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A COMMON FIXED POINT THEOREM FOR SIX WEAKLY COMPATIBLE AND COMMUTING MAPS IN *b*-METRIC SPACES

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Abstract: The aim of this paper is to obtain a common fixed point theorem for six weakly compatible and commuting maps in b-metric spaces. Moreover, as a bi-product we obtain several new common fixed point theorems.

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1. Introduction and Preliminaries

Fixed point theory has attracted many researchers since 1922 with the admired Banach fixed point theorem. This theorem supplies a method for solving a variety of applied dilemma in mathematical sciences and engineering. Fixed point theory has many applications in various branches of mathematics such as nonlinear analysis, differential equation, integral equations, etc. Also, it has been used in many other branches of science, such as chemistry, biology, economics, computer science, engineering, and many others. A large literature on this subject exists, and this is a very active area of research at present.

The concept of b-metric space was introduced by Bakhtin [3] and Czerwik [5]. In [5], S. Czerwik proved the contraction mapping principle in b-

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metric spaces that generalized the famous Banach contraction principle in metric spaces. Since then several papers have dealt with fixed point theory for single-valued and multi-valued operators in b-metric spaces (see [4]-[11] references therein). A b-metric space was also called a metric-type space in [6]. The fixed point theory in metric-type spaces was investigated in [6], [7].

The study of common fixed points of mappings satisfying certain contractive conditions has been at the centre of vigorous research activity, being the applications of fixed point very important in several areas of mathematics. The purpose of the present paper is to prove a common fixed point theorem for six commuting and weakly compatible mappings in b-metric spaces . Further, as a bi-product we obtain several new common fixed point theorems.

Definition 1. [2] Let X be a (nonempty) set and $s \geq 1$ a given real number. A function $d : X \times X \to \Re^+$(nonnegative real numbers) is called a b-metric provided that, for all $x, y, z \in X$, the following conditions are satisfied:

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space with parameter s.

We now give some examples of b-metric spaces.

Example 2. [4] The space $l_p(0 < p < 1)$,

$$l_p = \left\{(x_n) \in \Re : \sum |x_n|^p < \infty\right\},$$

together with the function $d : l_p \times l_p \to \Re$,

$$d(x, y) = (\sum |x_n - y_n|^p)^{\frac{1}{p}},$$

where $x = (x_n); y = (y_n) \in l_p$ is a b-metric space with $s = 2^{\frac{1}{p}}$.

Example 3. [4] The space $L_p(0 < p < 1)$ of all real functions $x(t), t \in [0, 1]$ such that $\int_0^1 |x(t)|^p \, dt < \infty$, is a b-metric space if we take $d(x, y) = (\int_0^1 |x(t) - y(t)|^p \, dt)^{\frac{1}{p}}$, for each $x, y \in L_p$.

Remark 4. We note that a metric space is evidently a b-metric space for $s = 1$. However, in general, a b-metric on X need not be a metric on X as shown in the following example:

Example 5. [2] Let $X = \{0, 1, 2\}$ and $d(2, 0) = d(0, 2) = m \geq 2, d(0, 1) = d(1, 2) = d(1, 0) = d(2, 1) = 1$ and $d(0, 0) = d(1, 1) = d(2, 2) = 0$.

Then $d(x, y) \leq \frac{m}{2}[d(x, z)+d(z, y)]$ for all $x, y, z \in X$. If $m > 2$, the ordinary triangle inequality does not hold.

Definition 6. [4] Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called a Cauchy sequence if for every $\epsilon > 0$, there exists $K(\epsilon) \in \mathbb{N}$, such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq K(\epsilon)$.

Definition 7. [4] Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is said to converge to $x \in X$ if for every $\epsilon > 0$, there exists $K(\epsilon) \in \mathbb{N}$, such that $d(x_n, x) < \epsilon$ for all $n \geq K(\epsilon)$. In this case, we write $\lim\limits_{n\to\infty} x_n = x$.

Definition 8. [4] The b-metric space (X, d) is complete if every Cauchy sequence in X converges in X.

