisfying $\varphi(w) \neq 0$ and $|\varphi(z)-\varphi(w)| \leq |z-w|^{\alpha}$ in a neighborhood of w. If $||\varphi \circ \psi_n||_{B_N} \leq |\varphi(w)|$ holds eventually for all $n \in \mathbb{N}$, then $\varphi(w)$ is an eigenvalue for $C_{\varphi,\psi}$ and $C^*_{\varphi,\psi}$ fails to be hypercyclic on $\mathcal{X}$.

Proof. Fix a compact set K in B_N and note that there exists a constant $\beta > 0$ such that

$$|1- < \psi_n(z), w > |^2 \leq \beta(1 - |\psi_n(z)|^2)$$

for all z in K and all n in $\mathbb{N}$. Clearly

$$|1- < \psi_n(z), w > |^2 = |w - \psi_n(z)|^2,$$

thus

$$|w - \psi_n(z)|^2 \leq \beta(1 - |\psi_n(z)|^2)$$

for every $z \in K$ and every $n \in \mathbb{N}$. Our hypothesis shows that there exists a neighborhood U_w of w such that if $\psi_n(z) \in U_w \cap B_N$, then

$$|\varphi \circ \psi_n(z) - \varphi(w)| \leq |\psi_n(z) - w|^{\alpha}. \tag{$*$}$$

Since $\psi_n \xrightarrow{k} w$, there exists N such that for all $n > N$, $\psi_n(z) \in U_w$. By using $(*)$, we obtain

$$\begin{aligned} |1 - \frac{1}{\varphi(w)}\varphi(\psi_n(z)| &\leq \frac{1}{|\varphi(w)|}|w - \psi_n(z)|^{\alpha} \\ &\leq \frac{1}{|\varphi(w)|}\beta^{\alpha/2}(1 - |\psi_n(z)|^2)^{\frac{\alpha}{2}} \\ &\leq \frac{2^{\alpha/2}\beta^{\alpha/2}}{|\varphi(w)|}(1 - |\psi_n(z)|)^{\alpha/2} \end{aligned}$$

for all $n > N$. So $\displaystyle\sum_{n=0}^{\infty}|1 - \frac{1}{\varphi(w)}\varphi(\psi_n(z))|$ converges uniformly on K. Thus $\displaystyle\prod_{n=0}^{\infty}\frac{1}{\varphi(w)}\varphi \circ \psi_n(z)$ also converges uniformly on K. Define

$$g(z) = \prod_{n=0}^{\infty}\frac{1}{\varphi(w)}\varphi \circ \psi_n(z).$$

This implies that g is nonzero holomorphic function on B_N. Since $||\varphi \circ \psi_n||_{B_N} \leq |\varphi(w)|$, $g \in H^\infty(B_N)$. Now we show that $H^\infty(B_N) \subset M(\mathcal{X})$. For this let $f \in H^\infty(B_N)$. Then there is a sequence $\{p_n\}_n$ of polynomials converging to f pointwise and for all n, $\|p_n\|_{B_N} \leq M$ for some $M > 0$. Since M_z is polynomially bounded on $\mathcal{X}$, we obtain $\|M_{p_n}\| \leq M$ for all n. But ball $B(\mathcal{X})$ is compact in the weak operator topology and so by passing to a subsequence if necessary, we may assume that for some $A \in B(\mathcal{X})$, $M_{p_n} \longrightarrow A$ in the weak operator topology. Using the fact that $M^*_{p_n} \longrightarrow A^*$ in the weak operator topology and acting these operators on e_λ we get

$$p_n(\lambda)e_\lambda = M^*_{p_n} e_\lambda \longrightarrow A^* e_\lambda$$

weakly. Since $p_n(\lambda) \longrightarrow f(\lambda)$ we see that $A^* e_\lambda = f(\lambda)e_\lambda$ from which we can conclude that $A = M_f$ and this implies that $f \in M(\mathcal{X})$. Thus $H^\infty(B_N) \subset M(\mathcal{X})$. But $1 \in \mathcal{X}$ and $M(\mathcal{X}) = H^\infty(B_N)$, thus indeed $g \in \mathcal{X}$. Also, $C_{\varphi,\psi} g = \varphi(w)g$, thus $\varphi(w)$ is an eigenvalue of $C_{\varphi,\psi}$. Now, clearly $C^*_{\varphi,\psi}$ is not hypercyclic on $\mathcal{H}$ and the proof is complete. □

Corollary 2.2. *Let w be the Denjoy-Wolff fixed point of a parabolic self-map ψ of the open unit disc U, and let there exists $\alpha > 0$ such that $|\varphi(z) - \varphi(w)| \leq |z - w|^\alpha$ in a neighborhood of w. If ψ' never vanishes and $\sum_{n=0}^{\infty}(1 - |\psi_n(z)|)^{\alpha/2}$ converges uniformly on compact subsets of U, then $C_{\psi',\psi}$ is hypercyclic on $H(U)$.*

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