Remark 9. In a b-metric space (X, d) the following assertions hold:

(i) A convergent sequence has a unique limit;

(ii) Every convergent sequence is Cauchy.

Definition 10. Two self maps S and T of a non empty set X are said to be commuting if $STx = TSx$ for all $x \in X$.

Definition 11. Two self maps S and T of a non empty set X are said to be weakly compatible if $STx = TSx$ whenever $Sx = Tx$.

2. Main Results

Theorem 12. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and F, G, H, J, K, L be self maps of X satisfying the following conditions $KL(X) \subseteq F(X)$, $HJ(X) \subseteq G(X)$ and $d(HJx, KLy) \leq \alpha d(Fx, Gy) +$*

$\beta(d(Fx, HJx) + d(Gy, KLy)) + \gamma(d(Fx, KLy) + d(Gy, HJx))$ for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $s(\alpha + 2\beta + \gamma + s\gamma) < 1$. Assume that pairs (KL, G) and (HJ, F) are weakly compatible. Pairs (K, L), (K, G), (L, G), (H, J), (H, F) and (J, F) are commuting pairs of maps. If one of $F(X)$ or $G(X)$ is a closed subspace of X, then F, G, H, J, K and L have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $KL(X) \subseteq F(X)$ and $HJ(X) \subseteq G(X)$, there exist x_1, x_2 in X such that $HJx_0 = Gx_1$ and $KLx_1 = Fx_2$. Again, there exist x_3, x_4 in X such that $HJx_2 = Gx_3$ and $KLx_3 = Fx_4$. By continuing this process, for each $n \in \mathbb{N} \cup \{0\}$, we can choose $x_n \in X$ such that $HJx_{2n} = Gx_{2n+1}$ and $KLx_{2n+1} = Fx_{2n+2}$. For each $n \in \mathbb{N} \cup \{0\}$, we let $y_{2n} = HJx_{2n} = Gx_{2n+1}$ and $y_{2n+1} = KLx_{2n+1} = Fx_{2n+2}$.
Consider,

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) =& d(HJx_{2n}, KLx_{2n+1}) \\
\leq& \alpha d(Fx_{2n}, Gx_{2n+1}) + \beta(d(Fx_{2n}, HJx_{2n}) + d(Gx_{2n+1}, KLx_{2n+1})) \\
&+ \gamma(d(Fx_{2n}, KLx_{2n+1}) + d(Gx_{2n+1}, HJx_{2n})) \\
=& \alpha d(y_{2n-1}, y_{2n}) + \beta(d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})) \\
&+ \gamma(d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})) \\
\leq& (\alpha + \beta + s\gamma)d(y_{2n-1}, y_{2n}) + (\beta + s\gamma)d(y_{2n}, y_{2n+1}) \\
\Rightarrow \;\; & d(y_{2n}, y_{2n+1}) \leq \frac{\alpha + \beta + s\gamma}{1 - \beta - s\gamma} d(y_{2n-1}, y_{2n}) = kd(y_{2n-1}, y_{2n}),
\end{aligned}$$

where $0 < k = \frac{\alpha+\beta+s\gamma}{1-\beta-s\gamma} < 1$.

Similarly, we obtain

$$d(y_{2n+1}, y_{2n+2}) \leq kd(y_{2n}, y_{2n+1}).$$

Therefore,

$$d(y_{n+1}, y_{n+2}) \leq kd(y_n, y_{n-1}) \leq ...k^{n+1} d(y_o, y_1),$$

for $n = 1, 2, 3, ...$

Now, for every $m, n \in \mathbb{N}$ such that $m > n$, we have

$$\begin{aligned}
d(y_n, y_m) \leq& sd(y_n, y_{n+1}) + s^2 d(y_{n+1}, y_{n+2}) + \cdots + s^{m-n} d(y_{m-1}, y_m) \\
\leq& (sk^n + s^2 k^{n+1} + \cdots + s^{m-n} k^{m-1}) d(y_1, y_o) \\
<& sk^n [1 + sk + (sk)^2 + ...] d(y_1, y_0)
\end{aligned}$$

$$= \frac{sk^n}{1-sk} d(y_1, y_0).$$

Letting $n, m \to \infty$, we have $\{y_n\}$ is a Cauchy sequence. Since X is a complete b-metric space, there exists a point z in X such that $lim\ y_n = z$.

$$\Rightarrow \lim\ HJx_{2n} = \lim Gx_{2n+1} = \lim KLx_{2n+1} = \lim Fx_{2n+2} = z.$$

Suppose that $F(X)$ is a closed subspace of X. It follows that $z = Fu$ for some $u \in X$.

Then, we have:

$$\begin{aligned} d(HJu, z) \leq & s[d(HJu, KLx_{2n-1}) + d(KLx_{2n-1}, z)] \\ \leq & s\alpha d(Fu, Gx_{2n-1}) + s\beta(d(Fu, HJu) + d(Gx_{2n-1}, KLx_{2n-1})) \\ & + s\gamma(d(Fu, KLx_{2n-1}) + d(Gx_{2n-1}, HJu)) + sd(KLx_{2n-1}, z). \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$\begin{aligned} d(HJu, z) \leq & s\alpha d(z, z) + s\beta(d(z, HJu) + d(z, z)) \\ & + s\gamma(d(z, z) + d(z, HJu)) + sd(z, z) \\ = & s(\beta + \gamma) d(HJu, z) \\ \Rightarrow & \ (1 - s(\beta + \gamma)) d(HJu, z) \leq 0. \end{aligned}$$

Therefore, $HJu = Fu = z$.

Now, since $HJ(X) \subseteq G(X)$, there exists a point v in X such that $z = Gv$. Then, we have

$$\begin{aligned} d(z, KLv) = & d(HJu, KLv) \\ \leq & \alpha d(Fu, Gv) + \beta(d(Fu, HJu) + d(Gv, KLv)) + \gamma(d(Fu, KLv) \\ & + d(Gv, HJu)) \\ = & \alpha d(z, z) + \beta(d(z, z) + d(z, KLv)) + \gamma(d(z, KLv) + d(z, z)) \\ = & (\beta + \gamma) d(z, KLv) \\ \Rightarrow & \ (1 - (\beta + \gamma)) d(z, KLv) \leq 0. \end{aligned}$$

Therefore, $KLv = Gv = z$ and so $HJu = Fu = KLv = Gv = z$.

Since F and HJ are weakly compatible maps, therefore $HJFu = FHJu$ and so $HJz = Fz$.

Now we claim that z is a fixed point of HJ. Consider

$$d(HJz, z) = d(HJz, KLv)$$

$$
\begin{aligned}
&\le \alpha d(Fz, Gv) + \beta(d(Fz, HJz) + d(Gv, KLv)) \\
&\quad + \gamma(d(Fz, KLv) + d(Gv, HJz)) \\
&= \alpha d(HJz, z) + \beta(d(HJz, HJz) + d(z, z) \\
&\quad + \gamma(d(HJz, z) + d(z, HJz)) \\
&= (\alpha + 2\gamma) d(HJz, z) \\
\Rightarrow \;\; & (1 - (\alpha + 2\gamma)) d(HJz, z) \le 0.
\end{aligned}
$$

Therefore, $HJz = z$. Hence $HJz = Fz = z$.

Similarly G and KL are weakly compatible maps, so we have $KLz = Gz$.

Now we claim that z is a fixed point of KL. Consider

$$
\begin{aligned}
d(z, KLz) &= d(HJz, KLz) \\
&\le \alpha d(Fz, Gz) + \beta(d(Fz, HJz) + d(Gz, KLz) \\
&\quad + \gamma(d(Fz, KLz) + d(Gz, HJz)) \\
&= \alpha d(z, KLz) + \beta(d(z, z) + d(KLz, KLz)) \\
&\quad + \gamma(d(z, KLz) + d(KLz, z)) \\
&= (\alpha + 2\gamma) d(z, KLz) \\
\Rightarrow \;\; & (1 - (\alpha + 2\gamma)) d(z, KLz) \le 0.
\end{aligned}
$$

Therefore, $KLz = z$. Hence $KLz = Gz = z$.

We have therefore proved that $HJz = KLz = Fz = Gz = z$. So z is common fixed point of F, G, HJ and KL.

By commuting conditions of pairs, we have

$$Kz = K(KLz) = K(LKz) = KL(Kz),$$

$$Kz = K(Fz) = F(Kz) \text{ and } Lz = L(KLz) = (LK)(Lz) = (KL)(Lz),$$

$$Lz = L(Fz) = F(Lz),$$

which shows that Kz and Lz are common fixed points of (KL, F).

Then $Kz = z = Lz = Fz = KLz$.

Similarly, $Hz = z = Jz = Gz = HJz$.

Therefore z is a common fixed point of F, G, H, J, K and L.

Uniqueness. Let w be another common fixed point of F, G, H, J, K and L. Then, we have

$$
\begin{aligned}
d(z, w) &= d(HJz, KLw) \\
&\le \alpha d(Fz, Gw) + \beta(d(Fz, HJz) + d(Gw, KLw))
\end{aligned}
$$

$$
\begin{aligned}
&+\gamma(d(Fz,KLw)+d(Gw,HJz))\\
=&\alpha d(z,w)+\beta(d(z,z)+d(w,w))+\gamma(d(z,w)+d(w,z))\\
=&(\alpha+2\gamma)d(z,w)\\
\Rightarrow\ &(1-(\alpha+2\gamma))d(z,w)\le 0.
\end{aligned}
$$

So, $z=w$. □

Corollary 13. *Let (X,d) be a complete b-metric space with constant $s\ge 1$ and F,G,H,L be self maps of X satisfying the following conditions $L(X)\subseteq F(X)$ and $H(X)\subseteq G(X)$ and*

$$
\begin{aligned}
&d(Hx,Ly)\\
&\le \alpha d(Fx,Gy)+\beta(d(Fx,Hx)+d(Gy,Ly))+\gamma(d(Fx,Ly)+d(Gy,Hx)),
\end{aligned}
$$

for all $x,y\in X$, where $\alpha,\beta,\gamma\ge 0$ and $s(\alpha+2\beta+\gamma+s\gamma)<1$. Assume that pairs (L,G) and (H,F) are weakly compatible. If one of $F(X)$ or $G(X)$ is a closed subspace of X, then F,G,H and L have a unique common fixed point in X.

Proof. Substitute K=J=I, the identity mapping in Theorem 12. □

Corollary 14. *Let (X,d) be a complete b-metric space with constant $s\ge 1$ and F,H,L be self maps of X satisfying the following conditions $L(X)\subseteq F(X)$ and $H(X)\subseteq F(X)$ and*

$$
\begin{aligned}
&d(Hx,Ly)\\
&\le \alpha d(Fx,Fy)+\beta(d(Fx,Hx)+d(Fy,Ly))+\gamma(d(Fx,Ly)+d(Fy,Hx)),
\end{aligned}
$$

for all $x,y\in X$, where $\alpha,\beta,\gamma\ge 0$ and $s(\alpha+2\beta+\gamma+s\gamma)<1$. Assume that pairs (L,F) and (H,F) are weakly compatible. If $F(X)$ is a closed subspace of X, then L,H, and F have a unique common fixed point in X.

Proof. Substitute K=J=I, the identity mapping and $F=G$ in Theorem 12. □

Corollary 15. *Let (X,d) be a complete b-metric space with constant $s\ge 1$ and A,B be self maps of X satisfying the following conditions*

$$d(Ax,By)\le \alpha d(x,y)+\beta(d(x,Ax)+d(y,By))+\gamma(d(x,By)+d(y,Ax))$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $s(\alpha + 2\beta + \gamma + s\gamma) < 1$. Then A and B have a unique common fixed point in X.

Proof. Substitute K=J=F=G=I, the identity mapping, $H = A$ and $L = B$ in Theorem 12. □

Corollary 16. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and T be a self map of X satisfying the following conditions*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta(d(x, Tx) + d(y, Ty)) + \gamma(d(x, Ty) + d(y, Tx))$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $s(\alpha + 2\beta + \gamma + s\gamma) < 1$. Then T has a unique common fixed point in X.

Proof. Substitute K=J=F=G=I, the identity mapping, $H = L = T$ in Theorem 12. □

Theorem 17. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and F, G, H, J, K, L be self maps of X satisfying the following conditions $KL(X) \subseteq F(X)$, $HJ(X) \subseteq G(X)$ and*

$$d(HJx, KLy) \leq q \max\Big\{d(Fx, Gy), d(Fx, HJx), d(Gy, KLy), \frac{1}{2}(d(Fx, KLy) + d(Gy, HJx))\Big\},$$

for all $x, y \in X$, where $0 < q < 1$ such that $s^2 q < 1$. Assume that pairs (KL, G) and (HJ, F) are weakly compatible. Pairs (K, L), (K, G), (L, G), (H, J), (H, F) and (J, F) are commuting pairs of maps. If one of $F(X)$ or $G(X)$ is a closed subspace of X, then F, G, H, J, K and L have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $KL(X) \subseteq F(X)$ and $HJ(X) \subseteq G(X)$, there exist x_1, x_2 in X such that $HJx_0 = Gx_1$ and $KLx_1 = Fx_2$. Again, there exist x_3, x_4 in X such that $HJx_2 = Gx_3$ and $KLx_3 = Fx_4$. By continuing this process, for each $n \in \mathbb{N} \cup \{0\}$, we can choose $x_n \in X$ such that $HJx_{2n} = Gx_{2n+1}$ and $KLx_{2n+1} = Fx_{2n+2}$. For each $n \in \mathbb{N} \cup \{0\}$, we let $y_{2n} = HJx_{2n} = Gx_{2n+1}$ and $y_{2n+1} = KLx_{2n+1} = Fx_{2n+2}$.

Consider,

$$d(y_{2n}, y_{2n+1}) = d(HJx_{2n}, KLx_{2n+1})$$

$$
\begin{aligned}
\leq & q \max\{d(Fx_{2n}, Gx_{2n+1}), d(Fx_{2n}, HJx_{2n}), d(Gx_{2n+1}, KLx_{2n+1}), \\
& \frac{1}{2}(d(Fx_{2n}, KLx_{2n+1}) + d(Gx_{2n+1}, HJx_{2n}))\} \\
= & q \max\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\
& \frac{1}{2}(d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n}))\} \\
\leq & q \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \\
& \frac{s}{2}(d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}))\} \\
\leq & qsd(y_{2n-1}, y_{2n}) = kd(y_{2n-1}, y_{2n}),
\end{aligned}
$$

where $0 < k = qs < 1$.

Similarly we obtain $d(y_{2n+1}, y_{2n+2}) \leq kd(y_{2n}, y_{2n+1})$.

Therefore,

$$
d(y_{n+1}, y_{n+2}) \leq kd(y_n, y_{n-1}) \leq ...k^{n+1}d(y_o, y_1),
$$

for $n = 1, 2, 3, ...$

Now, for every $m, n \in \mathbb{N}$ such that $m > n$, we have

$$
\begin{aligned}
d(y_n, y_m) \leq & sd(y_n, y_{n+1}) + s^2 d(y_{n+1}, y_{n+2}) + \cdots + s^{m-n} d(y_{m-1}, y_m) \\
\leq & (sk^n + s^2 k^{n+1} + \cdots + s^{m-n} k^{m-1}) d(y_1, y_o) \\
< & sk^n [1 + sk + (sk)^2 + ...] d(y_1, y_0) \\
= & \frac{sk^n}{1 - sk} d(y_1, y_0).
\end{aligned}
$$

Letting $n, m \to \infty$, we have $\{y_n\}$ is a Cauchy sequence. Since X is a complete b-metric space, there exists a point y^* in X such that lim $y_n = y^*$,

$$
\Rightarrow \lim HJx_{2n} = \lim Gx_{2n+1} = \lim KLx_{2n+1} = \lim Fx_{2n+2} = y^*.
$$

Suppose that $F(X)$ is a closed subspace of X. It follows that $y^* = Fu$ for some $u \in X$. Then, we have

$$
\begin{aligned}
d(HJu, y^*) \leq & s[d(HJu, KLx_{2n-1}) + d(KLx_{2n-1}, y^*)] \\
\leq & sq \max\{d(Fu, Gx_{2n-1}), d(Fu, HJu), d(Gx_{2n-1}, KLx_{2n-1})), \\
& \frac{1}{2}(d(Fu, KLx_{2n-1}) + d(Gx_{2n-1}, HJu))\} + sd(KLx_{2n-1}, y^*).
\end{aligned}
$$

Taking the limit as $n \to \infty$, we obtain

$$
d(HJu, y^*) \leq sq \max\{d(y^*, y^*), d(y^*, HJu), d(y^*, y^*),
$$

$$\frac{1}{2}(d(y^*,y^*)+d(y^*,HJu))\}+sd(y^*,y^*)$$
$$=sqd(HJu,y^*)$$
$$\Rightarrow\ (1-sq)d(HJu,y^*)\leq 0.$$

Therefore, $HJu=Fu=y^*$.

Again, since $HJ(X)\subseteq G(X)$, there exists a point v in X such that $y^*=Gv$. Then, we have

$$\begin{aligned}d(y^*,KLv)=&d(HJu,KLv)\\ \leq&q\max\{d(Fu,Gv),d(Fu,HJu)+d(Gv,KLv),\\ &\frac{1}{2}(d(Fu,KLv)+d(Gv,HJu))\}\\ =&q\max\{d(y^*,y^*),d(y^*,y^*),d(y^*,KLv)),\frac{1}{2}(d(y^*,KLv)+d(y^*,y^*))\\ =&qd(y^*,KLv)\\ \Rightarrow&\ (1-q)d(y^*,KLv)\leq 0.\end{aligned}$$

Therefore, $KLv=Gv=y^*$ and so $HJu=Fu=KLv=Gv=y^*$.

Since F and HJ are weakly compatible maps, therefore $HJFu=FHJu$ and so $HJy^*=Fy^*$.

Now we claim that y^* is a fixed point of HJ. Consider

$$\begin{aligned}d(HJy^*,y^*)=&d(HJy^*,KLv)\\ \leq&q\max\{d(Fy^*,Gv),d(Fy^*,HJy^*),d(Gv,KLv)),\\ &\frac{1}{2}(d(Fy^*,KLv)+d(Gv,HJy^*))\}\\ =&\ q\max\{d(HJy^*,y^*),d(HJy^*,HJy^*),d(y^*,y^*),\\ &\frac{1}{2}(d(HJy^*,y^*)+d(y^*,HJy^*))\}\\ =&qd(HJy^*,y^*)\\ \Rightarrow&\ (1-q)d(HJy^*,y^*)\leq 0.\end{aligned}$$

Therefore, $HJy^*=y^*$. Hence $HJy^*=Fy^*=y^*$.

Similarly G and KL are weakly compatible maps, so we have $KLy^*=Gy^*$.

Now we claim that y^* is a fixed point of KL. Consider

$$\begin{aligned}d(y^*,KLy^*)=&\ d(HJy^*,KLy^*)\\ \leq&q\max\{d(Fy^*,Gy^*),d(Fy^*,HJy^*),d(Gy^*,KLy^*),\end{aligned}$$

$$
\begin{aligned}
&\frac{1}{2}(d(Fy^*, KLy^*) + d(Gy^*, HJy^*))\} \\
=&q \max\{d(y^*, KLy^*), d(y^*, y^*), d(KLy^*, KLy^*), \\
&\frac{1}{2}(d(y^*, KLy^*) + d(KLy^*, y^*))\} \\
=&qd(y^*, KLy^*) \\
\Rightarrow \;& (1-q)d(y^*, KLy^*) \leq 0.
\end{aligned}
$$

Therefore, $KLy^* = y^*$. Hence $KLy^* = Gy^* = y^*$.
We have therefore proved that $HJy^* = KLy^* = Fy^* = Gy^* = y^*$. So y^* is common fixed point of F, G, HJ and KL.

By commuting conditions of pairs, we have

$$Ky^* = K(KLy^*) = K(LKy^*) = KL(Ky^*),$$

$$Ky^* = K(Fy^*) = F(Ky^*) \text{ and } Ly^* = L(KLy^*) = (LK)(Ly^*) = (KL)(Ly^*),$$

$$Ly^* = L(Fy^*) = F(Ly^*),$$

which show that Ky^* and Ly^* are common fixed points of (KL, F). Then $Ky^* = y^* = Ly^* = Fy^* = KLy^*$.

Similarly, $Hy^* = y^* = Jy^* = Gy^* = HJy^*$. Therefore y^* is a common fixed point of F, G, H, J, K and L.

Uniqueness. Let w be another common fixed point of F, G, H, J, K and L. Then, we have

$$
\begin{aligned}
d(y^*, w) =&d(HJy^*, KLw) \\
\leq&q \max\{d(Fy^*, Gw), d(Fy^*, HJy)^*, d(Gw, KLw), \\
&\frac{1}{2}(d(Fy^*, KLw) + d(Gw, HJ^*y))\} \\
=&q \max\{d(y^*, w), d(y^*, y^*), d(w, w), \frac{1}{2}(d(y^*, w) + d(w, y^*))\} \\
=&qd(y^*, w) \\
\Rightarrow \;& (1-q)d(y^*, w) \leq 0.
\end{aligned}
$$

So, $y^* = w$. $\square$

Corollary 18. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and F, G, H, L be self maps of X satisfying the following conditions $L(X) \subseteq F(X)$ and $H(X) \subseteq G(X)$ and*

$$d(Hx, Ly) \leq q \max\{d(Fx, Gy), d(Fx, Hx), d(Gy, Ly), \frac{1}{2}(d(Fx, Ly)+d(Gy, Hx))\},$$

for all $x, y \in X$, where $0 < q < 1$ such that $s^2 q < 1$. Assume that pairs (L, G) and (H, F) are weakly compatible. If one of $F(X)$ or $G(X)$ is a closed subspace of X, then F, G, H and L have a unique common fixed point in X.

Proof. Substitute K=J=I, the identity mapping in Theorem 17. □

Corollary 19. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and F, H, L be self maps of X satisfying the following conditions $L(X) \subseteq F(X)$ and $H(X) \subseteq F(X)$ and*

$$d(Hx, Ly) \leq q \max\{d(Fx, Fy), d(Fx, Hx), d(Fy, Ly), \frac{1}{2}(d(Fx, Ly)+d(Fy, Hx))\}$$

for all $x, y \in X$, where $0 < q < 1$ such that $s^2 q < 1$. Assume that pairs (L, F) and (H, F) are weakly compatible. If $F(X)$ is a closed subspace of X, then L, H, and F have a unique common fixed point in X.

Proof. Substitute K=J=I, the identity mapping and $F = G$ in Theorem 17. □

Corollary 20. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and A, B be self maps of X satisfying the following conditions*

$$d(Ax, By) \leq q \max\{d(x, y), d(x, Ax), d(y, By), \frac{1}{2}(d(x, By) + d(y, Ax))\},$$

for all $x, y \in X$, where $0 < q < 1$ such that $s^2 q < 1$. Then A and B have a unique common fixed point in X.

Proof. Substitute K=J=F=G=I, the identity mapping, $H = A$ and $L = B$ in Theorem 17. □

Corollary 21. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and T be a self map of X satisfying the following conditions*

$$d(Tx, Ty) \leq q \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(y, Tx))\},$$

for all $x, y \in X$, where $0 < q < 1$ such that $s^2 q < 1$. Then T has a unique common fixed point in X.

Proof. Substitute K=J=F=G=I, the identity mapping, $H = L = T$ in Theorem 17. □

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